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# Dynamical behaviors of Gilpin–Ayala competitive model with periodic coefficients on time scales

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**Abstract**

In the present paper we study the existence and stability problems of positive periodic solutions to a Gilpin–Ayala competitive model with periodic coefficients on time scales. Firstly, based on Schauder's fixed theorem, some sufficient conditions for the existence of positive periodic solution to the considered system are obtained. Furthermore, we establish asymptotic behavior by using the existence of periodic solutions. Since the considered system is based on an arbitrary time scale, our results are applicable to both discrete and continuous scenarios. We provide a specific example to verify the above results.

**Mathematics Subject Classification:** 34K13; 34D23

**Keywords:** Gilpin–Ayala competitive model; Schauder's fixed theorem; Positive periodic solution; Existence; Stability

## 1 Introduction

In 1973, Ayala, Gilpin and Eherenfeld [1] introduced the following competition population model:

$$\begin{aligned} \frac{dU(t)}{dt} &= a_1 U(t)\left[1 - \left(\frac{U(t)}{\beta_1}\right)^{\alpha_1} - b_{12}\frac{V(t)}{\beta_2}\right], \\ \frac{dV(t)}{dt} &= a_2 V(t)\left[1 - \left(\frac{V(t)}{\beta_2}\right)^{\alpha_2} - b_{21}\frac{U(t)}{\beta_1}\right], \end{aligned} \tag{1.1}$$

where $U(t)$ and $V(t)$ denote the population density at time $t$, $a_1$ and $a_2$ denote inherent growth ratios, $b_{12}$ and $b_{21}$ denote the measures of competition between species, $\beta_1$ and $\beta_2$ represent the maximum number of species in a completely noncompetitive environment, positive constants $\alpha_1$ and $\alpha_2$ can measure the degree of influence of nonlinear terms. System (1.1) is the so-called AG model which is a generalization of Lotka–Volterra system. After that, many results for the AG model and its generalizations have been obtained. Chen [2, 3] studied the permanence and extinction of nonautonomous GA competition model with delays. In [4], the authors investigated the GA competitive model with the effect of a toxic substance as follows:

$$
\begin{aligned}
x_1'(t) &= x_1(t)[a_1(t) - b_1(t)x_1^{\alpha_1}(t) - c_1(t)x_2^{\alpha_2}(t) - d_1(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)],\\
x_2'(t) &= x_2(t)[a_2(t) - b_2(t)x_1^{\alpha_1}(t) - c_2(t)x_2^{\alpha_2}(t) - d_2(t)x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)].
\end{aligned}
\tag{1.2}
$$

Based on a comparison theorem, some sufficient conditions for the extinction of system (1.2) have been obtained. For attraction, persistence and extinction, see [5–8]; for existence and stability, see [9, 10]; for the GA model with mixed delays and impulses, see [11–14].

In this article, we focus on the periodic solution problems of the AG model. Zhao [15] studied global exponential stability of positive periodic solutions for a class of multiple-species Gilpin–Ayala system with infinite distributed delays. Next, Zhao [16] further studied the following GA model with infinite distributed delays on time scales:

$$
\begin{aligned}
U^{\Delta}(t) &= a_1(t) - b_{11}(t)[e^{U(t)}]^{\alpha_1} - b_{12}(t)\int_{-\infty}^{0} k_1(s)e^{V(t+s)}\Delta s - \phi_1(t)e^{-U(t)},\\
V^{\Delta}(t) &= a_2(t) - b_{22}(t)[e^{V(t)}]^{\alpha_2} - b_{21}(t)\int_{-\infty}^{0} k_2(s)e^{U(t+s)}\Delta s - \phi_2(t)e^{-V(t)},
\end{aligned}
\tag{1.3}
$$

where $t \in \mathbb{T}$ which is a time sale and $\Delta$ is the delta (or Hilger) derivative. When $\mathbb{T} = \mathbb{R}$, letting $u(t) = e^{U(t)}$ and $v(t) = e^{V(t)}$, the system (1.3) is changed into the following form:

$$
\begin{aligned}
u'(t) &= u(t)\left[a_1(t) - b_{11}(t)[u(t)]^{\alpha_1} - b_{12}(t)\int_{-\infty}^{0} k_1(s)v(t+s)ds\right] - \phi_1(t),\\
v'(t) &= v(t)\left[a_2(t) - b_{22}(t)[v(t)]^{\alpha_2} - b_{21}(t)\int_{-\infty}^{0} k_2(s)u(t+s)ds\right] - \phi_2(t).
\end{aligned}
\tag{1.4}
$$

We find that if the system (1.3) has a solution $(U(t), V(t))^T$, then the system (1.4) has a positive solution $(e^{U(t)}, e^{V(t)})^T$ for $t \in \mathbb{T}$. However, a positive periodic solution of the system (1.4) is represented by $e$ exponential functions, not general positive functions. In this paper, we attempt to obtain the general form of a positive periodic solution for the system (1.4) on time scales. Hence, we study the following periodic GA model on time scales:

$$
\begin{aligned}
u^{\Delta}(t) &= a_1(t)u(\sigma(t)) - b_{11}(t)[u(t)]^{\alpha_1+1} - b_{12}(t)u(t)\int_{-\infty}^{0} k_1(s)v(t+s)\Delta s - \phi_1(t),\\
v^{\Delta}(t) &= a_2(t)v(\sigma(t)) - b_{22}(t)[v(t)]^{\alpha_2+1} - b_{21}(t)v(t)\int_{-\infty}^{0} k_2(s)u(t+s)\Delta s - \phi_2(t),
\end{aligned}
\tag{1.5}
$$

where $t \in \mathbb{T}$, which is a periodic time scale (see Definitions 2.1, 2.2), $\Delta$ is the delta (or Hilger) derivative, $u(t)$ and $v(t)$ denote the population densities at time $t$, $a_1$, $a_2 > 0$ denote inherent growth ratios, $b_{ij} > 0$ $(i,j = 1,2)$ denote measures of interaction between species, positive constants $\alpha_1$ and $\alpha_2$ can measure the degree of influence of nonlinear terms, $k_1(t)$, $k_2(t) > 0$ denote kernel functions corresponding to infinite distributed delays, $\phi_1(t)$, $\phi_2(t) > 0$ denote the measurement constants of nonlinear interference within species. The time scale dynamical system includes continuous and discrete systems, and its research has always been one of the hot topics in the study of differential dynamic systems, see [17–19].

The main contributions of this paper are listed as follows:

(1) We first study the GA model on time scales and obtain the existence of positive periodic solutions which have general forms which is different from corresponding results in [15, 16].

(2) We develop Schauder's fixed-point theorem for investigating differential systems on time scales.

(3) This study enriches and develops the research on the GA model, which can promote further research on the aforementioned system.

The remaining of the paper is organized as follows: Sect. 2 gives the preliminaries. In Sect. 3, some sufficient conditions for the existence of positive periodic solution of the system (1.5) are given. Section 4 gives asymptotic behavior of the system (1.5). In Sect. 5, an example is provided to show theoretical results. Finally, we provide some conclusions.

## 2 Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The specific meanings of the following symbols can be found in the book [20]: the backward jump operator is $\rho$, the forward jump operator is $\sigma$, regressive functions are denoted by $\mathcal{R}$ and positive regressive functions by $\mathcal{R}^+$. The interval $[x,y]_{\mathbb{T}}$ means $[x,y]\cap\mathbb{T}$. The intervals $(x,y]_{\mathbb{T}}$, $(x,y)_{\mathbb{T}}$, and $[x,y)_{\mathbb{T}}$ are defined similarly. Also $C_{rd}([a,\infty)_{\mathbb{T}})$ denotes the set of all rd-continuous functions on $[a,\infty)_{\mathbb{T}}$. For $s,t\in\mathbb{T}$, the exponential function $e_\delta(t,s)$ is defined by $e_\delta(t,s)=\exp\left(\int_s^t \xi_{\mu(\tau)}(\delta(\tau))\Delta\tau\right)$, where

$$\xi_{\mu(\tau)}(\delta(\tau))=\begin{cases}\frac{1}{\mu(\tau)}\log(1+\mu(\tau)\delta(\tau)), & \mu(\tau)>0,\\ \delta(\tau), & \mu(\tau)=0.\end{cases}$$

**Lemma 2.1** ([20]) *Let $\phi,\psi\in\mathcal{R}$. Then*

[i] $e_0(t,s)\equiv 1$ *and* $e_\phi(t,t)\equiv 1$;

[ii] $e_\phi(\rho(t),s)=(1-\mu(t)\phi(t))e_\phi(t,s)$;

[iii] $e_\phi(t,s)e_\psi(t,s)=e_{\phi\oplus\psi}(t,s)$.

[iv] $e_\phi(t,s)=\frac{1}{e_\phi(s,t)}=e_{\ominus\phi}(s,t)$;

[v] $e_\phi(t,s)e_\phi(s,r)=e_\phi(t,r)$.

**Definition 2.1** ([21]) A time scale $\mathbb{T}$ is periodic if there exists $\omega>0$ such that for each $\nu\in\mathbb{T}$ one has $\nu\pm\omega\in\mathbb{T}$. For $\mathbb{T}\neq\mathbb{R}$, the smallest such positive $\omega$ is the period of the time scale.

**Definition 2.2** ([21]) Let $\mathbb{T}\neq\mathbb{R}$ be a periodic time scale with period $\omega$. A function $f:\mathbb{T}\to\mathbb{R}$ is periodic with period $\mu$ if there exists a natural number $n$ such that $\mu=n\omega$ and $f(\nu\pm\mu)=f(\nu)$ for each $\nu\in\mathbb{T}$. When $\mathbb{T}=\mathbb{R}$, $f$ is a periodic function if $\kappa$ is the smallest positive number such that $f(\nu\pm\kappa)=f(\nu)$ for each $\nu\in\mathbb{T}$.

**Lemma 2.2** (Schauder's fixed point theorem [22]) *Let $\Xi$ be a convex, closed, and nonempty subset of a Banach space $\mathbb{B}$. Let $\Gamma:\Xi\to\Xi$ be a continuous mapping such that $\Gamma(\Xi)$ is a relatively compact subset of $\mathbb{B}$. Then $\Gamma$ has at least one fixed point in $\Xi$.*

## 3 Main results

Let

$$\mathbb{B} = \{z = (z_1, z_2)^T : z_1, z_2 \in C_{rd}(\mathbb{T}, \mathbb{R}),\ z_i(t+\omega) = z_i(t),\ i = 1, 2\}$$

with the norm $||z|| = \sup_{t\in\mathbb{T}} |z_1(t)| + \sup_{t\in\mathbb{T}} |z_2(t)|$, where $z \in \mathbb{B}$ and $\mathbb{T}$ is a periodic time scale. Under the above norm, $\mathbb{B}$ is a Banach space. For $L_1, L_2 > 0$, let

$$P_\omega(L_1, L_2) = \{z = (u, v)^T \in \mathbb{B} : L_1 \le u(t) \le L_2,\ L_1 \le v(t) \le L_2,\ t \in \mathbb{T}\}.$$

**Lemma 3.1** *Let $-a_1, -a_2 \in \mathcal{R}^+$, and $z = (u, v)^T \in P_\omega(L_1, L_2)$. The system (1.5) has a periodic solution $z = (u, v)^T \in P_\omega(L_1, L_2)$ if only if*

$$\begin{aligned} u(t) &= \frac{1}{e_{\ominus(-a_1)}(t, t-\omega) - 1} \int_{t-\omega}^{t} \Big[ b_{11}(s)[u(s)]^{\alpha_1+1} \\ &\quad + b_{12}(s)u(s) \int_{-\infty}^{0} k_1(\tau)v(s+\tau)\Delta\tau + \phi_1(s) \Big] e_{\ominus(-a_1)}(t, s)\Delta s, \\ v(t) &= \frac{1}{e_{\ominus(-a_2)}(t, t-\omega) - 1} \int_{t-\omega}^{t} \Big[ b_{22}(s)[v(s)]^{\alpha_2+1} \\ &\quad + b_{21}(s)v(s) \int_{-\infty}^{0} k_2(\tau)u(s+\tau)\Delta\tau + \phi_2(s) \Big] e_{\ominus(-a_2)}(t, s)\Delta s. \end{aligned} \tag{3.1}$$

*Proof* Change the first equation of the system (1.5) into the following form:

$$u^\Delta(t) - a_1(t)u(\sigma(t)) = -b_{11}(t)[u(t)]^{\alpha_1+1} - b_{12}(t)u(t) \int_{-\infty}^{0} k_1(s)v(t+s)\Delta s - \phi_1(t). \tag{3.2}$$

Multiplying both sides of (3.2) by $e_{-a_1}(t, 0)$ and integrating them from $t-\omega$ to $t$, we obtain

$$\begin{aligned} \int_{t-\omega}^{t} [e_{-a_1}(s, 0)u(s)]^\Delta \Delta s &= \int_{t-\omega}^{t} \Big[ b_{11}(s)[u(s)]^{\alpha_1+1} \\ &\quad + b_{12}(s)u(s) \int_{-\infty}^{0} k_1(\tau)v(s+\tau)\Delta\tau + \phi_1(s) \Big] e_{-a_1}(s, 0)\Delta s. \end{aligned} \tag{3.3}$$

Dividing both sides of (3.3) by $e_{-a_1}(t, 0)$, we get

$$\begin{aligned} u(t) &= \frac{1}{e_{\ominus(-a_1)}(t, t-\omega) - 1} \int_{t-\omega}^{t} \Big[ b_{11}(s)[u(s)]^{\alpha_1+1} \\ &\quad + b_{12}(s)u(s) \int_{-\infty}^{0} k_1(\tau)v(s+\tau)\Delta\tau + \phi_1(s) \Big] e_{\ominus(-a_1)}(t, s)\Delta s. \end{aligned}$$

Similar to the above proof, we obtain that the second equation of the system (3.1) holds. The proof is completed. □

*Remark* 3.1 We can verify that $e_{\ominus(-a_1)}(t,t-\omega)$ and $e_{\ominus(-a_2)}(t,t-\omega)$ do not depend on $t$. In fact, from the definition of $e_{\ominus a}(t,s)$, we have

$$\begin{aligned} e_{\ominus(-a_1)}(t,t-\omega) &= \exp\left(\int_{t-\omega}^{t}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s\right)\\ &= \exp\left(\int_{t-\omega}^{0}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s\right.\\ &\quad+\int_{0}^{\omega}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s\\ &\quad\left.+\int_{\omega}^{t}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s\right). \end{aligned}$$

Using the periodicity of $a_1$ and $\mu$, letting $s=u-\omega$, we have

$$\int_{t-\omega}^{0}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s=-\int_{t}^{\omega}\frac{\log(1+\ominus(-a_1(u))\mu(u))}{\mu(u)}\Delta u.$$

Thus,

$$e_{\ominus(-a_1)}(t,t-\omega)=exp\left(\int_{0}^{\omega}\frac{\log(1+\ominus(-a_1(s))\mu(s))}{\mu(s)}\Delta s\right).$$

We also obtain that $e_{\ominus(-a_2)}(t,t-\omega)$ does not depend on $t$.

*Remark* 3.2 Using the periodicity of the function $e_{\ominus(-a_1)}(t,s)$ for $s,t\in[0,\omega]_{\mathbb{T}}$, we can give its bounds as follows:

$$\begin{aligned} e_{\ominus(-a_1)}(t,s) &= \exp\left(\int_{s}^{t}\frac{\log(1+\ominus(-a_1(u))\mu(u))}{\mu(u)}\Delta u\right)\\ &= \exp\left(\int_{s}^{t}\frac{\log\frac{1}{1-a_1(u)\mu(u)}}{\mu(u)}\Delta u\right). \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_1 &:= \exp\left(-\int_{0}^{\omega}\frac{\log\frac{1}{1-a_1(u)\mu(u)}}{\mu(u)}\Delta u\right)\le e_{\ominus(-a_1)}(t,s)\\ &\le \exp\left(\int_{0}^{\omega}\frac{\log\frac{1}{1-a_1(u)\mu(u)}}{\mu(u)}\Delta u\right):=\lambda_2. \end{aligned}\tag{3.4}$$

Similar to the above proof, we also have

$$\lambda_3\le e_{\ominus(-a_2)}(t,s)\le\lambda_4,\tag{3.5}$$

where $\lambda_3=\exp\left(-\int_0^\omega\frac{\log\frac{1}{1-a_2(u)\mu(u)}}{\mu(u)}\Delta u\right)$, $\lambda_4=\exp\left(\int_0^\omega\frac{\log\frac{1}{1-a_2(u)\mu(u)}}{\mu(u)}\Delta u\right)$.

Now, we show that the system (1.5) has at least one positive periodic solution by the use of Schauder's fixed point theorem. For this, define the mapping $\Gamma:P_\omega(L_1,L_2)\to\mathbb{B}$ by

$$(\Gamma z)(t)=((\Gamma u)(t),(\Gamma v)(t))^T,\ t\in\mathbb{T},\ z=(u,v)^T,$$

where

$$(\Gamma u)(t) = \frac{1}{e_{\ominus(-a_1)}(t,t-\omega)-1}\int_{t-\omega}^{t}\Big[b_{11}(s)[u(s)]^{\alpha_1+1} + b_{12}(s)u(s)\int_{-\infty}^{0}k_1(\tau)v(s+\tau)\Delta\tau + \phi_1(s)\Big]e_{\ominus(-a_1)}(t,s)\Delta s,$$
$$(\Gamma v)(t) = \frac{1}{e_{\ominus(-a_2)}(t,t-\omega)-1}\int_{t-\omega}^{t}\Big[b_{22}(s)[v(s)]^{\alpha_2+1} + b_{21}(s)v(s)\int_{-\infty}^{0}k_2(\tau)u(s+\tau)\Delta\tau + \phi_2(s)\Big]e_{\ominus(-a_2)}(t,s)\Delta s. \tag{3.6}$$

In view of Lemma 3.1, the fixed points of $\Gamma$ are solutions of the system (1.5). Since $P_\omega(L_1,L_2)$ is equicontinuous and uniformly bounded, by Arzelá–Ascoli theorem, $P_\omega(L_1, L_2)$ is compact. Obviously, for each $z \in P_\omega(L_1,L_2)$, then $\Gamma z \in \mathbb{B}$. So, $\Gamma$ is well defined. To apply Schauder's fixed point theorem, it suffices to prove that $\Gamma$ is continuous and $\Gamma(P_\omega(L_1,L_2)) \subset P_\omega(L_1,L_2)$. Letting $f(t) \in C_{rd}(\mathbb{T},\mathbb{R})$ be a bounded function, denote

$$f^M = \sup_{t\in\mathbb{T}}|f(t)|,\ \ f^l = \inf_{t\in\mathbb{T}}|f(t)|.$$

Throughout this paper, we need the following assumptions:

($H_1$) The functions $a_i(t)$, $\phi_i(t)$, $b_{ij}(t) \in C_{rd}(\mathbb{T},\mathbb{R})$ are all $\omega$-periodic, where $i,j = 1,2$.

($H_2$) The kernel functions $k_1(t)$ and $k_2(t)$ satisfy

$$\int_{-\infty}^{0}k_1(s)\Delta s = K_1,\ \ \int_{-\infty}^{0}k_2(s)\Delta s = K_2,$$

where $K_1$ and $K_2$ are given positive constants.

($H_3$) The following inequalities are satisfied:

$$\eta_1\lambda_1\omega\Big(b_{11}^l L_1^{\alpha_1+1} + b_{12}^l L_1^2 K_1 + \phi_1^l\Big) \geq L_1, \tag{3.7}$$

$$\eta_1\lambda_2\omega\Big(b_{11}^M L_2^{\alpha_1+1} + b_{12}^M L_2^2 K_1 + \phi_1^M\Big) \leq L_2, \tag{3.8}$$

$$\eta_2\lambda_3\omega\Big(b_{22}^l L_1^{\alpha_2+1} + b_{21}^l L_1^2 K_2 + \phi_2^l\Big) \geq L_1, \tag{3.9}$$

$$\eta_2\lambda_4\omega\Big(b_{22}^M L_2^{\alpha_2+1} + b_{21}^M L_2^2 K_2 + \phi_2^M\Big) \leq L_2, \tag{3.10}$$

where $\eta_1$ and $\eta_2$ are defined by (3.11), $\lambda_1$ and $\lambda_2$ are defined by (3.4), $\lambda_3$ and $\lambda_4$ are defined by (3.5), $K_1$ and $K_2$ are defined by ($H_2$), $L_1$ and $L_2$ are positive constants with $L_1 < L_2$.

($H_4$) The following inequalities are satisfied:

$$\eta_1\Big(2b_{11}^M L_2^{\alpha_1+1}\lambda_2 + 2\lambda_2\phi_1^M + b_{11}^M L_2^{\alpha_1+1}\zeta_1 + 2b_{12}^M L_2^2 K_1\lambda_2 + b_{12}^M L_2^2\zeta_1\Big) \leq M,$$

$$\eta_2\Big(2b_{22}^M L_2^{\alpha_2+1}\lambda_4 + 2\lambda_4\phi_2^M + b_{22}^M L_2^{\alpha_2+1}\zeta_2 + 2b_{21}^M L_2^2 K_2\lambda_4 + b_{21}^M L_2^2\zeta_2\Big) \leq M,$$

where $M > 0$ is a given constant, $\zeta_1$ and $\zeta_2$ are defined by (3.25) and (3.26), respectively.

**Lemma 3.2** *If the assumptions* $(H_1)$ *and* $(H_2)$ *hold, then the operator* $\Gamma$ *is continuous on* $P_\omega(L_1,L_2)$.

*Proof* From Remark 3.1, let

$$\eta_1=\frac{1}{e_{\ominus(-a_1)}(t,t-\omega)-1},\ \eta_2=\frac{1}{e_{\ominus(-a_2)}(t,t-\omega)-1}, \tag{3.11}$$

where $\eta_1,\eta_2>0$ are constants. Let $z_1,z_2\in P_\omega(L_1,L_2)$, where $z_1=(u_1,v_1)^T$, $z_2=(u_2,v_2)^T$. From the first equation of the system (3.6), we have

$$\begin{aligned}&|(\Gamma u_1)(t)-(\Gamma u_2)(t)|\\&\le \eta_1\lambda_2 b_{11}^M\int_{t-\omega}^t |[u_1(s)]^{\alpha_1+1}-[u_2(s)]^{\alpha_1+1}|\Delta s\\&\quad+\eta_1\lambda_2 b_{12}^M\int_{t-\omega}^t\left|u_1(s)\int_{-\infty}^0 k_1(\tau)v_1(s+\tau)\Delta\tau-u_2(s)\int_{-\infty}^0 k_1(\tau)v_2(s+\tau)\Delta\tau\right|\Delta s.\end{aligned} \tag{3.12}$$

Using the mean value theorem, we have

$$\begin{aligned}|[u_1(s)]^{\alpha_1+1}-[u_2(s)]^{\alpha_1+1}|&=(\alpha_1+1)\xi^{\alpha_1}|u_1-u_2|\\&\le(\alpha_1+1)L_2^{\alpha_1}||z_1-z_2||,\end{aligned} \tag{3.13}$$

where $\xi$ is between $u_1$ and $u_2$. By assumption $(\mathrm{H}_2)$, we have

$$\begin{aligned}&\left|u_1(s)\int_{-\infty}^0 k_1(\tau)v_1(s+\tau)\Delta\tau-u_2(s)\int_{-\infty}^0 k_1(\tau)v_2(s+\tau)\Delta\tau\right|\\&\le\left|u_1(s)\int_{-\infty}^0 k_1(\tau)v_1(s+\tau)\Delta\tau-u_2(s)\int_{-\infty}^0 k_1(\tau)v_1(s+\tau)\Delta\tau\right|\\&\quad+\left|u_2(s)\int_{-\infty}^0 k_1(\tau)v_1(s+\tau)\Delta\tau-u_2(s)\int_{-\infty}^0 k_1(\tau)v_2(s+\tau)\Delta\tau\right|\\&\le K_1L_2||u_1-u_2||+K_1L_2||v_1-v_2||\\&\le 2K_1L_2||z_1-z_2||.\end{aligned} \tag{3.14}$$

From (3.12)–(3.14), we arrive at

$$|(\Gamma u_1)(t)-(\Gamma u_2)(t)|\le\left(\eta_1\lambda_2 b_{11}^M(\alpha_1+1)L_2^{\alpha_1}\omega+2\eta_1\lambda_2 b_{12}^M K_1L_2\omega\right)||z_1-z_2||. \tag{3.15}$$

Furthermore, from the second equation of the system (3.6), we have

$$\begin{aligned}&|(\Gamma v_1)(t)-(\Gamma v_2)(t)|\\&\le \eta_2\lambda_4 b_{22}^M\int_{t-\omega}^t |[v_1(s)]^{\alpha_2+1}-[v_2(s)]^{\alpha_2+1}|\Delta s\\&\quad+\eta_2\lambda_4 b_{21}^M\int_{t-\omega}^t\left|v_1(s)\int_{-\infty}^0 k_2(\tau)u_1(s+\tau)\Delta\tau-v_2(s)\int_{-\infty}^0 k_2(\tau)u_2(s+\tau)\Delta\tau\right|\Delta s.\end{aligned} \tag{3.16}$$

Using the mean value theorem, we have

$$\begin{aligned} |[v_1(s)]^{\alpha_1+1} - [v_2(s)]^{\alpha_2+1}| &= (\alpha_2+1)\zeta^{\alpha_2}|v_1 - v_2| \\ &\le (\alpha_2+1)L_2^{\alpha_2}||z_1 - z_2||, \end{aligned} \tag{3.17}$$

where $\zeta$ is between $v_1$ and $v_2$. By assumption $(H_2)$, we have

$$\begin{aligned} &\left| v_1(s)\int_{-\infty}^{0} k_2(\tau)u_1(s+\tau)\Delta\tau - v_2(s)\int_{-\infty}^{0} k_2(\tau)u_2(s+\tau)\Delta\tau \right| \\ &\le \left| v_1(s)\int_{-\infty}^{0} k_2(\tau)u_1(s+\tau)\Delta\tau - v_2(s)\int_{-\infty}^{0} k_2(\tau)u_1(s+\tau)\Delta\tau \right| \\ &\quad + \left| v_2(s)\int_{-\infty}^{0} k_2(\tau)u_1(s+\tau)\Delta\tau - v_2(s)\int_{-\infty}^{0} k_2(\tau)u_2(s+\tau)\Delta\tau \right| \\ &\le K_2L_2||v_1 - v_2|| + K_2L_2||u_1 - u_2|| \\ &\le 2K_2L_2||z_1 - z_2||. \end{aligned} \tag{3.18}$$

From (3.16)–(3.18), we arrive at

$$|(\Gamma v_1)(t) - (\Gamma v_2)(t)| \le \left( \eta_2\lambda_4 b_{22}^M(\alpha_2+1)L_2^{\alpha_2}\omega + 2\eta_2\lambda_4 b_{21}^M K_2L_2\omega \right)||z_1 - z_2||. \tag{3.19}$$

Thanks to (3.15) and (3.19),

$$\begin{aligned} |(\Gamma z_1)(t) - (\Gamma z_2)(t)| \le \Big( &\eta_1\lambda_2 b_{11}^M(\alpha_1+1)L_2^{\alpha_1}\omega + 2\eta_1\lambda_2 b_{12}^M K_1L_2\omega \\ &+ \eta_2\lambda_4 b_{22}^M(\alpha_2+1)L_2^{\alpha_2}\omega + 2\eta_2\lambda_4 b_{21}^M K_2L_2\omega \Big)||z_1 - z_2||. \end{aligned}$$

Therefore, $\Gamma$ is continuous on $P_\omega(L_1, L_2)$. □

**Lemma 3.3** *Suppose that the assumptions* $(H_1)$*–*$(H_3)$ *hold, then*

$$L_1 \le (\Gamma\vartheta_1)(t) \le L_2 \text{ and } L_1 \le (\Gamma\vartheta_2)(t) \le L_2 \text{ for all } \vartheta \in P_\omega(L_1, L_2),$$

*where* $\vartheta = (\vartheta_1, \vartheta_2)^T$.

*Proof* Since $\vartheta \in P_\omega(L_1, L_2)$, then

$$L_1 \le \vartheta_1 \le L_2, \quad L_1 \le \vartheta_2 \le L_2.$$

For $\vartheta \in P_{\omega}(L_1, L_2)$, from (3.7)–(3.10), we have

$$\begin{aligned}
(\Gamma\vartheta_1)(t) &= \frac{1}{e_{\ominus(-a_1)}(t,t-\omega)-1} \\
&\quad \times \int_{t-\omega}^{t} \Big[ b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1} + b_{12}(s)\vartheta_1(s) \\
&\quad \times \int_{-\infty}^{0} k_1(\tau)\vartheta_2(s+\tau)\Delta\tau + \phi_1(s) \Big] e_{\ominus(-a_1)}(t,s)\Delta s \\
&\geq \eta_1\lambda_1\omega\Big( b_{11}^{l} L_1^{\alpha_1+1} + b_{12}^{l} L_1^{2} K_1 + \phi_1^{l} \Big) \\
&\geq L_1,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
(\Gamma\vartheta_1)(t) &= \frac{1}{e_{\ominus(-a_1)}(t,t-\omega)-1} \\
&\quad \times \int_{t-\omega}^{t} \Big[ b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1} + b_{12}(s)\vartheta_1(s) \\
&\quad \times \int_{-\infty}^{0} k_1(\tau)\vartheta_2(s+\tau)\Delta\tau + \phi_1(s) \Big] e_{\ominus(-a_1)}(t,s)\Delta s \\
&\leq \eta_1\lambda_2\omega\Big( b_{11}^{M} L_2^{\alpha_1+1} + b_{12}^{M} L_2^{2} K_1 + \phi_1^{M} \Big) \\
&\leq L_2,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
(\Gamma\vartheta_2)(t) &= \frac{1}{e_{\ominus(-a_2)}(t,t-\omega)-1} \\
&\quad \times \int_{t-\omega}^{t} \Big[ b_{22}(s)[\vartheta_2(s)]^{\alpha_2+1} + b_{21}(s)\vartheta_2(s) \\
&\quad \times \int_{-\infty}^{0} k_2(\tau)\vartheta_1(s+\tau)\Delta\tau + \phi_2(s) \Big] e_{\ominus(-a_2)}(t,s)\Delta s \\
&\geq \eta_2\lambda_3\omega\Big( b_{22}^{l} L_1^{\alpha_2+1} + b_{21}^{l} L_1^{2} K_2 + \phi_2^{l} \Big) \\
&\geq L_1,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
(\Gamma\vartheta_2)(t) &= \frac{1}{e_{\ominus(-a_2)}(t,t-\omega)-1} \\
&\quad \times \int_{t-\omega}^{t} \Big[ b_{22}(s)[\vartheta_2(s)]^{\alpha_2+1} + b_{21}(s)\vartheta_2(s) \\
&\quad \times \int_{-\infty}^{0} k_2(\tau)\vartheta_1(s+\tau)\Delta\tau + \phi_2(s) \Big] e_{\ominus(-a_2)}(t,s)\Delta s \\
&\leq \eta_2\lambda_4\omega\Big( b_{22}^{M} L_2^{\alpha_2+1} + b_{21}^{l} L_2^{2} K_2 + \phi_2^{M} \Big) \\
&\leq L_2.
\end{aligned} \tag{3.23}$$

Due to (3.20)–(3.23), the proof is completed. □

**Lemma 3.4** *Assume that* $(H_1)$, $(H_2)$, *and* $(H_4)$ *hold, then*

$$|(\Gamma\vartheta_1)(t_2) - (\Gamma\vartheta_1)(t_1)| \le M|t_2 - t_1|$$

*and*

$$|(\Gamma\vartheta_2)(t_2) - (\Gamma\vartheta_2)(t_1)| \le M|t_2 - t_1|$$

*for all* $t_1, t_2 \in \mathbb{T}$, $\vartheta \in P_\omega(L_1, L_2)$.

*Proof* Let $t_1, t_2 \in [0, \omega]_{\mathbb{T}}$ with $t_1 < t_2$, $\vartheta \in P_\omega(L_1, L_2)$, then

$$\begin{aligned} &|(\Gamma\vartheta_1)(t_2) - (\Gamma\vartheta_1)(t_1)| \\ &\le \eta_1 \bigg| \int_{t_2-\omega}^{t_2} \Big( b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1} + \phi_1(s) \Big) e_{\ominus(-a_1)}(t_2, s)\Delta s \\ &\quad - \int_{t_1-\omega}^{t_1} \Big( b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1} + \phi_1(s) \Big) e_{\ominus(-a_1)}(t_1, s)\Delta s \bigg| \\ &\quad + \eta_1 \bigg| \int_{t_2-\omega}^{t_2} b_{12}(s)\vartheta_1(s) \int_{-\infty}^{0} k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_2, s)\Delta s \\ &\quad - \int_{t_1-\omega}^{t_1} b_{12}(s)\vartheta_1(s) \int_{-\infty}^{0} k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_1, s)\Delta s \bigg|. \end{aligned} \tag{3.24}$$

From the theory of time scales, we have

$$\begin{aligned} &|e_{\ominus(-a_1)}(t_2, s) - e_{\ominus(-a_1)}(t_1, s)| \\ &= \left| \exp\left( \int_s^{t_2} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) - \exp\left( \int_s^{t_1} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) \right| \\ &= \exp\left( \int_s^{t_2} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) \left| 1 - \exp\left( \int_{t_2}^{t_1} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) \right|, \end{aligned}$$

thus,

$$\begin{aligned} &\int_{t_1}^{t_1+\omega} |e_{\ominus(-a_1)}(t_2, s) - e_{\ominus(-a_1)}(t_1, s)|\Delta s \\ &\le \omega \exp\left( \int_0^{\omega} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) \left( \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \right)^M |t_2 - t_1| \\ &= \zeta_1 |t_2 - t_1|, \end{aligned} \tag{3.25}$$

where $\zeta_1 = \omega \exp\left( \int_0^{\omega} \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \Delta u \right) \left( \frac{\log \frac{1}{1-a_1(u)\mu(u)}}{\mu(u)} \right)^M$. Similar to the above proof, we also have

$$\int_{t_1}^{t_1+\omega} |e_{\ominus(-a_2)}(t_2, s) - e_{\ominus(-a_2)}(t_1, s)|\Delta s \le \zeta_2 |t_2 - t_1|, \tag{3.26}$$

where $\zeta_2 = \omega \exp\left(\int_0^\omega \frac{\log \frac{1}{1-a_2(u)\mu(u)}}{\mu(u)}\Delta u\right)\left(\frac{\log \frac{1}{1-a_2(u)\mu(u)}}{\mu(u)}\right)^M$. In view of (3.25) and (3.26), we have

$$\begin{aligned}
&\left|\int_{t_2-\omega}^{t_2}\left(b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1}+\phi_1(s)\right)e_{\ominus(-a_1)}(t_2,s)\Delta s\right.\\
&\quad\left.-\int_{t_1-\omega}^{t_1}\left(b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1}+\phi_1(s)\right)e_{\ominus(-a_1)}(t_1,s)\Delta s\right|\\
&\le\int_{t_1}^{t_2}\left(b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1}+\phi_1(s)\right)e_{\ominus(-a_1)}(t_2,s)\Delta s\\
&\quad+\int_{t_1+\omega}^{t_2+\omega}\left(b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1}+\phi_1(s)\right)e_{\ominus(-a_1)}(t_2,s)\Delta s\\
&\quad+\int_{t_1}^{t_1+\omega}\left(b_{11}(s)[\vartheta_1(s)]^{\alpha_1+1}+\phi_1(s)\right)|e_{\ominus(-a_1)}(t_2,s)-e_{\ominus(-a_1)}(t_1,s)|\Delta s\\
&\le 2\left(b_{11}^M L_2^{\alpha_1+1}+\phi_1^M\right)\lambda_2|t_2-t_1|+b_{11}^M L_2^{\alpha_1+1}\zeta_1|t_2-t_1|\\
&=\left(2b_{11}^M L_2^{\alpha_1+1}\lambda_2+2\lambda_2\phi_1^M+b_{11}^M L_2^{\alpha_1+1}\zeta_1\right)|t_2-t_1|.
\end{aligned}\tag{3.27}$$

We also have

$$\begin{aligned}
&\left|\int_{t_2-\omega}^{t_2}b_{12}(s)\vartheta_1(s)\int_{-\infty}^{0}k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_2,s)\Delta s\right.\\
&\left.-\int_{t_1-\omega}^{t_1}b_{12}(s)\vartheta_1(s)\int_{-\infty}^{0}k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_1,s)\Delta s\right|\\
&\le\int_{t_1}^{t_2}b_{12}(s)\vartheta_1(s)\int_{-\infty}^{0}k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_2,s)\Delta s\\
&\quad+\int_{t_1+\omega}^{t_2+\omega}b_{12}(s)\vartheta_1(s)\int_{-\infty}^{0}k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\, e_{\ominus(-a_1)}(t_2,s)\Delta s\\
&\quad+\int_{t_1}^{t_1+\omega}b_{12}(s)\vartheta_1(s)\int_{-\infty}^{0}k_1(\tau)\vartheta_2(s+\tau)\Delta\tau\,|e_{\ominus(-a_1)}(t_2,s)-e_{\ominus(-a_1)}(t_1,s)|\Delta s\\
&\le 2b_{12}^M L_2^2 K_1\lambda_2|t_2-t_1|+b_{11}^M L_2^{\alpha_1+1}\zeta_1|t_2-t_1|\\
&=\left(2b_{12}^M L_2^2 K_1\lambda_2+b_{12}^M L_2^2\zeta_1\right)|t_2-t_1|.
\end{aligned}\tag{3.28}$$

From (3.24), (3.27), (3.28), and assumption ($H_4$), we have

$$\begin{aligned}
&|(\Gamma\vartheta_1)(t_2)-(\Gamma\vartheta_1)(t_1)|\\
&\le\eta_1\left(2b_{11}^M L_2^{\alpha_1+1}\lambda_2+2\lambda_2\phi_1^M+b_{11}^M L_2^{\alpha_1+1}\zeta_1+2b_{12}^M L_2^2K_1\lambda_2+b_{12}^M L_2^2\zeta_1\right)|t_2-t_1|\\
&\le M|t_2-t_1|.
\end{aligned}\tag{3.29}$$

Similar the above proof, we also have

$$
\begin{aligned}
&|(\Gamma\vartheta_2)(t_2)-(\Gamma\vartheta_2)(t_1)|\\
&\le \eta_2\Big(2b_{22}^M L_2^{\alpha_2+1}\lambda_4+2\lambda_4\phi_2^M+b_{22}^M L_2^{\alpha_2+1}\zeta_2+2b_{21}^M L_2^2K_2\lambda_4+b_{21}^M L_2^2\zeta_2\Big)|t_2-t_1| \qquad (3.30)\\
&\le M|t_2-t_1|.
\end{aligned}
$$

Due to (3.29) and (3.30), the proof is completed. □

**Theorem 3.1** *Suppose that the assumptions* $(H_1)$–$(H_4)$ *hold. Then the system* (1.5) *has at least one positive* $\omega$*-periodic solution on* $P_\omega(L_1,L_2)$.

*Proof* From Lemma 3.1, it is easy to see that the system (1.5) has a solution $z$ on $P_\omega(L_1,L_2)$ if only if the operator $\Gamma$ defined by (3.6) has a fixed point. In view of Lemmas 3.2–3.4, all the conditions of Schauder's fixed point theorem are satisfied. Therefore, $\Gamma$ has at least one fixed point on $P_\omega(L_1,L_2)$ and this fixed point is a positive periodic solution of the system (1.5). □

## 4 Asymptotic behavior of positive periodic solutions

**Definition 4.1** For a periodic solution $z$ of the system (1.5), we define the following asymptotic behavior: if, for a given constant $\delta$, there exists a positive constant $\varepsilon=\varepsilon(\delta)$ such that

$$||z(t)-\tilde z(t)||\le\delta \text{ for all } t\in[0,\infty)_{\mathbb{T}},$$

whenever $||z(0)-\tilde z(0)||\le\varepsilon$, where $\tilde z$ is a solution of the system (1.5), $\delta>\Lambda_1$, $\Lambda_1$ is defined by (4.6).

**Theorem 4.1** *If all the conditions of Theorem* 3.1 *hold, then the solution of the system* (1.5) *has the asymptotic behavior which is defined by Definition* 4.1.

*Proof* Since all the conditions of Theorem 3.1 hold, the system (1.5) has a positive periodic solution $u(t)=(u_1(t),v_1(t))^T$. Let $\tilde u(t)=(\tilde u_1(t),\tilde v_1(t))^T$ be another solution of the system (1.5). Similarly as in the proof of Lemma 3.2, we have

$$
\begin{aligned}
|u_1(t)-\tilde u_1(t)|\le{}&\eta_1\lambda_2 b_{11}^M\int_{t-\omega}^t|[u_1(s)]^{\alpha_1+1}-[\tilde u_1(s)]^{\alpha_1+1}|\Delta s\\
&+\eta_1\lambda_2 b_{12}^M\int_{t-\omega}^t\Big|u_1(s)\int_{-\infty}^0k_1(\tau)v_1(s+\tau)\Delta\tau \qquad (4.1)\\
&-\tilde u_1(s)\int_{-\infty}^0k_1(\tau)\tilde v_1(s+\tau)\Delta\tau\Big|\Delta s.
\end{aligned}
$$

Obviously, we have

$$
\begin{aligned}
|[u_1(s)]^{\alpha_1+1}-[\tilde u_1(s)]^{\alpha_1+1}|={}&|[u_1(s)]^{\alpha_1+1}-[\tilde u_1(s)]^{\alpha_1+1}\\
&+u_1(0)-\tilde u_1(0)-u_1(0)+\tilde u_1(0)| \qquad (4.2)\\
\le{}&2L_2^{\alpha_1+1}+2L_2+|u_1(0)-\tilde u_1(0)|.
\end{aligned}
$$

We also have

$$\left|u_1(s)\int_{-\infty}^{0} k_1(\tau)v_1(s+\tau)\Delta\tau - \tilde{u}_1(s)\int_{-\infty}^{0} k_1(\tau)\tilde{v}_1(s+\tau)\Delta\tau\right| \le 2K_1L_2^2. \tag{4.3}$$

From (4.1)–(4.3), we obtain

$$\begin{aligned}|u_1(t)-\tilde{u}_1(t)| \le{}& 2\omega\eta_1\lambda_2 b_{11}^M L_2^{\alpha_1+1} + 2\omega\eta_1\lambda_2 b_{11}^M L_2 + \omega\eta_1\lambda_2 b_{11}^M |u_1(0)-\tilde{u}_1(0)| \\ &+ 2\omega\eta_1\lambda_2 b_{12}^M K_1 L_2^2.\end{aligned} \tag{4.4}$$

Similar to the above proof, we have

$$\begin{aligned}|v_1(t)-\tilde{v}_1(t)| \le{}& 2\omega\eta_2\lambda_4 b_{22}^M L_2^{\alpha_2+1} + 2\omega\eta_2\lambda_2 b_{22}^M L_2 + \omega\eta_2\lambda_4 b_{22}^M |v_1(0)-\tilde{v}_1(0)| \\ &+ 2\omega\eta_2\lambda_4 b_{21}^M K_2 L_2^2.\end{aligned} \tag{4.5}$$

Due to (4.4) and (4.5), then

$$||u(t)-\tilde{u}(t)|| \le \Lambda_1 + \Lambda_2 ||u(0)-\tilde{u}(0)||, \tag{4.6}$$

where

$$\begin{aligned}\Lambda_1 ={}& 2\omega\eta_1\lambda_2 b_{11}^M L_2^{\alpha_1+1} + 2\omega\eta_1\lambda_2 b_{11}^M L_2 + 2\omega\eta_1\lambda_2 b_{12}^M K_1 L_2^2 \\ &+ 2\omega\eta_2\lambda_4 b_{22}^M L_2^{\alpha_2+1} + 2\omega\eta_2\lambda_2 b_{22}^M L_2 + 2\omega\eta_2\lambda_4 b_{21}^M K_2 L_2^2, \\ \Lambda_2 ={}& \omega\eta_1\lambda_2 b_{11}^M + \omega\eta_2\lambda_4 b_{22}^M.\end{aligned}$$

Now, by choosing $\varepsilon \le \frac{\delta-\Lambda_1}{\Lambda_2}$, in view of (4.6), we get

$$||u(t)-\tilde{u}(t)|| \le \delta \text{ for all } t \in [0,\infty)_{\mathbb{T}}.$$

Hence, the solution of the system (1.5) has the asymptotic behavior which is defined by Definition 4.1. □

## 5 An example

When $\mathbb{T} = \mathbb{R}$, then the system (1.5) is changed into the following form:

$$\begin{aligned} u'(t) &= u(t)\left[a_1(t) - b_{11}(t)[u(t)]^{\alpha_1} - b_{12}(t)\int_{-\infty}^{0} k_1(s)v(t+s)ds\right] - \phi_1(t), \\ v'(t) &= v(t)\left[a_2(t) - b_{22}(t)[v(t)]^{\alpha_2} - b_{21}(t)\int_{-\infty}^{0} k_2(s)u(t+s)ds\right] - \phi_2(t), \end{aligned} \tag{5.1}$$

where $t \in \mathbb{R}$. Let

$$\begin{aligned} &a_1(t) = 9.5 + 2\cos\frac{20\pi}{9}t, \qquad a_2(t) = 10 + \sin\frac{20\pi}{9}t, \\ &b_{11}(t) = 0.06 + 0.03\cos\frac{20\pi}{9}t, \qquad b_{12}(t) = 0.04 + 0.02\sin\frac{20\pi}{9}t, \\ &b_{22}(t) = 0.05 + 0.01\cos\frac{20\pi}{9}t, \qquad b_{21}(t) = 0.02 + 0.01\sin\frac{20\pi}{9}t, \end{aligned}$$

$$k_1(t) = \frac{2}{\pi}\frac{1}{1+t^2}, \qquad k_2(t) = e^{2t},$$
$$\phi_1(t) = \frac{2+\cos\frac{20\pi}{9}t}{100}, \qquad \phi_2(t) = \frac{3+\cos\frac{20\pi}{9}t}{100},$$
$$\alpha_1 = 0.2, \qquad \alpha_2 = 0.5.$$

By simple calculation, we have

$$\begin{aligned}
&\omega = 0.9,\ \eta_1 \approx 1.23\times 10^{-26},\ \eta_2 \approx 6.57\times 10^{-24},\ \lambda_1 \approx 1.2\times 10^{-26},\ \lambda_2 \approx 8.12\times 10^{25},\\
&\lambda_3 \approx 5.3\times 10^{-28},\ \lambda_4 \approx 1.7\times 10^{27},\ b_{11}^l = 0.03,\ b_{11}^M = 0.09,\ b_{12}^l = 0.02,\ b_{12}^M = 0.06,\\
&b_{21}^l = 0.01,\ b_{21}^M = 0.03,\ b_{22}^l = 0.04,\ b_{22}^M = 0.06,\\
&\phi_1^l = 0.01,\ \phi_1^M = 0.03,\ \phi_2^l = 0.02,\ \phi_2^M = 0.04,\ K_1 = 1,\ K_2 = 0.5.
\end{aligned}$$

Choosing $L_1 = 1.5\times 10^{-55}$, $L_2 = 1$, we get

$$\begin{aligned}
\eta_1\lambda_1\omega\left(b_{11}^l L_1^{\alpha_1+1} + b_{12}^l L_1^2 K_1 + \phi_1^l\right) &\approx 1.415\times 10^{-52} \ge L_1,\\
\eta_1\lambda_2\omega\left(b_{11}^M L_2^{\alpha_1+1} + b_{12}^M L_2^2 K_1 + \phi_1^M\right) &\approx 0.162 \le L_2,\\
\eta_2\lambda_3\omega\left(b_{22}^l L_1^{\alpha_2+1} + b_{21}^l L_1^2 K_2 + \phi_2^l\right) &\approx 1.306\times 10^{-52} \ge L_1,\\
\eta_2\lambda_4\omega\left(b_{22}^M L_2^{\alpha_2+1} + b_{21}^M L_2^2 K_2 + \phi_2^M\right) &\approx 0.157 \le L_2.
\end{aligned}$$

So all the conditions of Theorems 3.1 and 4.1 are satisfied, thus the solution of the system (5.1) has the asymptotic behavior which is defined by Definition 4.1. A simulation of the dynamic behavior of the system (5.1) is given in Fig. 1.

When $\mathbb{T} = \mathbb{Z}$, then the system (1.5) is changed into the following form:

$$\begin{aligned}
\Delta u(k) &= a_1(k)u(k+1) - b_{11}(k)[u(k)]^{\alpha_1+1} - b_{12}(k)u(k)\sum_{s=-\infty}^{0} k_1(s)v(k+s) - \phi_1(k),\\
\Delta v(k) &= a_2(k)v(k+1) - b_{22}(t)[v(k)]^{\alpha_2+1} - b_{21}(k)v(k)\sum_{s=-\infty}^{0} k_2(s)u(k+s) - \phi_2(k),
\end{aligned} \tag{5.2}$$

where $k\in\mathbb{Z}$, $\Delta u(k) = u(k+1) - u(k)$, $\Delta v(k) = v(k+1) - v(k)$. Let

$$\begin{aligned}
&a_1(k) = 10 + 1.5\cos\frac{5\pi}{2}k,\ a_2(k) = 9 + 1.2\sin\frac{5\pi}{2}k,\\
&b_{11}(k) = 0.05 + 0.01\cos\frac{5\pi}{2}k,\ b_{12}(k) = 0.05 + 0.02\sin\frac{5\pi}{2}k,\\
&b_{22}(k) = 0.07 + 0.02\cos\frac{5\pi}{2}k,\ b_{21}(k) = 0.04 + 0.01\sin\frac{5\pi}{2}k,\ k_1(s) = k_2(s) = \frac{1}{100}e^s,\\
&\phi_1(k) = \frac{3+2\cos\frac{5\pi}{2}k}{100},\ \phi_2(t) = \frac{4+3\cos\frac{5\pi}{2}k}{100},\ \alpha_1 = 0.4,\ \alpha_2 = 0.6.
\end{aligned}$$

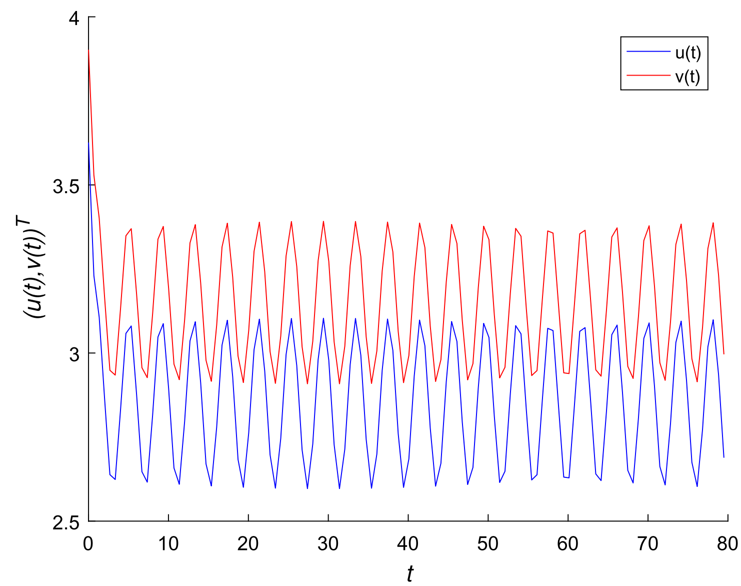


**Figure 1** Existence and asymptotic behavior of a positive periodic solution to the system (5.1)

By simple calculation, we have

$$\begin{aligned}&\omega = 0.8,\ \eta_1 \approx 5.62 \times 10^{-8},\ \eta_2 \approx 4.17 \times 10^{-6},\ \lambda_1 \approx 5.621 \times 10^{-8},\ \lambda_2 \approx 1.62 \times 10^{7},\\ &\lambda_3 \approx 4.171 \times 10^{-6},\ \lambda_4 \approx 1.82 \times 10^{5},\ b_{11}^l = 0.04,\ b_{11}^M = 0.06,\ b_{12}^l = 0.03,\ b_{12}^M = 0.07,\\ &b_{21}^l = 0.03,\ b_{21}^M = 0.05,\ b_{22}^l = 0.05,\ b_{22}^M = 0.09,\\ &\phi_1^l = 0.01,\ \phi_1^M = 0.05,\ \phi_2^l = 0.01,\ \phi_2^M = 0.07,\ K_1 = K_2 \approx 0.038.\end{aligned}$$

Choosing $L_1 = 1.25 \times 10^{-20}$, $L_2 = 1$, we get

$$\begin{aligned}\eta_1\lambda_1\omega\left(b_{11}^l L_1^{\alpha_1+1} + b_{12}^l L_1^2 K_1 + \phi_1^l\right) &\approx 2.53 \times 10^{-17} \geq L_1,\\ \eta_1\lambda_2\omega\left(b_{11}^M L_2^{\alpha_1+1} + b_{12}^M L_2^2 K_1 + \phi_1^M\right) &\approx 0.073 \leq L_2,\\ \eta_2\lambda_3\omega\left(b_{22}^l L_1^{\alpha_2+1} + b_{21}^l L_1^2 K_2 + \phi_2^l\right) &\approx 1.39 \times 10^{-15} \geq L_1,\\ \eta_2\lambda_4\omega\left(b_{22}^M L_2^{\alpha_2+1} + b_{21}^M L_2^2 K_2 + \phi_2^M\right) &\approx 0.021 \leq L_2.\end{aligned}$$

So all the conditions of Theorems 3.1 and 4.1 are satisfied, thus the solution of the system (5.2) has the asymptotic behavior which is defined by Definition 4.1. A simulation of the dynamic behavior of the system (5.2) is given in Fig. 2.

## 6 Conclusions

The GA model has been one of the hot research topics in the past few decades. This ecosystem contains a large number of dynamic properties and ecological significance. This paper deals with a classic GA model with periodic coefficients and distributed delays on time

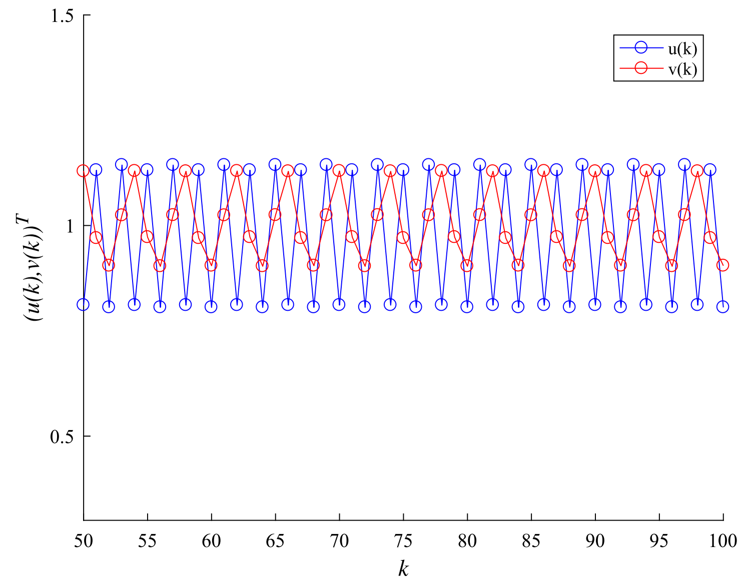


**Figure 2** Existence and asymptotic behavior of a positive periodic solution to the system (5.2)

scales. Using Schauder's fixed theorem, we obtain sufficient criteria for the existence of a positive periodic solution to the system (1.5). Furthermore, we obtian the asymptotic behavior of solution by using inequality techniques.

There are still many issues to be studied for the system (1.5). For example, in the system (1.5), there could be pulse or random terms.

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**Author contributions**
BD dealt with the writing-original draft preparation. SZ and XL dealt with the conceptualization, supervision, and methodology. All authors read and approved the final manuscript.

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**Data availability**
Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

## Declarations

**Ethics approval and consent to participate**
Not applicable.

**Competing interests**
The authors declare no competing interests.

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