Advances in Continuous and Discrete Models

RESEARCH Open Access

Global Hopf bifurcation and positive periodic solutions of a delayed diffusive predator–prey model with weak Allee effect for predator

Xiaohui Zhang[1], Xiaofeng Xu[2*] and Ming Liu[3]

*Correspondence: xxf_ray@163.com
[2]Engineering Research Center of Agricultural Microbiology Technology, Ministry of Education & Heilongjiang Provincial Key Laboratory of Ecological Restoration and Resource Utilization for Cold Region & School of Mathematical Science, Heilongjiang University, Harbin 150080, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate a delayed diffusive predator–prey model with weak Allee effect for predator. First, we discuss the existence and uniqueness of positive steady state and the local Hopf bifurcation. Next, we obtain the permanence and uniform boundedness of positive periodic solutions of the delayed reaction–diffusion system. The existence range of positive periodic solutions is further compressed by using iterative approach. For the system without delay, we prove the global attractiveness of unique positive steady state by using Lyapunov function, which ensures that the periods of positive periodic solutions are uniformly bounded. Then we obtain the global Hopf bifurcation and extended existence of positive periodic solutions by using the global Hopf bifurcation theorem of partial functional differential equation. Finally, the validity of the conclusion is verified through numerical simulation.

Keywords: Predator–prey; Global Hopf bifurcation; Weak Allee effect; Diffusion; Delay

1 Introduction

Periodic fluctuation is a prevalent phenomenon in the real world, and numerous biological species have been observed to exhibit periodic oscillations. Nicholson's blowfly experiments in 1954 demonstrated that the number of blowflies in petri dishes fluctuates in roughly 30-day cycles [1, 2]. Finerty's data reveal that Canadian snow rabbits and lynx populations fluctuate in roughly 10-year cycles, whereas lemmings and Arctic foxes fluctuate in roughly 4-year cycles [3]. Spruce budworms have been erupting periodically for about 40 years since the availability of statistics in North America [4–7]. Additionally, several major pine caterpillars in China, such as Larch caterpillars, Dendrolimus tabulaeformis, and Dendrolimus punctatus, also display periodic fluctuations [8].

How to explain the periodic fluctuations of these species? Scholars usually use periodic solutions of differential equations to describe these periodic fluctuations. Hopf bifurcation is widely recognized as an important mechanism for generating periodic solutions within aperiodic systems. Hence it has garnered significant attention from researchers over recent decades [9–18]. Local Hopf bifurcation can establish the existence of periodic solutions when the bifurcation parameter falls within a sufficiently small neighborhood. However, during the process of establishing the mathematical model, appropriate simplification and hypothesis are essential. Due to the influence of neglected factors, the bifurcation parameter value of the model should vary within a certain range, and this range of changes is likely to be relatively large. In these cases the local Hopf bifurcation does not explain the periodic fluctuations well, and we need to use the global Hopf bifurcation.

Due to their importance in the field of population ecology, predator–prey systems have been extensively investigated [19–26]. In small-scale populations, some challenges, such as small yield, dysfunctional social settlements, difficulty finding mates, or difficulty in resisting natural enemies, often affect the population sizes. The positive relationship between any component of individual fitness and the number or density of the same species is referred to as the Allee effect [27]. The Allee effect is a fundamental principle in species ecology, and research has shown that it significantly influences the integrity and dynamics of populations. Thus there have been extensive investigations of predator–prey models incorporating the Allee effect (see [28–31]). The Allee effect can be categorized into strong and weak Allee effects. In cases with a strong Allee effect, there exists a threshold below which fertility is lower than mortality, resulting in further population decline and eventual extinction. On the other hand, weak Allee effects do not have this threshold; instead, population growth remains positive even when close to zero, although at a slower rate. Intuitively, higher animals are more susceptible to experiencing Allee effects compared to lower animals due to their smaller population sizes.

A Leslie–Gower model with Holling III functional response and extra foods for predator is as follows:

$$\begin{cases} u_t = ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a + u^2}, \\ v_t = \delta v\left(1 - \dfrac{hv}{m + u}\right), \end{cases}$$

where, $u = u(t)$ and $v = v(t)$ are the population densities of prey and predator at time t, respectively, $r > 0$ and $\delta > 0$ are the intrinsic growth rates of the prey and predator, respectively, $k > 0$ is the carrying capacity of the prey population in the absence of predation, $b > 0$ is the maximum value of the per capita reduction rate of the prey due to existence of predator, $a > 0$ is the half-saturation constant of prey, $h > 0$ is the proportionality coefficient of prey density to the carrying capacity for the predator, and $m > 0$ is another constant food resource of predator.

In view of the movement of species from high-density areas to low-density areas and digestion time for predator, we obtain the following delayed reaction–diffusion model:

$$\begin{cases} u_t = d_1 \Delta u + ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a + u^2}, \\ v_t = d_2 \Delta v + \delta v\left(1 - \dfrac{hv}{m + u_\tau}\right), \end{cases}$$

where $u = u(x, t)$ and $v = v(x, t)$ are the population densities of prey and predator at location x and time t, respectively, $u_\tau = u(x, t - \tau)$, $\tau > 0$ is the digestion time for predator, and the positive constants d_1 and d_2 are the diffusion coefficients of prey and predator, respectively.

Moreover, considering that the predator is affected by a weak Allee effect, we replace the intrinsic growth rate δ by $\frac{\delta v}{b_1+v}$, where $b_1 > 0$ describes the strength of the corresponding Allee effect. In this paper, we consider the following delayed reaction–diffusion model under Neumann boundary condition:

$$\begin{cases} u_t = d_1\Delta u + ru\Big(1 - \dfrac{u}{k}\Big) - \dfrac{bu^2 v}{a+u^2}, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v + \delta v\Big(\dfrac{v}{b_1+v} - \dfrac{hv}{m+u_\tau}\Big), & \\ u_\nu = v_\nu = 0, \quad x \in \partial\Omega, t > 0, & \\ u(x,0) \geq 0, v(x,0) \geq 0, \quad x \in \bar{\Omega}, & \end{cases} \tag{1}$$

where $\Omega = (0, l\pi)$, and ν is the outward unit normal vector on $\partial\Omega$.

The main purpose of this paper is to investigate the global Hopf bifurcation and extended existence of positive periodic solutions for Eq. (1) using the global Hopf bifurcation theorem presented by Wu [32]. We would like to mention that due to the existence of spatial factors, the existence of global Hopf bifurcation in a partial functional differential equation is very difficult to obtain. The results achieved are also relatively few (see [33–37]). Wu utilized degree theory to establish the global Hopf bifurcation theorem of abstract partial functional differential equations, which avoids the spatial decomposition theory of solution operators and complex linear functional differential equations. However, the prerequisite for the application of this theorem is to establish the complete structure of constant and nonconstant steady states and nearby local Hopf bifurcation. Moreover, some properties of periodic solutions, such as uniform boundedness, for the system with delay and diffusion need to be proved. It is very difficult to prove these properties in delayed reaction–diffusion models with real background.

The organizational structure of this paper is as follows. In Sect. 2, we give our main results about the global Hopf bifurcation of Eq. (1). In Sect. 3, we provide a detailed proof of the conclusions of Sect. 2. In Sect. 4, we present some numerical simulations.

2 Main results

The positive constant steady states (u_0, v_0) of Eq. (1) satisfy

$$r\Big(1 - \frac{u_0}{k}\Big) - \frac{bu_0v_0}{a+u_0^2} = 0, \quad \frac{1}{b_1+v_0} - \frac{h}{m+u_0} = 0, \tag{2}$$

that is,

$$\begin{aligned} &\Phi(u_0) = r\Big(1 - \frac{u_0}{k}\Big) - \frac{bu_0}{a+u_0^2}\Big(\frac{m+u_0}{h} - b_1\Big) = 0, \\ &u_0 > b_1h - m, \qquad v_0 = \frac{m+u_0}{h} - b_1. \end{aligned} \tag{3}$$

Theorem 2.1 *If one of (a)–(e) holds, then $\Phi(u_0) = 0$ has a unique positive root u_0. In this case, further assuming that $u_0 > b_1h - m$, Eq. (1) has a unique positive constant steady state*

(u_0, v_0).

(a) $Ł \le 0$;

(b) $Ł > 0, \quad B_1 \le 0$;

(c) $Ł > 0, \quad B_1 > 0, \quad C_1 \ge 0$;

(d) $Ł > 0, \quad B_1 > 0, \quad C_1 < 0, \quad \Psi(u_1) > 0$;

(e) $Ł > 0, \quad B_1 > 0, \quad C_1 < 0, \quad \Psi(u_2) > 0$,

where

$$
\begin{aligned}
&A_1 = -\frac{rh}{k}, && B_1 = rh - b,\\
&C_1 = bb_1h - \frac{rah}{k} - bm, && D_1 = arh,\\
&Ł = 4B_1^2 - 12A_1C_1, && \Psi(u) = A_1u^3 + B_1u^2 + C_1u + D_1,\\
&u_1 = \frac{-B_1 - \sqrt{B_1^2 - 3A_1C_1}}{3A_1}, && u_2 = \frac{-B_1 + \sqrt{B_1^2 - 3A_1C_1}}{3A_1}.
\end{aligned}
$$

For convenience, we introduce

$$(\aleph_1): \text{ one of } (a)\text{–}(e) \text{ holds, and } u_0 > b_1h - m.$$

Under condition $(\aleph_1)$, Eq. (1) has a unique positive constant steady state. Next, we take τ as the bifurcation parameter to discuss the stability of the positive constant steady state (u_0, v_0). Define the real-valued Sobolev space

$$X = \{(u,v) \in H^2(0, l\pi) \times H^2(0, l\pi) | u_x = v_x = 0, x = 0, l\pi\}$$

and the abstract space $\mathcal{C} = C([-\tau, 0], X)$. The linearization of Eq. (1) at (u_0, v_0) can be rewritten as a differential equation in the phase space $\mathcal{C}$,

$$\dot{U}_t = D\Delta U(x,t) + AU(x,t) + BU(x,t-\tau), \tag{4}$$

where $U(x,t) = (u(x,t), v(x,t))^T$, $D = diag(d_1, d_2)$, and

$$
\begin{aligned}
A &= \begin{pmatrix} -\vartheta(u_0) & -\gamma(u_0) \\ 0 & -\delta\kappa(u_0) \end{pmatrix},\\
B &= \begin{pmatrix} 0 & 0 \\ \frac{\delta}{h}\kappa(u_0) & 0 \end{pmatrix}.
\end{aligned}
$$

The characteristic equation of the linearized system is

$$\lambda^2 + E_n\lambda + F_n + He^{-\lambda\tau} = 0, \quad n \in N_0, \tag{5}$$

where $N_0 = N \cup \{0\}$, N is the set of positive integers, and

$$\begin{aligned} E_n &= (d_1 + d_2)\left(\frac{n}{l}\right)^2 + \vartheta(u_0) + \delta\kappa(u_0), \\ F_n &= d_1 d_2 \left(\frac{n}{l}\right)^4 + \left(d_1\delta\kappa(u_0) + d_2\vartheta(u_0)\right)\left(\frac{n}{l}\right)^2 + \delta\vartheta(u_0)\kappa(u_0), \\ H &= \frac{\delta}{h}\kappa(u_0)\gamma(u_0), \end{aligned} \tag{6}$$

with

$$\begin{aligned} \vartheta(u_0) &= -r\left[1 - \frac{2\left(u_0^3 + ak\right)}{k(a + u_0^2)}\right], \\ \kappa(u_0) &= \left(1 - \frac{hb_1}{m + u_0}\right)^2, \\ \gamma(u_0) &= \frac{bu_0^2}{a + u_0^2}. \end{aligned} \tag{7}$$

Then we intoduce

$$(\aleph_2): \ \vartheta(u_0) > \max\left\{ -\delta\kappa(u_0), -\frac{d_1}{d_2}\delta\kappa(u_0), -\frac{\gamma(u_0)}{h} \right\}.$$

If $(\aleph_1)$–$(\aleph_2)$ hold, then based on Eq. (6), we have $E_0 = \vartheta(u_0) + \delta\kappa(u_0)$, $F_0 = \delta\vartheta(u_0)\kappa(u_0)$. Obviously, for any $n \in N_0$, E_n and F_n are increasing functions. Suppose $(\aleph_2)$ holds. Then $E_0 > 0$ and $F_0 + H > 0$. It follows that $E_n > 0$ and $F_n + H > 0$. From this we can conclude that the roots of Eq. (5) all have negative real parts. Then we can get the following lemmas.

Lemma 2.2 *If* $(\aleph_1)$ *and* $(\aleph_2)$ *are satisfied, then all the roots of Eq.* (5) *have negative real parts when* $\tau = 0$.

Lemma 2.3 *If* $(\aleph_1)$ *and* $(\aleph_2)$ *hold, then* $\lambda = 0$ *is not a root of Eq.* (5) *for all* $\tau \geq 0$.

Lemma 2.2 shows that the positive constant steady state of Eq. (1) is asymptotically stable at $\tau = 0$. Lemma 2.3 shows that as the bifurcation parameter τ increases from zero, to change the stability, the characteristic roots must cross the imaginary axis in the right half-plane, that is, there must exist a pair of purely imaginary roots of Eq. (5) when τ takes some value. Below we determine the value of τ.

For convenience, we make the following assumption

$$(\aleph_3): \ h\vartheta(u_0) < \gamma(u_0).$$

Lemma 2.4 *If* $(\aleph_1)$–$(\aleph_3)$ *hold, then there exists a positive integer* N_* *such that Eq.* (5) *has a pair of purely imaginary roots* $\lambda^{\pm} = \pm i\omega_n$ $(\omega_n > 0)$ *for* $n \in I_1$ *and* $\tau = \tau_n^j$, *where* $I_1 = \{0, 1, 2, \ldots, N_* - 1\}$, *and*

$$\begin{aligned} \omega_n^2 &= \frac{2F_n - E_n^2 + \sqrt{(E_n^2 - 2F_n)^2 - 4(F_n^2 - H^2)}}{2}, \\ \tau_n^j &= \frac{1}{\omega_n}(\arccos\frac{\omega^2 - F_n}{H} + 2j\pi), \quad n \in I_1, j \in N_0. \end{aligned}$$

Lemma 2.5 *Suppose that* $(\aleph_1)$–$(\aleph_3)$ *hold. For* $n \in I_1$, *let* $\lambda(\tau) = \varrho(\tau) + i\varsigma(\tau)$ *be the root of Eq.* (5) *satisfying* $\varrho(\tau_n^j) = 0$ *and* $\varsigma(\tau_n^j) = \omega_n$. *Then* $\mathtt{Sign}\, \varrho'(\tau_n^j) = 1$.

From the above lemmas we derive the following:

Theorem 2.6 *Assume that* $(\aleph_1)$–$(\aleph_2)$ *hold.*

(i) *If* $h\vartheta(u_0) \geq \gamma(u_0)$, *then the positive steady state* (u_0, v_0) *of Eq.* (1) *is locally asymptotically stable for any* $\tau \geq 0$.

(ii) *Assume* $h\vartheta(u_0) < \gamma(u_0)$ *holds,*

  (a) *When* $\tau \in (0, \tau_0)$, *the positive steady state* (u_0, v_0) *of Eq.* (1) *is locally asymptotically stable; when* $\tau > \tau_0 = \min_{n\in I_1}\{\tau_n^0\}$, *the positive steady state* (u_0, v_0) *of Eq.* (1) *is unstable;*

  (b) *When* $\tau = \tau_n^j$, *Eq.* (1) *produces a Hopf bifurcation at the positive steady state* (u_0, v_0).

Now we use the comparison principle to prove the uniform boundedness of positive periodic solutions and determine the range of positive periodic solutions.

Lemma 2.7 *Assume that* $(\aleph_1)$ *holds. Then Eq.* (1) *with* $u(x,0) \not\equiv 0$, $v(x,0) \not\equiv 0$ *has the permanence properties, that is,*

$$\begin{aligned}
&\lim_{t\to\infty} \sup \max_{x\in\bar\Omega} u(x,t) \leq k := \widetilde{u}_0,\\
&\lim_{t\to\infty} \sup \max_{x\in\bar\Omega} v(x,t) \leq \frac{m+k}{h} - b_1 := \widetilde{v}_0,\\
&\lim_{t\to\infty} \inf \min_{x\in\bar\Omega} u(x,t) \geq \frac{arhk}{arh + bk(m+k)} := \widehat{u}_0,\\
&\lim_{t\to\infty} \inf \min_{x\in\bar\Omega} v(x,t) \geq \frac{m+\widehat{u}_0}{h} - b_1 := \widehat{v}_0,
\end{aligned} \tag{8}$$

with

$$\widehat{u}_0 > b_1 h - m.$$

To avoid contradictions, we further compress the range in which the positive periodic solution exists. Then we get the following:

Lemma 2.8 *Assume that* $(\aleph_1)$ *holds. Then Eq.* (1) *with* $u(x,0) \not\equiv 0$, $v(x,0) \not\equiv 0$ *has the permanence properties, that is,*

$$\begin{aligned}
&\lim_{t\to\infty} \sup \max_{x\in\bar\Omega} u(x,t) \leq \widetilde{u}\ , \quad \lim_{t\to\infty} \sup \max_{x\in\bar\Omega} v(x,t) \leq \widetilde{v},\\
&\lim_{t\to\infty} \inf \min_{x\in\bar\Omega} u(x,t) \geq \widehat{u}\ , \quad \lim_{t\to\infty} \inf \min_{x\in\bar\Omega} v(x,t) \geq \widehat{v},
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
&\widetilde{u} = \lim_{n\to\infty} \widetilde{u}_n = \frac{kr(a+\widetilde{u}^2)}{r(a+\widetilde{u}^2) + kb\widehat{v}}\ , \quad \widetilde{v} = \lim_{n\to\infty} \widetilde{v}_n = \frac{m+\widetilde{u}}{h} - b_1,\\
&\widehat{u} = \lim_{n\to\infty} \widehat{u}_n = \frac{kr(a+\widehat{u}^2)}{r(a+\widehat{u}^2) + kb\widetilde{v}}\ , \quad \widehat{v} = \lim_{n\to\infty} \widehat{v}_n = \frac{m+\widehat{u}}{h} - b_1,
\end{aligned}$$

and

$$\begin{aligned} \widetilde{u}_n &= \frac{kr(a+\widetilde{u}_{n-1}^2)}{r(a+\widetilde{u}_{n-1}^2)+kb\widehat{v}_{n-1}}, \quad \widetilde{v}_n = \frac{m+\widetilde{u}_{n-1}}{h} - b_1, \\ \widehat{u}_n &= \frac{kr(a+\widehat{u}_{n-1}^2)}{r(a+\widehat{u}_{n-1}^2)+kb\widetilde{v}_n}, \quad \widehat{v}_n = \frac{m+\widehat{u}_n}{h} - b_1, \end{aligned} \tag{10}$$

with $\widetilde{u}_0$, $\widehat{u}_0$, $\widetilde{v}_0$, *and* $\widehat{v}_0$ *defined in Lemma* 2.7.

Now we use the Lyapunov function to prove that the positive constant steady state of a system without time delay is globally attracted:

Lemma 2.9 *Assume that* $(\aleph_1)$–$(\aleph_3)$ *hold. Then the positive constant steady state* (u_0, v_0) *is globally asymptotically stable if the following condition is satisfied*:

$(\aleph_4)$ *for all* $(u, v) \in G$, *we have*

$$\begin{aligned} &v(uu_0 - a) < \left[\frac{r}{kb} - \left|\frac{u_0(a-km)}{2kma}\right|\right] a(a+u_0^2), \\ &h > \left[\frac{1}{(b_1+v_0)} + \left|\frac{bu_0(a-km)}{2kma}\right|\right](m+k). \end{aligned} \tag{11}$$

Finally, we use Wu's theory [32] to prove that Eq. (1) undergoes global Hopf bifurcation. We normalize the delay by time scale $\bar{t} \to \frac{t}{\tau}$ and omit the bar symbol. Then Eq. (1) becomes

$$\begin{cases} u_t = \tau d_1 \Delta u + \tau \left[ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a+u^2}\right], & \\ & x \in \Omega, t > 0, \\ v_t = \tau d_2 \Delta v + \tau \delta v \left(\dfrac{v}{b_1+v} - \dfrac{hv}{m+u(x,t-1)}\right), & \\ u_\nu = v_\nu = 0, \quad x \in \partial\Omega, t > 0, & \\ u(x,0) \geq 0, v(x,0) \geq 0, \quad x \in \bar{\Omega}. & \end{cases} \tag{12}$$

In the phase space $\mathcal{C} = C([-1,0], X)$, the linearized form of Eq. (12) at (u_0, v_0) is

$$\dot{U}_t = \tau D \Delta U(x,t) + \tau A U(x,t) + \tau B U(x,t-1). \tag{13}$$

The characteristic equation of the linearized system is

$$\lambda^2 + \tau E_n \lambda + \tau^2 F_n + \tau^2 H e^{-\lambda\tau} = 0, \quad n \in N_0. \tag{14}$$

Apparently, if $(\aleph_1)$–$(\aleph_3)$ hold, then Eq. (13) has a pair of purely imaginary roots $\pm i\omega_n \tau_n^j$, and Eq. (12) undergoes the Hopf bifurcation at (u_0, v_0) when $\tau = \tau_n^j$. By Theorem 2.9, when $\tau = 0$, the positive constant steady state (u_0, v_0) is globally asymptotically stable. So Eq. (1) has no positive nonconstant periodic solution with period 1. In a similar way, Eq. (12) has no positive nonconstant periodic solution with period 1.

To state Wu's [32] global Hopf bifurcation theorem, we define:

(i) $E = C(S^1, X)$ is a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C} : |z| = 1\}$;

(ii) Let $E^G := \{x \in E : gx = x \text{ for all } g \in G\}$. Then $E^G = X$, and E has an isotypical direct sum decomposition $E = E^G \bigoplus_{k=1}^{\infty} E_k$, where $E_k = \{e^{ikt}x : x \in X\}$ for $k \geq 1$.

Then, according to [32], Eq. (1) can be transformed into a continuously differentiable, completely continuous, and G-invariant integral equation.

According to Theorem 2.1, we get that Eq. (1) has a unique positive constant steady state $U_0 = (u_0, v_0)$. Lemma 2.3 tells us that $\lambda = 0$ is not a root of Eq. (5) satisfying hypothesis H(1) in [24, Sect. 6.5]. Lemma 2.4 tells us that Eq. (5) has a pair of purely imaginary roots $\lambda^{\pm} = \pm i\omega_n$ $(\omega_n > 0)$ when $\tau = \tau_n^j$, and hence assumption H(2) in [24, Sect. 6.5] is satisfied. For sufficiently small $\varepsilon_0, \varsigma_0 > 0$, we define the local steady-state manifold

$$M = \{(U_0, \tau, \omega) : |\tau - \tau_n^j| < \varepsilon_0, |\omega - \omega_n| < \varsigma_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+.$$

Then for

$$(\tau, \omega) \in [\tau_n^j - \varepsilon_0, \tau_n^j + \varepsilon_0] \times [\omega_n - \varsigma_0, \omega_n + \varsigma_0],$$

$\pm i\omega_n$ is an eigenvalue of Eq. (5) if and only if $\tau = \tau_n^j$ and $\omega = \omega_n$. From [36, Lemma 6.5.3] we conclude that $(U_0, \tau_n^j, \omega_n)$ is an isolated singular point in M.

Let $\mu_k(U_0, \tau_n^j, \omega_n)$ $(k = 1, 2, \dots)$ be the number of generalized crossing defined in [36, Sect. 6.5]. Then according to Lemma 2.5, if $\lambda(\tau) = \varrho(\tau) + i\varsigma(\tau)$ are the eigenvalues of Eq. (5) satisfying $\lambda(\tau_n^j) = \pm i\omega_n$, then $\mu_1(U_0, \tau_n^j, \omega_n) = 1$. Thus we get the local topological Hopf bifurcation Eq. (1) at $\tau = \tau_n^j$.

Next, we study the global nature of the Hopf bifurcation. Let

$$\begin{aligned} S = \mathrm{Cl}\{&(z, \tau, \omega) \in E \times \mathbb{R} \times \mathbb{R}_+ : z = (z_1(\cdot, \omega t), z_2(\cdot, \omega t)) = (u(\cdot, t), v(\cdot, t)) \\ &\text{is a nontrivial } \frac{2\pi}{\omega}\text{-periodic solution of Eq. (1)}\}. \end{aligned}$$

Then according to the local bifurcation theorem, $(U_0, \tau_n^j, \omega_n) \in S$. We define the complete steady-state manifold

$$M^* = \{(U_0, \tau) : \tau \in \mathbb{R}\} \subset E^G \times \mathbb{R}$$

and

$$\mathfrak{C}_n^j = \mathfrak{C}_n^j(U_0, \tau_n^j, \omega_n),$$

the connected component of S for which $(U_0, \tau_n^j, \omega_n)$ belongs to. Now we can state Wu's global Hopf bifurcation result.

Lemma 2.10 [24, Theorem 6.5.5] *For each connected component $\mathfrak{C}_n^j$, at least one of the following holds*:

(i) $\mathfrak{C}_n^j$ *is unbounded, i.e.*,

$$\sup\{\max_{t \in \mathbb{R}} |z(t)| + |\tau| + \omega + \omega^{-1} : (z, \tau, \omega) \in \mathfrak{C}_n^j = \infty;$$

(ii) $\mathfrak{C}_n^j \cap M^* \times \mathbb{R}_+$ *is finite, and for all* $k \geq 1$,

$$\sum_{(U_0,\tau_n^j,\omega_n)\in\mathfrak{C}_n^j\cap M^*\times\mathbb{R}_+} \mu_k(U_0,\tau_n^j,\omega_n) = 0.$$

At present, we are well prepared to present the following global Hopf bifurcation results.

Theorem 2.11 *Assume that* $(\aleph_1)$–$(\aleph_4)$ *hold. Then Eq.* (1) *has at least one positive periodic orbit when* $\tau > \tau_1 = \min_{n\in I_1}\{\tau_n^1\}$.

3 Proofs of main results

3.1 Proof of Theorem 2.1

Multiply the first equation of Eq. (3) by $(a + u^2)h$ and denote

$$\Psi = A_1u^3 + B_1u^2 + C_1u + D_1 \tag{15}$$

with

$$\begin{aligned} A_1 &= -\frac{rh}{k}, & C_1 &= bb_1h - \frac{rah}{k} - bm,\\ B_1 &= rh - b, & D_1 &= arh. \end{aligned} \tag{16}$$

Apparently, if Eq. (15) exists positive roots greater than $b_1h - m$, then Eq. (1) has positive constant steady states. Here we explore the roots of Eq. (15) by differentiation:

$$\Psi' = 3A_1u^2 + 2B_1u + C_1. \tag{17}$$

After analyzing the distribution of zeros of Eq. (17), we can conclude as follows. For convenience, we denote

$$Ł = 4B_1^2 - 12A_1C_1.$$

The axis of symmetry of Ψ' is $\varpi = -\frac{B_1}{3A_1}$, and the zeros of Eq. (17) are

$$u_1 = \frac{-B_1 - \sqrt{B_1^2 - 3A_1C_1}}{3A_1}, \qquad u_2 = \frac{-B_1 + \sqrt{B_1^2 - 3A_1C_1}}{3A_1}.$$

(a) Since $A_1 \leq 0$, Ψ' is a quadratic function of the opening downward. If $Ł \leq 0$, then $\Psi' \leq 0$, and Ψ is a decreasing function. Since $\Psi(0) = D_1 > 0$, Ψ has an intersection point with the u-axis as u increases, that is, Eq. (1) has a unique positive constant steady state.

(b) If $Ł > 0$ and $B_1 \leq 0$, then $\varpi \leq 0$.

1. Assume that $u_1 < 0$ and $u_2 \leq 0$. Then $\Psi' \leq 0$ when $u > 0$. The situation is the same as in (a), without going into details.
2. Assume that $u_1 < 0$ and $u_2 > 0$. Then $\Psi' > 0$ when $u \in (u_1, u_2)$, and $\Psi' \leq 0$ when $u \geq u_2$. So Ψ is an increasing function for $u \in [0, u_2]$ and a decreasing function for $u > u_2$, that is, Eq. (1) has a unique positive constant steady state.

(c) Let Ł > 0 and $B_1 > 0$.
   1. Assume that $C_1 > 0$. Then $u_1 < 0$ and $u_2 > 0$. The situation is the same as in (b).2, without going into details.
   2. Assume that $C_1 = 0$, and thus $u_1 > 0$ and $u_2 = 0$. Then $\Psi' > 0$ when $u \in (0, u_1)$, and $\Psi' \leq 0$ when $u \geq u_1$. So Ψ is an increasing function for $u \in [0, u_1]$ and a monotonically decreasing function for $u > u_1$. That is, Eq. (1) has the unique positive constant steady state.

(d) If Ł > 0, $B_1 > 0$ and $C_1 < 0$, we have $u_1 > 0$ and $u_2 > 0$. Then we have $\Psi' > 0$ when $u \in (u_1, u_2)$ and $\Psi' \leq 0$ when $u \geq u_2$ and $u \leq u_1$. So Ψ is an increasing function for $u \in [0, u_2]$ and a decreasing function for $u \geq u_2$ and $u \leq u_1$. If $\Psi(u_1) > 0$, then $\Psi(u_2) > 0$. We can see from the above that Eq. (1) has a unique positive constant steady state.

(e) Similarly, if $\Psi(u_2) < 0$, then $\Psi(u_1) < 0$. We can see from the above that Eq. (1) has a unique positive constant steady state.

This completes the proof of Theorem 2.1.

3.2 Proof of Lemma 2.4

Now assume that $(\aleph_1)$ and $(\aleph_2)$ hold and let $\lambda = i\omega(\omega > 0)$ be a purely imaginary root of Eq. (5). Substituting it into Eq. (5) and separating the real and imaginary parts, we have

$$\begin{aligned} &-\omega^2 + F_n = -H\cos\omega\tau, \\ &E_n\omega = H\sin\omega\tau. \end{aligned} \tag{18}$$

Squaring and adding both equations of Eq. (18), we have

$$\Upsilon(w^2) = \omega^4 + (E_n^2 - 2F_n)\omega^2 + (F_n^2 - H^2) = 0, \tag{19}$$

where

$$E_n^2 - 2F_n = (d_1^2 + d_2^2)\left(\frac{n}{l}\right)^4 + 2\big(d_1\vartheta(u_0) + d_2\delta\kappa(u_0)\big)\left(\frac{n}{l}\right)^2 + \vartheta^2(u_0) + \delta^2\kappa^2(u_0) > 0.$$

If $(\aleph_3)$ holds, then

$$F_0 - H = \delta\vartheta(u_0)\kappa(u_0) - \frac{\delta}{h}\kappa(u_0)\gamma(u_0) < 0.$$

There exists $N_* \in N$ such that $F_n - H < 0$ for $n < N_*$, and thereby we get $F_n^2 - H^2 < 0$. Obviously, Eq. (19) has a positive root, that is, Eq. (5) has a unique pair of purely imaginary roots $\pm i\omega$. On the contrary, if $n \geq N_*$, then $F_n^2 - H^2 \geq 0$, and Eq. (5) has no purely imaginary root.

Moreover, when $n < N_*$, let $\omega_n^2 = z_n$, where

$$z_n = \frac{2F_n - E_n^2 + \sqrt{(E_n^2 - 2F_n)^2 - 4(F_n^2 - H^2)}}{2}. \tag{20}$$

Then according to Eq. (18), we have

$$\sin(\omega\tau) = \frac{E_n\omega}{H} > 0, \qquad \cos(\omega\tau) = \frac{\omega^2 - F_n}{H}.$$

It follows that

$$\omega_n^2 = \frac{2F_n - E_n^2 + \sqrt{(E_n^2 - 2F_n)^2 - 4(F_n^2 - H^2)}}{2},$$
$$\tau_n^j = \frac{1}{\omega_n}(\arccos \frac{\omega^2 - F_n}{H} + 2j\pi), \quad n \in I_1, j \in N_0. \tag{21}$$

3.3 Proof of Lemma 2.5

Differentiating λ on both sides of Eq. (5) gives

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{\lambda}\left(-\frac{2\lambda + E_n}{\lambda^2 + E_n\lambda + F_n} - \tau\right). \tag{22}$$

Substituting $\lambda = i\omega$ into Eq. (22), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n^j} = \frac{1}{i\omega}\left(-\frac{2i\omega + E_n}{i\omega^2 + E_n i\omega + F_n} - \tau\right).$$

Then

$$\mathtt{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n^j} = \frac{\sqrt{(E_n^2 - 2F_n)^2 - 4(F_n^2 - H^2)}}{4E_n^2\omega_n^2 + (F_n - \omega_n^2)^2} > 0.$$

Therefore

$$\mathtt{Sign}\, \varrho'(\tau_n^j) = \mathtt{Sign}\left(\mathtt{Re}\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n^j} = 1.$$

3.4 Proof of Lemma 2.6

Lemma 2.2 shows that the eigenequation has no root in the right half complex plane when $\tau = 0$. If $h\vartheta(u_0) \geq \gamma(u_0)$, then

$$F_0 - H = \delta\vartheta(u_0)\kappa(u_0) - \frac{\delta}{h}\kappa(u_0)\gamma(u_0) > 0.$$

Then we have $F_n - H > 0$ for all $n \in N_0$. Therefore Eq. (19) has no positive root, and Eq. (5) has no purely imaginary root for all $\tau \geq 0$. This means that the positive constant steady state of Eq. (1) is asymptotically stable for all $\tau \geq 0$.

If $h\vartheta(u_0) < \gamma(u_0)$, then from Lemma 2.4 we know that Eq. (5) has a pair of purely imaginary roots $\pm i\omega_n$ when $\tau = \tau_n^j$. Lemma 2.5 shows that Eq. (5) has no positive root for $\tau < \tau_0$ and has at least two positive roots for $\tau > \tau_0$. Moreover, Eq. (1) produces a subcritical Hopf bifurcation at the positive steady state (u_0, v_0) when $\tau = \tau_n^j$.

3.5 Proof of Lemma 2.7

From the first equation of Eq. (1) we see that for $t > 0$,

$$u_t \leq d_1\triangle u + ru\left(1 - \frac{u}{k}\right)$$

and $u(x,t) \le \widetilde{P}(x,t)$ in $\bar{\Omega} \times [0,\infty)$, where $\widetilde{P}(x,t)$ is the solution of the equation

$$\begin{cases} \widetilde{P}_t = d_1 \Delta \widetilde{u} + r\widetilde{P}\Big(1 - \dfrac{\widetilde{P}}{k}\Big), & x \in \Omega, t > 0, \\ \widetilde{P}_\nu = 0, & x \in \partial\Omega, t > 0, \\ \widetilde{P}(x,0) = u(x,0), & x \in \bar{\Omega}. \end{cases}$$

As $t \to \infty$, we have $\lim\limits_{t\to\infty} \widetilde{P}(x,t) = k$. Then

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le k = \widetilde{u}_0$$

for arbitrarily small $\varepsilon_1 > 0$, and there exists $T_1 > 0$ such that $u(x,t) \le \widetilde{u}_0 + \varepsilon_1$ for $t > T_1$.

From the second equation of Eq. (1) we obtain that for $t > T_1$,

$$v_t \le d_2 \triangle v + \delta v\Big(\frac{1}{b_1 + v} - \frac{hv}{m + \widetilde{u}_0 + \varepsilon_1}\Big),$$

and $v(x,t) < \widetilde{Q}(x,t)$ in $\bar{\Omega} \times [T_1,\infty)$, where $\widetilde{Q}(x,t)$ is the solution of the equation

$$\begin{cases} \widetilde{Q}_t = d_2 \Delta \widetilde{Q} + \delta\widetilde{Q}\Big(\dfrac{1}{b_1 + v} - \dfrac{h\widetilde{Q}}{m + \widetilde{u}_0 + \varepsilon_1}\Big), & x \in \Omega, t > T_1, \\ \widetilde{Q}_\nu = 0, & x \in \partial\Omega, t > T_1, \\ \widetilde{Q}(x,T_1) = v(x,T_1), & x \in \bar{\Omega}. \end{cases}$$

As $t \to \infty$, we have $\lim\limits_{t\to\infty} \widetilde{Q}(x,t) = \frac{m+\widetilde{u}_0+\varepsilon_1}{h} - b_1$. Then

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \frac{m + \widetilde{u}_0 + \varepsilon_1}{h} - b_1.$$

Since ε_1 is arbitrarily small,

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \frac{m + \widetilde{u}_0}{h} - b_1 = \widetilde{v}_0$$

for arbitrarily small $\varepsilon_2 > 0$, and there exists $T_2 > T_1$ such that $v(x,t) \le \widetilde{v}_0 + \varepsilon_2$ for $t > T_2$.

From the first equation of Eq. (1) we obtain that for $t > T_2$,

$$u_t \ge d_1 \triangle u + ru\Big[1 - \Big(\frac{1}{k} + \frac{b(\widetilde{v}_0 + \varepsilon_2)}{ar}\Big)u\Big],$$

and $u(x,t) \ge \widehat{P}(x,t)$ in $\bar{\Omega} \times [T_2,\infty)$, where $\widehat{P}(x,t)$ is the solution of the equation

$$\begin{cases} \widehat{P}_t = d_1 \triangle \widehat{P} + r\widehat{P}\Big[1 - \Big(\dfrac{1}{k} + \dfrac{b(\widetilde{v}_0 + \varepsilon_2)}{ar}\Big)\widehat{P}\Big], & x \in \Omega, t > T_2, \\ \widehat{P}_\nu = 0, & x \in \partial\Omega, t > T_2, \\ \widehat{P}(x,T_2) = u(x,T_2), & x \in \bar{\Omega}. \end{cases}$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} \widehat{P}(x,t) = \Big(\frac{1}{k} + \frac{b(\widetilde{v}_0 + \varepsilon_2)}{ar}\Big)^{-1}.$$

Then

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u(x,t) \geq \Big(\frac{1}{k} + \frac{b(\widetilde{v}_0 + \varepsilon_2)}{ar}\Big)^{-1}.$$

Since ε_2 is arbitrarily small,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u(x,t) \geq \Big(\frac{1}{k} + \frac{b\widetilde{v}_0}{ar}\Big)^{-1} = \widehat{u}_0$$

for arbitrarily small $\varepsilon_3 > 0$, and there exists $T_3 > T_2$ such that $u(x,t) \geq \widehat{u}_0 - \varepsilon_3$ for $t > T_3$.

From the second equation of Eq. (1) we obtain that for $t > T_3$,

$$v_t \geq d_2 \triangle v + \delta v^2 \Big(\frac{1}{b_1 + v} - \frac{h}{m + \widehat{u}_0 - \varepsilon_3}\Big)$$

and $v(x,t) \geq \widehat{Q}(x,t)$ in $\bar{\Omega} \times [T_3, \infty)$, where $\widehat{Q}(x,t)$ is the solution of the equation

$$\begin{cases} \widehat{Q}_t = d_2 \triangle \widehat{Q} + \delta \widehat{Q}^2 \Big(\frac{1}{b_1 + \widehat{Q}} - \frac{h}{m + \widehat{u}_0 - \varepsilon_3}\Big), & x \in \Omega, t > T_3, \\ \widehat{Q}_\nu = 0, \quad x \in \partial\Omega, t > T_3, \\ \widehat{Q}(x, T_3) = v(x, T_3), \quad x \in \bar{\Omega}. \end{cases} \tag{23}$$

We define

$$f(v) = \delta v^2 \Big(\frac{1}{b_1 + v} - \frac{h}{m + \widehat{u}_0 - \varepsilon_3}\Big) \tag{24}$$

and

$$(H_4): \ \widehat{u}_0 > b_1 h - m.$$

Assuming that (H_4) holds, let M and ε be positive constants satisfying

$$M \geq \max\{\rho_1, \zeta_1\}, \quad \varepsilon \leq \min\{\rho_1, \zeta_2\},$$

where

$$\rho_1 = \frac{m + \widehat{u}_0 - \varepsilon_3}{h} - b_1 \ , \ \zeta_1 = \max_{x\in\bar{\Omega}} u(x,0) \ , \ \zeta_2 = \min_{x\in\bar{\Omega}} v(x,0).$$

Then we have

$$\delta M^2 \Big(\frac{1}{b_1 + M} - \frac{h}{m + \widehat{u}_0 - \varepsilon_3}\Big) \leq 0, \quad \delta \varepsilon^2 \Big(\frac{1}{b_1 + \varepsilon} - \frac{h}{m + \widehat{u}_0 - \varepsilon_3}\Big) \geq 0,$$

and $\varepsilon \le v \le M$. In that way, M and ε are a couple of upper and lower solutions of Eq. (23). From the boundedness of the partial derivative of $f(v)$ with respect to v we obtain that $f(v)$ satisfies the Lipschitz condition on $[\varepsilon, M]$. Denote the Lipschitz constant by K and define two sequences $\{\widetilde{V}_1^{(j)}\}$ and $\{\widehat{V}_1^{(j)}\}$ as follows:

$$\begin{aligned}
\widetilde{V}_1^{(j)} &= \widetilde{V}_1^{(j-1)} + \frac{1}{K}\left[\delta \widetilde{V}_1^2\left(\frac{1}{b_1+\widetilde{V}_1} - \frac{h}{m+\widehat{u}_0-\varepsilon_3}\right)\right], \\
\widehat{V}_1^{(j)} &= \widehat{V}_1^{(j-1)} + \frac{1}{K}\left[\delta \widehat{V}_1^2\left(\frac{1}{b_1+\widehat{V}_1} - \frac{h}{m+\widehat{u}_0-\varepsilon_3}\right)\right],
\end{aligned} \quad j \in N,$$

where $\widetilde{V}_1^{(0)} = M$, $\widehat{V}_1^{(0)} = \varepsilon$, and K is the Lipschitz constant. It is well known that $\lim\limits_{j\to\infty}(\widetilde{V}_1^{(j)}, \widehat{V}_1^{(j)}) = (\widetilde{V}_1, \widehat{V}_1)$ with $\varepsilon \le \widehat{V}_1 \le \widetilde{V}_1 \le M$. Obviously, $\widetilde{V}_1$ and $\widehat{V}_1$ satisfy

$$\delta \widetilde{V}_1^2\left(\frac{1}{b_1+\widetilde{V}_1} - \frac{h}{m+\widehat{u}_0-\varepsilon_3}\right) = 0 \;,\; \delta \widehat{V}_1^2\left(\frac{1}{b_1+\widehat{V}_1} - \frac{h}{m+\widehat{u}_0-\varepsilon_3}\right) = 0.$$

Then we have $\widetilde{V}_1 = \widehat{V}_1 = v_0$. As $t \to \infty$, we have

$$\lim_{t\to\infty} \widehat{v}(x,t) = \frac{m+\widehat{u}_0-\varepsilon_3}{h} - b_1,$$

and thus

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \ge \frac{m+\widehat{u}_0-\varepsilon_3}{h} - b_1.$$

Since ε_3 is arbitrarily small, we have

$$\lim_{t\to\infty} \inf \min_{x\in\bar{\Omega}} v(x,t) \ge \frac{m+\widehat{u}_0}{h} - b_1 = \widehat{v}_0.$$

In summary, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le k = \widetilde{u}_0, \\
&\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \frac{m+\widetilde{u}_0}{h} - b_1 = \widetilde{v}_0, \\
&\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \ge \left(\frac{1}{k} + \frac{b\widetilde{v}_0}{ar}\right)^{-1} = \widehat{u}_0, \\
&\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \ge \frac{m+\widehat{u}_0}{h} - b_1 = \widehat{v}_0,
\end{aligned}$$

where

$$\widehat{u}_0 > b_1 h - m,$$

and the limit set of (u,v) belongs to $G_0 = [\widehat{u}_0, \widetilde{u}_0] \times [\widehat{v}_0, \widetilde{v}_0]$.

3.6 Proof of Lemma 2.8

Let $\big(u(x,t),v(x,t)\big)$ be the solution of system Eq. (1), with $u(x,0)\not\equiv 0$, $v(x,0)\not\equiv 0$ for $x\in\bar{\Omega}$ and $t>0$.

For arbitrarily small $\varepsilon_4>0$, there exists $T_4>T_3$ such that $v(x,t)\geq\widehat{v}_0-\varepsilon_4$ for $t>T_4$. According to the first equality of Eq. (8), we obtain that for $t>T_4$,

$$u_t\leq d_1\Delta u+ru\Big(1-\frac{u}{k}\Big)-\frac{bu^2(\widehat{v}_0-\varepsilon_4)}{a+\widetilde{u}_0^2}=d_1\Delta u+ru\left[1-\left(\frac{1}{k}+\frac{b(\widehat{v}_0-\varepsilon_4)}{r(a+\widetilde{u}_0^2)}\right)u\right],$$

and $\widetilde{P}(x,t)\leq\widetilde{U}(x,t)$ in $\bar{\Omega}\times[T_4,\infty)$, where $\widetilde{U}(x,t)$ is the solution of the equation

$$\begin{cases}\dfrac{\partial\widetilde{U}}{\partial t}=d_1\Delta\widetilde{U}+r\widetilde{U}\left[1-\left(\dfrac{1}{k}+\dfrac{b(\widehat{v}_0-\varepsilon_4)}{r(a+\widetilde{u}_0^2)}\right)\widetilde{U}\right], & x\in\Omega, t>T_4,\\ \widetilde{U}_\nu=0, \quad x\in\partial\Omega, t>T_4, & \\ \widetilde{U}(x,T_4)=u(x,T_4), \quad x\in\bar{\Omega}. & \end{cases}$$

From the above we have $\widehat{P}(x,t)\leq u(x,t)\leq\widetilde{P}(x,t)\leq\widetilde{U}(x,t)$. As $t\to\infty$, we have

$$\limsup_{t\to\infty}\sup\max_{x\in\bar{\Omega}}u(x,t)\leq\frac{kr(a+\widetilde{u}_0^2)}{r(a+\widetilde{u}_0^2)+kb(\widehat{v}_0-\varepsilon_4)}.$$

Since ε_4 is arbitrarily small, we have

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}}u(x,t)\leq\frac{kr(a+\widetilde{u}_0^2)}{r(a+\widetilde{u}_0^2)+kb\widehat{v}_0}=\widetilde{u}_1$$

for arbitrarily small $\varepsilon_5>0$, and there exists $T_5>T_4$ such that $u(x,t)\leq\widetilde{u}_1+\varepsilon_5$ for $t>T_5$.

According to the second equality of Eq. (8), we obtain that for $t>T_5$,

$$v_t\leq d_2\Delta v+\delta v\Big(\frac{1}{b_1+v}-\frac{hv}{m+\widetilde{u}_1+\varepsilon_5}\Big),$$

and $\widetilde{Q}(x,t)\leq\widetilde{V}(x,t)$ in $\bar{\Omega}\times[T_5,\infty)$, where $\widetilde{V}(x,t)$ is the solution of the equation

$$\begin{cases}\dfrac{\partial\widetilde{V}}{\partial t}=d_2\Delta\widetilde{V}+\delta\widetilde{V}\Big(\dfrac{1}{b_1+v}-\dfrac{h\widetilde{V}}{m+\widetilde{u}_1+\varepsilon_5}\Big), & x\in\Omega, t>T_5,\\ \widetilde{V}_\nu=0, \quad x\in\partial\Omega, t>T_5 & \\ \widetilde{V}(x,T_5)=u(x,T_5), \quad x\in\bar{\Omega}. & \end{cases}$$

From the above we have $\widehat{Q}(x,t)\leq v(x,t)\leq\widetilde{Q}(x,t)\leq\widetilde{V}(x,t)$. Thus, as $t\to\infty$ and $\varepsilon_5\to 0$, we have

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}}v(x,t)\leq\frac{m+\widetilde{u}_1}{h}-b_1=\widetilde{v}_1$$

for arbitrarily small $\varepsilon_6>0$, and there exists $T_6>T_5$ such that $v(x,t)\leq\widetilde{v}_1+\varepsilon_6$ for $t>T_6$.

According to the third equality of Eq. (8), we obtain that for $t > T_6$,

$$u_t \geq d_1\Delta u + ru\Big(1-\frac{u}{k}\Big) - \frac{bu^2(\widetilde{v}_1+\varepsilon_6)}{a+\widehat{u}_0^2} = d_1\Delta u + ru\bigg[1-\bigg(\frac{1}{k}+\frac{b(\widetilde{v}_1+\varepsilon_6)}{r(a+\widehat{u}_0^2)}\bigg)u\bigg],$$

and $\widehat{U}(x,t) \leq \widehat{P}(x,t) \leq u(x,t)$ in $\bar{\Omega}\times[T_6,\infty)$, where $\widehat{U}(x,t)$ is the solution of the equation

$$\begin{cases} \dfrac{\partial \widehat{U}}{\partial t} = d_1\Delta\widehat{U} + r\widehat{U}\bigg[1-\bigg(\dfrac{1}{k}+\dfrac{b(\widetilde{v}_1+\varepsilon_6)}{r(a+\widetilde{u}_0^2)}\bigg)\widehat{U}\bigg], & x\in\Omega, t>T_6, \\ \widehat{U}_\nu = 0, & x\in\partial\Omega, t>T_6, \\ \widehat{U}(x,T_6) = u(x,T_6), & x\in\bar{\Omega}. \end{cases}$$

From the above we have $\widehat{U}(x,t) \leq \widehat{P}(x,t) \leq u(x,t) \leq \widetilde{P}(x,t)$. As $t\to\infty$ and $\varepsilon_6 \to 0$, we have

$$\lim_{t\to\infty}\inf\min_{x\in\bar{\Omega}} u(x,t) \geq \frac{kr(a+\widehat{u}_0^2)}{r(a+\widehat{u}_0^2)+kb(\widetilde{v}_1+\varepsilon_6)} = \widehat{u}_1$$

for arbitrarily small $\varepsilon_7 > 0$, and there exists $T_7 > T_6$ such that $u(x,t) \leq \widetilde{u}_1 - \varepsilon_7$ for $t > T_7$.

According to the last equality of Eq. (8), we obtain that for $t > T_7$,

$$v_t \geq d_2\Delta v + \delta v^2\Big(\frac{1}{b_1+v} - \frac{h}{m+\widehat{u}_1-\varepsilon_7}\Big),$$

and $\widehat{V}(x,t) \leq \widehat{Q}(x,t) \leq v(x,t)$ in $\bar{\Omega}\times[T_7,\infty)$, where $\widehat{V}(x,t)$ is the solution of the equation

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} = d_2\Delta\widehat{V} + \delta\widehat{V}^2\Big(\dfrac{1}{b_1+\widehat{V}} - \dfrac{h}{m+\widehat{u}_1-\varepsilon_7}\Big), & x\in\Omega, t>T_7, \\ \widehat{V}_\nu = 0, & x\in\partial\Omega, t>T_7, \\ \widehat{V}(x,T_7) = u(x,T_7), & x\in\bar{\Omega}. \end{cases}$$

From the above we have $\widehat{V}(x,t) \leq \widehat{Q}(x,t) \leq v(x,t) \leq \widetilde{Q}(x,t)$. As $t\to\infty$ and $\varepsilon_7 \to 0$, we have

$$\lim_{t\to\infty}\inf\min_{x\in\bar{\Omega}} v(x,t) \geq \frac{m+\widehat{u}_1}{h} - b_1 = \widehat{v}_1$$

for arbitrarily small $\varepsilon_8 > 0$, and there exists $T_8 > T_7$ such that $v(x,t) \leq \widetilde{v}_1 - \varepsilon_8$ for $t > T_8$. Then we have $v(x,t) \leq \widetilde{v}_1$.

In summary, we get

$$\begin{cases} \lim\limits_{t\to\infty}\sup\max\limits_{x\in\bar{\Omega}} u(x,t) \leq \dfrac{kr(a+\widetilde{u}_0^2)}{r(a+\widetilde{u}_0^2)+kb\widehat{v}_0} = \widetilde{u}_1, \\ \lim\limits_{t\to\infty}\sup\max\limits_{x\in\bar{\Omega}} v(x,t) \leq \dfrac{m+\widetilde{u}_1}{h} - b_1 = \widetilde{v}_1, \\ \lim\limits_{t\to\infty}\inf\min\limits_{x\in\bar{\Omega}} u(x,t) \geq \dfrac{kr(a+\widehat{u}_0^2)}{r(a+\widehat{u}_0^2)+kb\widetilde{v}_1} = \widehat{u}_1, \\ \lim\limits_{t\to\infty}\inf\min\limits_{x\in\bar{\Omega}} v(x,t) \geq \dfrac{m+\widehat{u}_1}{h} - b_1 = \widehat{v}_1, \end{cases}$$

and the limit set of (u,v) belongs to $G_1 = [\widehat{u}_1,\widetilde{u}_1]\times[\widehat{v}_1,\widetilde{v}_1]$.

Moreover, we have

$$\widetilde{u}_1 = \frac{kr(a+\widetilde{u}_0^2)}{r(a+\widetilde{u}_0^2)+kb\widehat{v}_0} \le k = \widetilde{u}_0,$$

so we have

$$\widetilde{v}_1 = \frac{m+\widetilde{u}_1}{h} - b_1 \le \frac{m+\widetilde{u}_0}{h} - b_1 = \widetilde{v}_0,$$

then

$$\begin{aligned}\widehat{u}_1 = \frac{kr(a+\widehat{u}_0^2)}{r(a+\widehat{u}_0^2)+kb\widetilde{v}_1} &\ge \frac{kr(a+\widehat{u}_0^2)}{r(a+\widehat{u}_0^2)+kb\widetilde{v}_0}\\ &\ge \frac{kra+(k-1)r\widehat{u}_0^2}{ra+kb\widetilde{v}_0} \ge \frac{kar}{ar+kb\widetilde{v}_0} = \widehat{u}_0,\end{aligned}$$

and therefore

$$\widehat{v}_0 = \frac{m+\widehat{u}_0}{h} - b_1 \ge \frac{m+\widehat{u}_1}{h} - b_1 = \widehat{v}_1.$$

In summary, we have $\widehat{u}_0 \le \widehat{u}_1 \le u \le \widetilde{u}_1 \le \widetilde{u}_0$ and $\widehat{v}_0 \le \widehat{v}_1 \le v \le \widetilde{v}_1 \le \widetilde{v}_0$.

Continuing with this compression process, we have

$$\begin{cases}\limsup\limits_{t\to\infty}\max\limits_{x\in\bar{\Omega}} u(x,t) \le \dfrac{kr(a+\widetilde{u}_{n-1}^2)}{r(a+\widetilde{u}_{n-1}^2)+kb\widehat{v}_{n-1}} = \widetilde{u}_n,\\ \limsup\limits_{t\to\infty}\max\limits_{x\in\bar{\Omega}} v(x,t) \le \dfrac{m+\widetilde{u}_{n-1}}{h} - b_1 = \widetilde{v}_n,\\ \liminf\limits_{t\to\infty}\min\limits_{x\in\bar{\Omega}} u(x,t) \ge \dfrac{kr(a+\widehat{u}_{n-1}^2)}{r(a+\widehat{u}_{n-1}^2)+kb\widetilde{v}_n} = \widehat{u}_n,\\ \liminf\limits_{t\to\infty}\min\limits_{x\in\bar{\Omega}} v(x,t) \ge \dfrac{m+\widehat{u}_n}{h} - b_1 = \widehat{v}_n,\end{cases} \tag{25}$$

with $\widehat{u}_0 \le \widehat{u}_1 \le \widehat{u}_2 \le \cdots \le \widehat{u}_{n+1} \le u \le \widetilde{u}_{n+1} \le \widetilde{u}_n \le \widetilde{u}_{n-1} \le \cdots \le \widetilde{u}_0$ and $\widehat{v}_0 \le \widehat{v}_1 \le \widehat{v}_2 \le \cdots \le \widehat{v}_{n+1} \le v \le \widetilde{v}_{n+1} \le \widetilde{v}_n \le \widetilde{v}_{n-1} \le \cdots \le \widetilde{v}_0$.

Taking the limits in Eq. (25), we have

$$\begin{aligned}\widetilde{u} = \lim_{n\to\infty}\widetilde{u}_n = \frac{kr(a+\widetilde{u}^2)}{r(a+\widetilde{u}^2)+kb\widehat{v}} \quad &, \quad \widetilde{v} = \lim_{n\to\infty}\widetilde{v}_n = \frac{m+\widetilde{u}}{h} - b_1,\\ \widehat{u} = \lim_{n\to\infty}\widehat{u}_n = \frac{kr(a+\widehat{u}^2)}{r(a+\widehat{u}^2)+kb\widetilde{v}} \quad &, \quad \widehat{v} = \lim_{n\to\infty}\widehat{v}_n = \frac{m+\widehat{u}}{h} - b_1,\end{aligned} \tag{26}$$

and then we get

$$\begin{aligned}&\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} u(x,t) \le \widetilde{u}, \quad \limsup_{t\to\infty}\max_{x\in\bar{\Omega}} v(x,t) \le \widetilde{v},\\ &\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u(x,t) \ge \widehat{u}, \quad \liminf_{t\to\infty}\min_{x\in\bar{\Omega}} v(x,t) \ge \widehat{v},\end{aligned}$$

and the limit set of (u,v) belongs to $G = [\widehat{u},\widetilde{u}] \times [\widehat{v},\widetilde{v}]$.

3.7 Proof of Lemma 2.9

Next, we prove the global asymptotic stability of the unique positive constant steady state of Eq. (1) at $\tau = 0$, i.e., Theorem 2.9.

First, we introduce a suitable Lyapunov function from [32],

$$W(u,v) = (u-u_0) - u_0 \ln\left(\frac{u}{u_0}\right) + \rho\left[(v-v_0) - v_0 \ln\left(\frac{v}{v_0}\right)\right],$$

where ρ is the coefficient to be determined.

The function W is continuous in $G = [\widehat{u}, \widetilde{u}] \times [\widehat{v}, \widetilde{v}]$, equal to zero at equilibrium (u_0, v_0), and always greater than zero at other u, v. Then (u_0, v_0) is the global minimum of W, and the time derivative along W of the solution of Eq. (1) is

$$\begin{aligned}
\frac{dW}{dt} =&(u-u_0)\left[r\left(1-\frac{u}{k}\right) - \frac{buv}{a+u^2}\right] + \delta\alpha(v-v_0)\left(\frac{v}{b_1+v} - \frac{hv}{m+u}\right)\\
=&(u-u_0)\left[r\left(1-\frac{1}{k}\right)(u-u_0) - \frac{u_0}{k} - \frac{buv}{a+u^2}(u-u_0) - \frac{buv}{a+u^2}\right]\\
&+ \delta\alpha\big((v-v_0)+v_0\big)\left(\frac{1}{b_1+v} - \frac{h}{m+u}\right)\\
=&\left(-\frac{r}{k} - \frac{bv}{a+u^2}\right)(u-u_0)^2 + \left(\frac{bu_0v_0}{a+u_0^2} - \frac{bu_0v_0}{a+u^2}\right)(u-u_0)\\
&+ \alpha\delta\left(\frac{1}{b_1+v} - \frac{h}{m+u}\right)(v-v_0)^2 + \alpha\delta\left[\left(\frac{1}{b_1+v} - \frac{1}{b_1+v_0}\right)\right.\\
&\left.+\left(\frac{h}{m+u_0} - \frac{h}{m+u}\right)\right]v_0(v-v_0) - \frac{bu_0}{a+u^2}(u-u_0)(v-v_0)\\
=&\left[-\frac{r}{k} - \frac{bv}{a+u^2} + \frac{bu_0v(u+u_0)}{(a+u_0^2)(a+u^2)}\right](u-u_0)^2 + \left[\frac{\alpha\delta v_0 h}{(m+u)(m+u_0)} - \frac{bu_0}{a+u^2}\right]\\
&(u-u_0)(v-v_0) + \alpha\delta\left[\frac{1}{b_1+v} - \frac{h}{m+u} - \frac{v_0}{(b_1+v)(b_1+v_0)}\right](v-v_0)^2\\
=&\left[\frac{bv(uu_0-a)}{(a+u^2)(a+u_0^2)} - \frac{r}{k}\right](u-u_0)^2 + \left[\frac{\alpha\delta v_0 h}{(m+u)(m+u_0)} - \frac{bu_0}{a+u^2}\right](u-u_0)(v-v_0)\\
&+ \alpha\delta\left[\frac{b_1}{(b_1+v)(b_1+v_0)} - \frac{h}{m+u}\right](v-v_0)^2.
\end{aligned}$$

Now denote

$$H(\alpha,u) = \frac{\alpha\delta v_0 h}{(m+u)(m+u_0)} - \frac{bu_0}{a+u^2}.$$

According to Eq. (8), we get $u(x,t) \leq k$, and thus

$$H(\alpha,u) \leq \frac{\alpha\delta v_0 h}{(m+u)(m+u_0)} - \frac{bu_0}{a+ku}.$$

Recall that

$$\alpha = \frac{bmu_0 + bu_0^2}{\delta v_0 hk}.$$

In that way, we have

$$
\begin{aligned}
H(u) &\le \frac{(bmu_0 + bu_0^2)}{k(m+u)(m+u_0)} - \frac{bu_0}{a+ku} \\
&= \frac{(bmu_0 + bu_0^2)(a+ku) - bu_0k(m+u)(m+u_0)}{k(m+u)(m+u_0)(a+ku)} \\
&= \frac{bu_0(a-km)}{(m+u)(a+ku)k} \\
&\le \frac{bu_0|a-km|}{kma}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{dW}{dt} &\le \left[\frac{bv(uu_0-a)}{(a+u^2)(a+u_0^2)} - \frac{r}{k}\right](u-u_0)^2 + \frac{bu_0|a-km|}{kma}|u-u_0||v-v_0| \\
&\quad + \alpha\delta\left[\frac{b_1}{(b_1+v)(b_1+v_0)} - \frac{h}{m+u}\right](v-v_0)^2 \\
&\le \left[\frac{bv(uu_0-a)}{(a+u^2)(a+u_0^2)} - \frac{r}{k}\right](u-u_0)^2 + \left|\frac{bu_0(a-km)}{kma}\right|\frac{(u-u_0)^2+(v-v_0)^2}{2} \\
&\quad + \alpha\delta\left[\frac{b_1}{(b_1+v)(b_1+v_0)} - \frac{h}{m+u}\right](v-v_0)^2 \\
&\le \left[\frac{bv(uu_0-a)}{a(a+u_0^2)} - \frac{r}{k}\right](u-u_0)^2 + \left|\frac{bu_0(a-km)}{kma}\right|\frac{(u-u_0)^2+(v-v_0)^2}{2} \\
&\quad + \alpha\delta\left[\frac{b_1}{b_1(b_1+v_0)} - \frac{h}{m+k}\right](v-v_0)^2 \\
&= \left[\frac{bv(uu_0-a)}{a(a+u_0^2)} - \frac{r}{k} + \left|\frac{bu_0(a-km)}{2kma}\right|\right](u-u_0)^2 \\
&\quad + \alpha\delta\left[\frac{1}{b_1+v_0} - \frac{h}{m+k} + \left|\frac{bu_0(a-km)}{2kma}\right|\right](v-v_0)^2 \\
&= D(u,v)(u-u_0)^2 + R(v-v_0)^2.
\end{aligned}
$$

If $(\aleph_4)$ is established, then $D(u,v) < 0$ and $R < 0$. It is clear that $\frac{dW}{dt} < 0$, that is, the positive constant steady state (u_0, v_0) is globally asymptotically stable.

3.8 Proof of Theorem 2.11

From Lemma 2.7 we know that the projection of $\mathfrak{C}_n^j$ onto the z-space is bounded. Notice that

$$2j\pi < \omega_n \tau_n^j < 2(j+1)\pi, \quad j \in \mathbb{N}.$$

It follows that

$$\frac{1}{j+1} < \frac{2\pi}{\omega_n \tau_n^j} < \frac{1}{j}, \quad j \in \mathbb{N}.$$

Assume that $(z, \tau, \omega) \in \mathfrak{C}_n^j$ for $j \in \mathbb{N}$. Then $\frac{1}{j+1} < \frac{2\pi}{\omega} < \frac{1}{j}$ by Lemma 2.8. This fact shows that the projection of $\mathfrak{C}_n^j$ onto the T-space is bounded if τ is bounded.

Then from Lemma 2.10, since $\mu_1(U_0, \tau_n^j, \omega_n) > 0$ for all τ_n^j $(n \in I_1, j \in N_0)$, we get that each connected component $\mathfrak{C}_n^j$ is unbounded. In particular, according to the proof of Lemma 2.2, system (1) has no positive periodic solutions when $\tau = 0$. Thus the projections of $\mathfrak{C}_n^j$ for $j \in \mathbb{N}$ onto the τ-space include $\mathfrak{C}_n^j$.

4 Numerical simulation

In this section, we use numerical simulations to verify the theoretical results of Eq. (1). We choose the following parameters:

$$d_1 = 0.3, \quad r = 0.11, \quad a = 4.74, \quad \delta = 2, \quad b = 0.04, \quad h = 0.3801,$$
$$d_2 = 0.3, \quad m = 0.01, \quad b_1 = 0.901, \quad k = 6.8, \quad l = 6.$$

Then Ł = –0.0012 < 0, and by Theorem 2.1(a) we obtain that Eq. (3) has a unique positive root. Moreover, $u_0 = 3.0054 > b_1 h - m = 0.3325$, which ensures that Eq. (1) has a unique positive steady state (3.0054, 7.0323). By calculation we have $\vartheta(u_0) = 0.0295$, $\Gamma = \max\left\{ -\delta\kappa(u_0), -\frac{d_1}{d_2}\delta\kappa(u_0), -\frac{\gamma(u_0)}{h}\right\} = -0.0690$, so that $(\aleph_2)$ is satisfied. Then we have $h\vartheta(u_0) = 0.0112$, $\gamma(u_0) = 0.0262$, so that $(\aleph_3)$ is satisfied. In addition, we define $A_n = F_n^2 - H^2$. Then we know that $A_n < 0$ for $n \leq N_* = 2$ and $A_n > 0$ for $n > N_* = 2$:

$$A_0 = -0.0096, \quad A_1 = -0.0082, \quad A_2 = -0.0016,$$
$$A_3 = 0.0178, \quad A_4 = 0.0653, \quad A_5 = 0.1674, \ldots .$$

Hence for $n \leq N_* = 2$, Eq. (5) has a unique pair of purely imaginary roots $\pm i\omega$. By Eq. (21) we obtain that

$$\begin{aligned} &\tau_{0,0}^+ = 31.6469, \quad \tau_{0,1}^+ = 132.4325, \quad \tau_{0,2}^+ = 233.2181, \ldots, \\ &\tau_{1,0}^+ = 37.0069, \quad \tau_{1,1}^+ = 146.7688, \quad \tau_{1,2}^+ = 256.5307, \ldots, \\ &\tau_{2,0}^+ = 110.3625, \quad \tau_{2,1}^+ = 362.6447, \quad \tau_{2,2}^+ = 614.9269, \ldots . \end{aligned} \tag{27}$$

From Theorem 2.6 we have $\tau_0 = 31.6469$ and $\tau_1 = 132.4325$. Hence the positive steady state (3.0054, 7.0323) of Eq. (1) is locally asymptotically stable for $\tau \in (0, \tau_0)$ (see Fig. 1(a, d)) and is unstable for $\tau > \tau_0$ (see Fig. 1(b, e)). It then follows from Theorem 2.11 thate system (1) has at least one positive periodic orbit when $\tau > \tau_1$.

To verify the extended existence of bifurcating periodic solutions, we choose $\tau = 220$ faraway from Hopf bifurcation points in Eq. (27). The corresponding numerical simulation results are shown in Fig. 1(c, f).

5 Conclusions and discussion

In this paper, we establish the dynamic behaviors of a delayed diffusive predator–prey model with weak Allee effect for predator under the Neumann boundary condition. The stability of the positive steady state and the existence of Hopf bifurcation are obtained by taking the time delay τ as the bifurcation parameter. Particularly, we give the existence condition of the global Hopf bifurcation by using Wu's theory. This shows that the model has at least one periodic solution when the bifurcation parameter varies in a very large range. This can better explain the phenomenon of periodic fluctuations of biological populations.

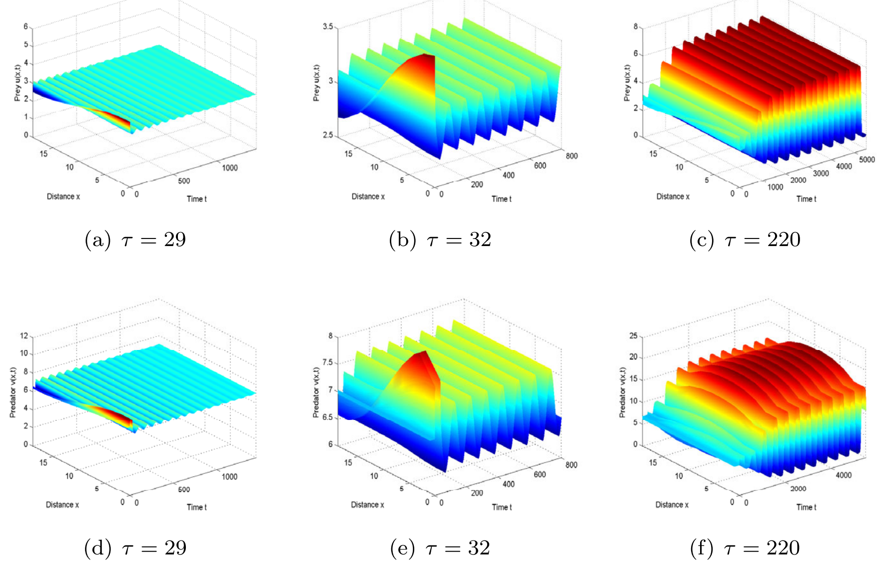

Figure 1 The unique positive steady state (3.0054, 7.0323) of Eq. (1) is asymptotically stable when $\tau = 29 < \tau_0 = 31.6469$, and system (1) has one positive periodic solution for $\tau = 32 > \tau_0$ and $\tau = 133 > \tau_1 = 132.4325$. (a) and (d) show the behaviors for u and v for $\tau = 29$; (b) and (e) show the behaviors for u and v for $\tau = 32$, respectively; (c) and (f) show the behaviors for u and v for $\tau = 220$, respectively

In our method, the key step is compressing the range in which positive periodic solutions exist. Otherwise, we cannot find a set of parameter values that satisfy all the conditions of global Hopf bifurcation. This method is valid for the Leslie–Gower models or Gause models with Holling III functional response but may be not applicable to the models with other functional response. These problems need further study.

Acknowledgements
Not applicable.

Author contributions
All authors participated in writing and coordination of the manuscript, and all authors read and approved the final manuscript.

Funding
This work is supported by the Natural Science Foundation of Heilongjiang Province (No. LH2022A002) and the National Natural Science Foundation of China (No. 12371160).

Data availability
All data generated or analyzed during this study are included in this paper (and its supplementary information files).

Declarations

Competing interests
The authors declare that they have no competing interests.

Author details
[1]School of Mathematical Sciences, Heilongjiang University, Harbin, Heilongjiang 150080, P.R. China. [2]Engineering Research Center of Agricultural Microbiology Technology, Ministry of Education & Heilongjiang Provincial Key Laboratory of Ecological Restoration and Resource Utilization for Cold Region & School of Mathematical Science, Heilongjiang University, Harbin 150080, P.R. China. [3]Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, P.R. China.

Received: 23 August 2024 Accepted: 12 January 2025 Published online: 27 January 2025

References

1. Nicholson, A.: An outline of the dynamics of animal populations. Aust. J. Zool. **2**, 9–65 (1954)
2. Gurney, W., Blythe, S., Nisbet, R.: Nicholson's blowflies revisited. Nature **287**, 17–21 (1980)
3. Charnov, E., Finerty, J.: Vole population-cycles: a case for kin-selection? Oecologia **45**(1), 1–2 (1980)
4. Fleming, R., Shoemaker, C.: Evaluating models for spruce budworm-forest management: comparing output with regional field data. Ecol. Appl. **2**(4), 460–477 (1992)
5. Royama, T., MacKinnon, W., Kettela, E., Carter, N., Hartling, L.: Analysis of spruce budworm outbreak cycles in New Brunswick, Canada, since 1952. Ecology **86**(5), 1212–1224 (2005)
6. Morin, H., Laprise, D., Bergeron, Y.: Chronology of spruce budworm outbreaks near Lake Duparquet, Abitibi region, Quebec. Can. J. For. Res. **23**, 1497–1506 (1993)
7. Alfaro, R., Berg, J., Axelson, J.: Periodicity of western spruce budworm in Southern British Columbia, Canada. For. Ecol. Manag. **315**(1), 72–79 (2014)
8. Hou, T.: Chinese Pine Caterpillars. China Science and Technology Press, Bei Jing (1987)
9. Wei, J.: Bifurcation analysis in a scalar delay differential equation. Nonlinearity **20**, 2483–2498 (2007)
10. Wei, J., Li, M.: Global existence of periodic solutions in a trineuron network model with delays. Physica D **98**, 106–119 (2004)
11. Yan, X., Li, W.: Bifurcation and global periodic solutions in a delayed facultative mutualism system. Physica D **227**, 51–69 (2007)
12. Ma, Y.: Global Hopf bifurcation in the Leslie–Gower predator–prey model with two delays. Nonlinear Anal., Real World Appl. **13**, 370–375 (2012)
13. Chen, S., Shi, J., Wei, J.: Global stability and Hopf bifurcation in a delayed diffusive Leslie–Gower predator–prey system. Int. J. Bifurc. Chaos **22**, 1250061 (2012)
14. Li, Y., Xiao, D.: Bifurcations of a predator–prey system of Holling and Leslie types. Chaos Solitons Fractals **34**, 606–620 (2007)
15. Ni, W., Wang, M.: Dynamics and patterns of a diffusive Leslie–Gower prey–predator model with strong Allee effect in prey. J. Differ. Equ. **261**, 4244–4274 (2016)
16. Zhou, J., Shi, J.: The existence, bifurcation and stability of positive stationary solutions of a diffusive Leslie–Gower predator–prey model with Holling-type II functional responses. J. Math. Anal. Appl. **405**, 618–630 (2013)
17. Liu, M., Xu, X., Zhang, C.: Stability and global Hopf bifurcation for neutral BAM neural network. Neurocomputing **145**, 122–130 (2014)
18. Song, Y., Wei, J.: Local Hopf bifurcation and global periodic solutions in a delayed predator–prey system. J. Math. Anal. Appl. **301**, 1–21 (2005)
19. Jiang, J., Song, Y.: Delay-induced Bogdanov–Takens bifurcation in a Leslie–Gower predator–prey model with nonmonotonic functional response. Commun. Nonlinear Sci. Numer. Simul. **19**, 2454–2465 (2014)
20. Cantrell, R., Cosner, C.: On the dynamics of predator–prey models with the Beddington–DeAngelis functional response. J. Math. Anal. Appl. **257**, 206–222 (2001)
21. Fan, M., Wang, K.: Periodic solutions of a discrete time nonautonomous ratio-dependent predator–prey system. Math. Comput. Model. **35**, 951–961 (2002)
22. Chen, Y.: Multiple periodic solutions of delayed predator–prey systems with type IV functional responses. Nonlinear Anal., Real World Appl. **5**, 45–53 (2004)
23. Fan, Y., Li, W., Wang, L.: Periodic solutions of delayed ratio-dependent predator–prey models with monotonic or non-monotonic functional response. Nonlinear Anal., Real World Appl. **5**, 247–263 (2004)
24. Skalski, G., Gilliam, J.: Functional responses with predator interference: viable alternatives to the Holling II model. Ecology **82**, 3083–3092 (2001)
25. Sarkar, K., Khajanchi, S.: Spatiotemporal dynamics of a predator-prey system with fear effect. J. Franklin Inst. **360**(11), 7380–7414 (2023)
26. Hossain, M., Pal, N., Samanta, S.: An eco-epidemiological model with the impact of fear. Chaos Solitons Fractals **134**, 109718 (2020)
27. Stephens, P., Sutherland, W., Freckleton, R.: What is the Allee effect? Oikos **87**(1), 185–190 (1999)
28. Yang, P., Wang, Y.: Hopf–zero bifurcation in an age-dependent predator–prey system with Monod–Haldane functional response comprising strong Allee effect. J. Differ. Equ. **269**(11), 9583–9618 (2020)
29. Wang, H., Guo, S., Li, S.: Stationary solutions of advective Lotka–Volterra models with a weak Allee effect and large diffusion. Nonlinear Anal., Real World Appl. **56**, 103171 (2020)
30. Wang, J., Shi, J., Wei, J.: Dynamics and pattern formation in a diffusive predator–prey system with strong Allee effect in prey. J. Differ. Equ. **251**(4–5), 1276–1304 (2011)
31. Wang, M., Kot, M.: Speeds of invasion in a model with strong or weak Allee effects. Math. Biosci. **171**(1), 83–97 (2001)
32. Wu, J.: Theory and Applications of Partial Functional Differential Equations. Springer, New York (1996)
33. Xu, X., Liu, M.: Global Hopf bifurcation of a general predator–prey system with diffusion and stage structures. J. Differ. Equ. **269**, 8370–8386 (2020)
34. Liu, M., Cao, J., Xu, X.: Global Hopf bifurcation in a phytoplankton–zooplankton system with delay and diffusion. Int. J. Bifurc. Chaos **31**, 2150114 (2021)
35. Xu, X., Wei, J.: Bifurcation analysis of a spruce budworm model with diffusion and physiological structures. J. Differ. Equ. **262**, 5206–5230 (2017)
36. Zhang, Q., Liu, M., Xu, X.: Global Hopf bifurcation of a delayed diffusive Gause-type predator–prey system with the fear effect and Holling III functional response. Math. Model. Nat. Phenom. **19**(5), 1–31 (2024)
37. Wang, K., Xu, X., Liu, M.: Global Hopf bifurcation of a diffusive modified Leslie–Gower predator–prey model with delay and Michaelis–Menten type prey harvesting. Qual. Theory Dyn. Syst. **23**(2), 81 (2024)

Publisher's Note