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Stability of a delayed prey–predator model with fear, stochastic effects and Beddington–DeAngelis functional response

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Abstract

In view of the importance of predator-dependent functional response and fear of prey induced by powerful predators, we construct a delayed prey–predator model with fear and Beddington–DeAngelis functional response. The existence, uniqueness, and global asymptotic stability of equilibrium points are investigated and some criteria are established. Next, Hopf bifurcation analysis is executed, and the critical values of such bifurcation parameters as fear and delay for the determinate system are obtained. Then we extend it to a random environment and study the boundedness of expectation of solutions and the global asymptotic stability. Finally, the main findings are validated by numerical examples. It is worth noting that the specific influences of fear by predator, time delay, and white noise are explored numerically. Simulation figures intuitively exhibit that fear, delay, and white noise bring serious influences on the stability of the system. Fear from predator leads to a lower equilibrium state of prey and predator, and it can change the system stability from unstable to stable after exceeding a certain critical value. The time delay has a significant impact on the system stability by producing Hopf bifurcations accompanied by limit cycles, and even lead to multiple stabilities. Larger white noise can change the system stability from stable to unstable.

Keywords: Delay; Fear; Prey; Predator model; Hopf bifurcation; Stability

1 Introduction

For a predator–prey system, functional response is an important index of assessing the speed of feeding prey by per predator, and it often brings large influence to the system stability and bifurcation dynamics. The original functional response is assumed that the feeding rate of predator is proportional to the amount of prey and predator species, i.e., it is linear [1]. Graphically, it is a straight line passing through the origin. By analysis, it is not difficult to find that the assumption is not reasonable. For each predator, the feeding rate is limited and there is a maximum, that is, it is impossible to be proportional to the prey amount if the prey is superabundant. At this situation, the predation rate should be related to the predator density, so it is modified to the Holling II functional response, which well describes the predation rate up to a limited range [2]. Whereas for large predators in a population system, there always exists mutual interference and competition among indi-

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viduals. It will reduce the predation rate of predators, and hence Holling-II type is changed to the Beddington–DeAngelis functional response, which is first introduced by the authors [3, 4]. The main difference between them is the competition and interference among predator individuals, which is inferred by experimental data [5]. Usually there are three kinds of predator-dependent functional response, i.e., Beddington–DeAngelis, Crowley–Martin [6], and Hassell–Varley [7]. Compared with other ratio-dependent responses, the Beddington–DeAngelis type is very popular since even in the case of low density, it will not appear the singular phenomena. By grouping effect of predation, the Beddington–DeAngelis functional response can be found in [8]. In the last few decades, there have been many research works on the dynamics of system with Beddington–DeAngelis functional response [9–11]. The traditional prey–predator system with Beddington–DeAngelis functional response is listed below:

$$\begin{cases} \dfrac{dx}{dt} = x\left(r_1 - a_1 x - \dfrac{b_1 y}{1 + \mu_1 x + \mu_2 y}\right), \\ \dfrac{dy}{dt} = y\left(-r_2 - a_2 y + \dfrac{b_2 x}{1 + \mu_1 x + \mu_2 y}\right), \end{cases}$$

where x denotes the prey biomass and y denotes the predator biomass at time t. For the biological meanings of all parameters, the readers are referred to [3, 4].

Direct predation is a popular phenomenon that has been studied for a long time. Apart from the direct killing by predator, the fear of prey to the powerful predator is also a crucial factor affecting the system dynamics, and sometimes it even changes the prey's demography [12–14]. It is natural that when the prey perceives the predator's signal or predation cue, it will always present some instinctive reactions like reducing its foraging activity, shifting to another safer place and presenting physiological stress resulting in the decrease of fecundity. For example, in Reference [15], an experiment was carried out between the garter snake (predator) and salamander (prey). The experimental results showed that when the salamander perceived the danger through chemical cues of being predated by garter snake, the salamander reduced its foraging activity. For a system with no direct predation, in order to explore how the fear from predator would affect the population reproduction, in 2011, Zanette, White, and Allen [16] executed an experiment on songbirds during the whole breeding period. They found that due to the anti-predation response, the female declined foraging activity and laid few eggs resulting in the decrease of the birth numbers of songbirds and survival numbers of descendants, which led to about 40 percent reduction of offsprings. Recently, more works about how fear affects the system dynamics have been reported [17–19]. Incorporating the effect of fear from predator, we get the following model:

$$\begin{cases} \dfrac{dx}{dt} = x\left(\dfrac{r_1}{1 + ky} - a_1 x - \dfrac{b_1 y}{1 + \mu_1 x + \mu_2 y}\right), \\ \dfrac{dy}{dt} = y\left(-r_2 - a_2 y + \dfrac{b_2 x}{1 + \mu_1 x + \mu_2 y}\right), \end{cases} \tag{1.1}$$

where k represents the level to which the fear affects the birth rate of prey. In addition, we know that time delay is inevitable in most of the biological processes. There exists a time lag in almost all proceedings of population dynamics [20]. For example, for predator, it will take some definite time to digest the prey. There is a time lag to convert the prey into the growth of predator [21]. For young predator to mature, it needs some time. Almost identically, in the process of the predator consuming prey and breeding its progeny, it will take a long period of time. This is so-called gestation delay [22]. For delayed fear effect on the system dynamics, it is more usual. After the prey feeling risk from predator, it will assess the level of fear and take some measures like decreasing foraging activity or shifting to another safer zone, and so on. These counter predation manners are not immediate and will take some time to accomplish. That is, there is a time lag between the prey's perceiving risk and presenting some anti-predation measures [23–25]. The negative influence of fear on the population dynamics cannot be seen right away; for example, it will take a long time to see the influence of fear from predator on the prey's birth number. On the other hand, from mathematical perspective, time delay can change the system stability, even lead to multiple stability [19, 24]. Therefore, it is necessary to incorporate the delay effect into model (1.1).

As mentioned above, incorporating the effect of delayed fear on the birth rate of prey (τ_1), as well as the effect of gestation delay of predators (τ_2), we get the delayed version of (1.1) as follows:

$$\begin{cases} \dfrac{dx}{dt} = x\left(\dfrac{r_1}{1+ky(t-\tau_1)} - a_1x - \dfrac{b_1y}{1+\mu_1x+\mu_2y}\right), \\ \dfrac{dy}{dt} = y\left(-r_2 - a_2y + \dfrac{b_2x(t-\tau_2)}{1+\mu_1x(t-\tau_2)+\mu_2y(t-\tau_2)}\right). \end{cases} \tag{1.2}$$

The initial data is as follows:

$$x(\theta) = \chi_1(\theta) > 0, y(\theta) = \chi_2(\theta) > 0, -\tau \le \theta \le 0,$$

where $(\chi_1, \chi_2)^T \in C([-\tau, 0], R_+^2)$ is positive and continuous defined on $[\tau, 0)$, $\tau = \max[\tau_1, \tau_2]$. We assume all parameters are positive to meet the biological requirements.

For the determinate model, the stability of equilibrium state is and is going to be an important topic. Some nice results have been reported, such as the Lyapunov-based stability for a prey–predator system [7, 19, 22, 23] and an epidemic system [26, 27], finite-time stabilization for an impulsive system [28, 29], and stochastic stabilization for a stochastic system [30, 31]. Based on the importance of system stability, in this paper we aim to study the local or global stability of the system and explore when the Hopf bifurcation occurs if the stability is lost. Our main contributions are as follows:

(1) A prey–predator model with fear and two delays is formulated, then it is extended to stochastic scenarios.
(2) The sufficient conditions of local or global stability of the model are established.
(3) The critical values of fear and time delay resulting in the occurrence of Hopf bifurcations are obtained.
(4) How the fear, time delay, and stochastic environment affect the system dynamics is numerically investigated.

The rest of this work is organized as follows. The existence and boundedness of solutions and the existence and stability of equilibrium are discussed in Sect. 2. The Hopf bifurcation analysis is carried out in Sect. 3. The dynamics of stochastic scenario is executed in Sect. 4. Some numerical examples are performed in Sect. 5. Finally, a brief conclusion is given to end this work in Sect. 6.

2 Properties of the equilibrium

By the theory of delayed differential equation, we conclude that system (1.2) has a unique positive solution under above conditions, so we begin with the existence and stability of the equilibrium state of (1.2).

2.1 Existence of the equilibrium

Let $\tilde{E}(\tilde{x},\tilde{y})$ be the equilibrium of system (1.2), then it satisfies the following equation:

$$\begin{cases} \dfrac{r_1}{1+k\tilde{y}} - a_1\tilde{x} &= \dfrac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}, \\ \dfrac{b_2\tilde{x}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} &= r_2 + a_2\tilde{y}. \end{cases} \tag{2.1}$$

By the second equation of (2.1), then

$$\tilde{x} = \frac{(r_2+a_2\tilde{y})(1+\mu_2\tilde{y})}{b_2-\mu_1(r_2+a_2\tilde{y})},$$

which is positive under the condition that $b_2-\mu_1(r_2+a_2\tilde{y})>0$. Substituting $\tilde{x}$ into (2.1), then $\tilde{y}$ should meet the following quartic equation:

$$ka_1a_2\mu_2^2y^4 + \varrho_1y^3 + \varrho_2y^2 + \varrho_3y + \varrho_4 = 0, \tag{2.2}$$

where

$$\begin{aligned} \varrho_1 &= a_1a_2\mu_2(\mu_2+k(1+\mu_1\tilde{x})) + ka_1\mu_2(a_2+r_2\mu_2) - kb_1\mu_1a_2, \\ \varrho_2 &= kb_1(b_2-\mu_1r_2) - b_1\mu_1a_2 + a_1(a_2\mu_2(1+\mu_1\tilde{x}) + kr_1r_2\mu_2 + (a_2+r_2\mu_2) \\ &\quad (\mu_2+k(1+\mu_1x))) + r_1a_2\mu_2^2, \\ \varrho_3 &= b_1(b_2-r_2\mu_1) + r_1a_2\mu_2(1+\mu_1\tilde{x}) + a_1((a_2+r_2\mu_2)(1+\mu_1\tilde{x}) \\ &\quad +r_2(\mu_2+k(1+\mu_1\tilde{x}))), \\ \varrho_4 &= (r_2(a_1+r_1\mu_1) - r_1b_2)(1+\mu_1\tilde{x}). \end{aligned}$$

We verify that if

$$r_2(a_1+r_1\mu_1) < r_1b_2 \quad \text{and} \quad b_1\mu_1 < a_1\mu_2,$$

then

$$\varrho_i > 0\ (i=1,2,3) \quad \text{and} \quad \varrho_4 < 0.$$

By Descartes' rule of sign, (2.2) has a unique positive solution $\tilde{y}$. Therefore, system (1.2) has a unique positive equilibrium. In the rest of this paper, we always denote the unique equilibrium of (1.2) by $\tilde{E}$ (for simplicity).

2.2 Stability of the equilibrium

Now we study the local asymptotic stability (LAS) and global asymptotic stability (GAS) of $\tilde{E}$.

Theorem 2.1 *System* (1.1) *with the initial data* $\chi_i(0) > 0 (i = 1, 2)$ *is LAS around* $\tilde{E}$ *under the condition that*

C_0 : $\dfrac{b_1\mu_1\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} < a_1$.

Proof By Talor's formula, we linearize (1.1) and obtain the variational matrix at $\tilde{E}$ as follows:

$$J|_{\tilde{E}} = \begin{pmatrix} -a_1\tilde{x} + \dfrac{b_1\mu_1\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} & -\dfrac{kr_1}{(1+k\tilde{y})^2} - \dfrac{b_1\tilde{x}(1+\mu_1\tilde{x})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} \\ \dfrac{b_2\tilde{y}(1+\mu_2\tilde{y})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} & -a_2\tilde{y} - \dfrac{b_2\mu_2\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} \end{pmatrix}.$$

The characteristic equation of $J|_{\tilde{E}}$ is

$$\begin{aligned} &\lambda^2 + \left(a_1\tilde{x} + a_2\tilde{y} + \frac{b_2\mu_2\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} - \frac{b_1\mu_1\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}\right)\lambda \\ &\quad + \left(a_1\tilde{x} - \frac{b_1\mu_1\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}\right)\left(a_2\tilde{y} + \frac{b_2\mu_2\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}\right) \\ &\quad + \left(\frac{kr_1}{(1+k\tilde{y})^2} + \frac{b_1\tilde{x}(1+\mu_1\tilde{x})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}\right)\frac{b_2\tilde{y}(1+\mu_2\tilde{y})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} = 0. \end{aligned} \tag{2.3}$$

Under condition C_0, we know that equation (2.3) has two negative roots. By the stability theory of functional differential equations [32, Theorem 4.4], system (1.1) is LAS around $\tilde{E}$. □

As to the GAS of $\tilde{E}$, we have the following conclusion.

Theorem 2.2 *For system* (1.2) *with the initial data* $\chi_i(\theta) > 0 (i = 1, 2), -\tau \le \theta \le 0$, *suppose that the following conditions hold*:

(C_1) $A := a_1 - \dfrac{b_1\mu_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} - \dfrac{b_2(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} > 0$,

(C_2) $B := a_2 - \dfrac{b_1}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} - \dfrac{r_1k}{1+k\tilde{y}} > 0$.

Then (1.2) *is GAS around the equilibrium* $\tilde{E}$.

Proof For $\tilde{E}(\tilde{x},\tilde{y})$, we make a transformation as $x(t) = \tilde{x}e^{X(t)}, y(t) = \tilde{y}e^{Y(t)}$, then (1.2) turns into

$$\begin{cases} \dfrac{dX(t)}{dt} = \dfrac{r_1}{1+k\tilde{y}e^{Y(t-\tau_1)}} - a_1\tilde{x}e^X - \dfrac{b_1\tilde{y}e^Y}{1+\mu_1\tilde{x}e^X+\mu_2\tilde{y}e^Y}, \\ \dfrac{dY(t)}{dt} = -r_2 - a_2\tilde{y}e^Y + \dfrac{b_2\tilde{x}e^{X(t-\tau_2)}}{1+\mu_1\tilde{x}e^{X(t-\tau_2)}+\mu_2\tilde{y}e^{Y(t-\tau_2)}}, \end{cases} \tag{2.4}$$

where $X(t)$ and $Y(t)$ are both positive on $t \in [-\tau, \infty)$. It is easy to know that the equilibrium state of system (1.2) is changed to zero $(X, Y) = (0, 0)$ (trivial equilibrium state) of (2.4). By the stability criteria of delayed equations [20], it is sufficient to find a functional $V(t)$ such that the Dini derivative $D^+V(t) < 0$. From (2.4) we have

$$\begin{aligned}
\frac{dX(t)}{dt} &= \frac{r_1}{1+k\tilde{y}e^{Y(t-\tau_1)}} - a_1\tilde{x}e^X - \frac{b_1\tilde{y}e^Y}{1+\mu_1\tilde{x}e^X+\mu_2\tilde{y}e^Y} \\
&= \frac{r_1}{1+k\tilde{y}e^{Y(t-\tau_1)}} - a_1\tilde{x}e^X - \frac{b_1\tilde{y}e^Y}{1+\mu_1\tilde{x}e^X+\mu_2\tilde{y}e^Y} \\
&\quad -\frac{r_1}{1+k\tilde{y}} + a_1\tilde{x} + \frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} \\
&= -a_1\tilde{x}(e^X-1) - \frac{r_1k\tilde{y}}{(1+k\tilde{y}e^{Y(t-\tau_1)})(1+k\tilde{y})}(e^{Y(t-\tau_1)}-1) \\
&\quad + \frac{b_1\mu_1\tilde{x}\tilde{y}(e^X-1) - b_1\tilde{y}(e^Y-1)}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})(1+\mu_1\tilde{x}e^X+\mu_2\tilde{y}e^Y)}.
\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
D^+|X(t)| &\le -a_1\tilde{x}|e^X-1|) + \frac{r_1k\tilde{y}}{(1+k\tilde{y}e^{Y(t-\tau_1)})(1+k\tilde{y})}|e^{Y(t-\tau_1)}-1| \\
&\quad + \frac{b_1\mu_1\tilde{x}\tilde{y}|e^X-1| + b_1\tilde{y}|e^Y-1|)}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})(1+\mu_1\tilde{x}e^X+\mu_2\tilde{y}e^Y)} \\
&\le -a_1\tilde{x}|e^X-1| + \frac{r_1k\tilde{y}}{1+k\tilde{y}}|e^{Y(t-\tau_1)}-1| + \frac{b_1\mu_1\tilde{x}\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}|e^X-1| \\
&\quad + \frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}|e^Y-1|.
\end{aligned} \tag{2.6}$$

Similarly, we have

$$\begin{aligned}
\frac{dY(t)}{dt} &= -r_2 - a_2y + \frac{b_2x(t-\tau_2)}{1+\mu_1x(t-\tau_2)+\mu_2y(t-\tau_2)} - \frac{\delta_2^2}{2} \\
&= -r_2 - a_2\tilde{y}e^Y + \frac{b_2\tilde{x}e^{X(t-\tau_2)}}{1+\mu_1\tilde{x}e^{X(t-\tau_2)}+\mu_2\tilde{y}e^{Y(t-\tau_2)}} \\
&\quad + r_2 + a_2\tilde{y} - \frac{b_2\tilde{x}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} \\
&= -a_2\tilde{y}(e^Y-1) + \frac{b_2\tilde{x}(1+\mu_2\tilde{y})(e^{X(t-\tau_2)}-1) - b_2\mu_2\tilde{x}\tilde{y}(e^{Y(t-\tau_2)}-1)}{(1+\mu_1\tilde{x}e^{X(t-\tau_2)}+\mu_2\tilde{y}e^{Y(t-\tau_2)})(1+\mu_1\tilde{x}+\mu_2\tilde{y})},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
D^+|Y(t)| &\le -a_2\tilde{y}|e^Y-1| + \frac{b_2\tilde{x}(1+\mu_2\tilde{y})|e^{X(t-\tau_2)}-1| - b_2\mu_2\tilde{x}\tilde{y}|e^{Y(t-\tau_2)}-1|}{(1+\mu_1\tilde{x}e^{X(t-\tau_2)}+\mu_2\tilde{y}e^{Y(t-\tau_2)})(1+\mu_1\tilde{x}+\mu_2\tilde{y})} \\
&\le -a_2\tilde{y}|e^Y-1| + \frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}|e^{X(t-\tau_2)}-1|.
\end{aligned} \tag{2.8}$$

Adding (2.6) and (2.8), then

$$\begin{aligned} D^{+}(|X(t)|+|Y(t)|) \le & -\left(a_1\tilde{x}-\frac{b_1\mu_1\tilde{x}\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)|e^{X}-1| \\ & -\left(a_2\tilde{y}-\frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)|e^{Y}-1| \\ & +\frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}|e^{X(t-\tau_2)}-1|+\frac{r_1k\tilde{y}}{1+k\tilde{y}}|e^{Y(t-\tau_1)}-1|. \end{aligned} \tag{2.9}$$

To eliminate the delay term, we define

$$W(t)=\int_t^{t+\tau_2}\frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}|e^{X(s-\tau_2)}-1|ds+\int_t^{t+\tau_1}\frac{r_1k\tilde{y}}{1+k\tilde{y}}|e^{Y(s-\tau_1)}-1|ds. \tag{2.10}$$

Differentiating (2.10) on t, then

$$\begin{aligned} \frac{dW(t)}{dt}= & \frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}(|e^{X(t)}-1|-|e^{X(t-\tau_2)}-1|) \\ & +\frac{r_1k\tilde{y}}{1+k\tilde{y}}(|e^{Y(t)}-1|-|e^{Y(t-\tau_1)}-1|). \end{aligned} \tag{2.11}$$

Let $V(t)=|X(t)|+|Y(t)|+W(t)$. Obviously, it is positive on $t\in[-\tau,\infty)$. Adding (2.9) and (2.11), we have

$$\begin{aligned} D^{+}V(t) \le & -\left(a_1\tilde{x}-\frac{b_1\mu_1\tilde{x}\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)|e^{X}-1| \\ & -\left(a_2\tilde{y}-\frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{r_1k\tilde{y}}{1+k\tilde{y}}\right)|e^{Y}-1|. \end{aligned} \tag{2.12}$$

Applying Taylor's formula, then

$$\begin{aligned} & \left(a_1\tilde{x}-\frac{b_1\mu_1\tilde{x}\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)|e^{X}-1| \\ & +\left(a_2\tilde{y}-\frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{r_1k\tilde{y}}{1+k\tilde{y}}\right)|e^{Y}-1| \\ \ge & \left(a_1\tilde{x}-\frac{b_1\mu_1\tilde{x}\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{b_2\tilde{x}(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)|X(t)|+ \\ & \left(a_2\tilde{y}-\frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{r_1k\tilde{y}}{1+k\tilde{y}}\right)|Y(t)| \\ = & A|X|+B|Y|. \end{aligned}$$

By $A>0, B>0$, then

$$D^{+}V(t)\le -A|X|-B|Y|<0.$$

Applying the stability theory of delayed functional equation [see Page 138 in 20], the equilibrium $\tilde{E}$ of system (1.2) is GAS. □

Remark 2.1 According to Theorem 2.1, if the condition C_0 does not hold, system (1.1) may change its stability and produce a fluctuation. Compared with reference [19], the competition and the gestation delay of predators are incorporated in system (1.2). The Hopf bifurcation caused by the change of stability of (1.2) is analyzed in the next section, which is not studied in [19], but it is the main work of this paper. It is meaningful to investigate the dynamic properties of species by analyzing their periodic fluctuations.

Remark 2.2 In the process of the proof of GAS, with the help of a transformation, the equilibrium point of (1.2) is transferred to zero (the trivial equilibrium point), which makes the subsequent computation easier. In addition, by constructing a suitable functional to eliminate the effect of time delays, the sufficient conditions in Theorem 2.2 have no delays, which means that, under certain constraints, the time delay of biological process has no effect on the GAS of system (1.2).

3 Hopf bifurcation analysis

We begin with linearizing system (1.2) by Taylor's formula. Make a transformation as $U(t) = x - \tilde{x}$, $V(t) = y - \tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are the equilibrium points of (1.2), then we have

$$\frac{d}{dt}\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = J_0 \begin{pmatrix} U(t) \\ V(t) \end{pmatrix} + J_1 \begin{pmatrix} U(t-\tau_1) \\ V(t-\tau_1) \end{pmatrix} + J_2 \begin{pmatrix} U(t-\tau_2) \\ V(t-\tau_2) \end{pmatrix},$$

where

$$J_0 = \begin{pmatrix} -a_1\tilde{x} + \dfrac{b_1\mu_1\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} & -\dfrac{b_1\tilde{x}(1+\mu_1\tilde{x})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} \\ 0 & -a_2\tilde{y} \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 0 & -\dfrac{r_1 k\tilde{x}}{(1+k\tilde{y})^2} \\ 0 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & 0 \\ \dfrac{b_2\tilde{y}(1+\mu_2\tilde{y})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} & -\dfrac{b_2\mu_2\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2} \end{pmatrix}.$$

The Jacobian matrix at the equilibrium status reads

$$J = J_0 + J_1 e^{-\lambda\tau_1} + J_2 e^{-\lambda\tau_2}.$$

Define

$$\rho_1 = -a_1\tilde{x} + \frac{b_1\mu_1\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}, \quad \rho_2 = -\frac{b_1\tilde{x}(1+\mu_1\tilde{x})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}, \quad \rho_3 = -a_2\tilde{y},$$
$$\rho_4 = -\frac{r_1 k\tilde{x}}{(1+k\tilde{y})^2}, \quad \rho_5 = \frac{b_2\tilde{y}(1+\mu_2\tilde{y})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}, \quad \rho_6 = -\frac{b_2\mu_2\tilde{x}\tilde{y}}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})^2}.$$

We have $\rho_i < 0$ under condition $C_0, i = 1, 2, \ldots, 6$. Then the characteristic equation of J is $|J - \lambda I| = 0$, that is,

$$\lambda^2 - (\rho_1 + \rho_3 + \rho_6 e^{-\lambda\tau_2})\lambda + \rho_1(\rho_3 + \rho_6 e^{-\lambda\tau_2}) - (\rho_2 + \rho_4 e^{-\lambda\tau_1})\rho_5 e^{-\lambda\tau_2} = 0. \tag{3.1}$$

By reorganizing, then

$$\lambda^2 - (\rho_1 + \rho_3)\lambda + \rho_1\rho_3 + (\rho_1\rho_6 - \rho_2\rho_5 - \rho_6\lambda)e^{-\lambda\tau_2} - \rho_4\rho_5 e^{-\lambda(\tau_1+\tau_2)} = 0.$$

For convenience, we rewrite it as

$$\lambda^2 + \sigma_1\lambda + \sigma_2 + (\sigma_3 + \sigma_4\lambda)e^{-\lambda\tau_2} + \sigma_5 e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{3.2}$$

where

$$\sigma_1 = -(\rho_1 + \rho_3), \quad \sigma_2 = \rho_1\rho_3, \quad \sigma_3 = \rho_1\rho_6 - \rho_2\rho_5, \quad \sigma_4 = -\rho_6, \quad \sigma_5 = -\rho_4\rho_5. \tag{3.3}$$

The negativity of $\rho_i (i = 1, 2, \ldots, 6)$ implies the positivity of σ_j, i.e., $\sigma_j > 0$ for all $j = 1, 2, \ldots, 5$.

3.1 Hopf bifurcation of (1.1)

System (1.1) is equivalent to the scenario of (1.2) with $\tau_1 = \tau_2 = 0$, then (4.2) turns into

$$\lambda^2 + (\sigma_1 + \sigma_4)\lambda + (\sigma_2 + \sigma_3 + \sigma_5) = 0. \tag{3.4}$$

Now we analyze the existence of Hopf bifurcation around $\tilde{E}$ on parameter k. Denote

$$u(k) = (\sigma_1 + \sigma_4)(k), \quad v(k) = (\sigma_2 + \sigma_3 + \sigma_5)(k),$$

then (3.4) becomes

$$\lambda^2 + u(k)\lambda + v(k) = 0. \tag{3.5}$$

For the discussion of Hopf bifurcation, we give the following Hopf bifurcation theorem.

Lemma 3.1 [33] *Suppose that system* (1.1) *is LAS around* $\tilde{E}$, *and* $\lambda = \varphi(k) \pm i\omega(k)$ *is a pair of complex eigenvalues of* (3.5). *If there exists a constant* $\tilde{k}$ *such that* $\varphi(\tilde{k}) = 0, \omega(\tilde{k}) > 0$ *and* $\frac{d\varphi}{dk}\Big|_{k=\tilde{k}} \neq 0$, *then* $\tilde{E}$ *changes its stability from stable to unstable, and there is a Hopf bifurcation around* $\tilde{E}$ *accompanied by a limit cycle at* $k = \tilde{k}$.

Take fear k as the Hopf bifurcation parameter, then we have the following results.

Theorem 3.1 *System* (1.1) *undergoes a Hopf bifurcation around* $\tilde{E}$ *when the bifurcation parameter* k *crosses the threshold value* $\tilde{k}$ *satisfying* $u(\tilde{k}) = 0, v(\tilde{k}) > 0$.

Proof Due to the condition $u(\tilde{k}) = 0, v(\tilde{k}) > 0$, there exist two purely imaginary roots $\lambda_j = \pm i\sqrt{v(\tilde{k})}$ for equation (3.5), and hence the roots of (3.5) have the form $\lambda_j = \varphi(k) \pm i\omega(k)$ in an open neighborhood of $\tilde{k}$, where $\varphi(k), \omega(k)$ are real valued respectively. By Lemma 3.1, system (1.1) changes its stability through Hopf bifurcation provided the following transversality condition holds:

$$\frac{d}{dk}(Re\lambda_j(k))\Big|_{k=\tilde{k}} = \frac{d\varphi(k)}{dk}\Big|_{k=\tilde{k}} \neq 0.$$

Putting $\lambda(k) = \varphi(k) + i\omega(k)$ in (3.5) and differentiating k, and separating the real and the imaginary parts, we have

$$\begin{cases} (2\varphi + u)\varphi'(k) - 2\omega\omega'(k) = u'(k)\varphi + v'(k), \\ 2\omega\varphi'(k) + (2\varphi + u)\omega'(k) = -u'(k)\omega. \end{cases}$$

That is,

$$\begin{cases} \varphi'(k)P_1 - \omega'(k)P_2 = P_3, \\ \varphi'(k)P_2 + \omega'(k)P_1 = P_4, \end{cases}$$

where $P_1 = 2\varphi + u, P_2 = 2\omega, P_3 = u'(k)\varphi + v'(k), P_4 = -u'(k)\omega$. Then

$$\varphi'(k) = \frac{P_1P_3 + P_2P_4}{P_1^2 + P_2^2}. \tag{3.6}$$

By the condition that $\varphi(\tilde{k}) = 0, \omega(\tilde{k}) = \pm\sqrt{v}$. When $\varphi(\tilde{k}) = 0, \omega(\tilde{k}) = \sqrt{v}$, we have $P_1 = 0, P_2 = 2\sqrt{v}, P_3 = v'(k), P_4 = u'(k)\sqrt{v}$. Then

$$\left.\frac{d\varphi(k)}{dk}\right|_{k=\tilde{k}} = \left.\frac{u'(k)}{2}\right|_{k=\tilde{k}} \neq 0.$$

Similarly, the conclusion holds provided $\varphi(\tilde{k}) = 0, \omega(\tilde{k}) = -\sqrt{v}$. □

Remark 3.1 For system (1.1), by the condition $u(k) = 0$, we have

$$a_1\tilde{x} + a_2\tilde{y} + \frac{b_2\mu_2\tilde{x}\tilde{y}}{(1 + \mu_1\tilde{x} + \mu_2\tilde{y})^2} - \frac{b_1\mu_1\tilde{x}\tilde{y}}{(1 + \mu_1\tilde{x} + \mu_2\tilde{y})^2} = 0. \tag{3.7}$$

Solving equation (3.7) together with the definition of $\tilde{E}$, we have

$$\tilde{k} = \frac{r_1 - \Upsilon}{\Upsilon\tilde{y}},$$

where

$$\Upsilon = \left(r_2 - \frac{b_2x(1 + \mu_1x + 2\mu_2y)}{(1 + \mu_1x + \mu_2y)^2} + \frac{b_1y(1 + 2\mu_1x + \mu_2y)}{(1 + \mu_1x + \mu_2y)^2}\right).$$

An easy computation yields

$$\left.\frac{d\varphi(k)}{dk}\right|_{k=\tilde{k}} = -\tilde{k}\Upsilon \neq 0.$$

Therefore system (1.1) has Hopf bifurcation at $\tilde{k} = \frac{r_1-\Upsilon}{\Upsilon\tilde{y}}$, which is indicated in Sect. 5 by a numerical example.

Take b_1 as the bifurcation parameter, then we have the following result.

Theorem 3.2 *System* (1.1) *undergoes a Hopf bifurcation around the equilibrium point* $\tilde{E}$ *when the bifurcation parameter* b_1 *crosses the threshold value* $\tilde{b}_1$ *where*

$$\tilde{b}_1 = \frac{(a_1\tilde{x} + a_2\tilde{y})(1 + \mu_1\tilde{x} + \mu_2\tilde{y})^2 + b_2\mu_2\tilde{x}\tilde{y}}{\mu_1\tilde{x}\tilde{y}}.$$

3.2 Hopf bifurcation of (1.2)

For the discussion of Hopf bifurcation of delayed differential equations, we give a useful lemma given by Yuan and Wei [34].

Lemma 3.2 *For the following exponential polynomial*:

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_r}) = \lambda^m + p_1^0\lambda^{r-1} + \cdots + p_{m-1}^0\lambda + p_m^0 \\ +(p_1^1\lambda^{m-1} + \cdots + p_{m-1}^1\lambda + p_m^1)e^{-\lambda\tau_1} + \cdots \\ +(p_1^r\lambda^{m-1} + \cdots + p_{m-1}^r\lambda + p_m^r)e^{-\lambda\tau_r} \end{aligned}$$

where $\tau_k \geq 0 (k = 1, 2, \dots, r)$ *and* $p_j^i (i = 0, 1, \dots, r, j = 1, 2, \dots, m)$ *are constants. Denote the zero of* $P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_r})$ *in the open half plane by* λ_0, ζ *is the sum of the orders of* λ_0. *Then* ζ *will vary as* $(\tau_1, \tau_2, \dots, \tau_r)$ *varies only if a zero appears on or across the imaginary axis.*

3.2.1 Model with one delay

In the subsection, we begin with one delay case. In system 1.2, let $\tau_1 = 0, \tau_2 > 0$, then (1.2) leads to

$$\begin{cases} dx = x\left(\dfrac{r_1}{1 + ky(t)} - a_1x - \dfrac{b_1y}{1 + \mu_1x + \mu_2y}\right), \\ dy = y\left(-r_2 - a_2y + \dfrac{b_2x(t-\tau_2)}{1 + \mu_1x(t-\tau_2) + \mu_2y(t-\tau_2)}\right)dt. \end{cases} \tag{3.8}$$

Now we discuss the dynamics of system (3.8).

Theorem 3.3 *For system* (1.1), *suppose that there is a unique positive equilibrium state* $\tilde{E}$ *which is LAS, then we have the following conclusions*:

(*i*) *Suppose that* H_1 *holds, then system* (3.8) *is LAS around* $\tilde{E}$ *for all* $\tau_2 > 0$;

(*ii*) *Suppose that* H_2 *and* H_3 *hold, then there is* $\tilde{\tau}_2$ *such that* $\tau_2 < \tilde{\tau}_2$, *system* (3.8) *is LAS around* $\tilde{E}$, *while* $\tau_2 > \tilde{\tau}_2$, *a Hopf bifurcation occurs at* $\tau_2 = \tilde{\tau}_2$;

(*iii*) *Suppose that* H_4 *and* H_5 *hold, then there exists an integer r such that system* (3.8) *changes from stable to unstable to stable and multiple Hopf bifurcations occur at* $\tau_2 = \tau_{2_j}^{(1)}$ *and* $\tau_2 = \tau_{2_j}^{(2)}, j = 1, 2, \dots, r$, *respectively, where conditions* $H_1 - H_5$ *are defined in the proof.*

Proof It is easy to get the characteristic equation of (3.8) as follows:

$$\lambda^2 + \sigma_1\lambda + \sigma_2 + (\sigma_3 + \sigma_5 + \sigma_4\lambda)e^{-\lambda\tau_2} = 0. \tag{3.9}$$

Denote the root of (3.9) by $\lambda(\tau_2) = \varphi(\tau_2) + i\omega(\tau_2)$. Since $\tilde{E}$ is LAS for non-delayed case, then $\varphi(0) < 0$. By the continuity of $\lambda(\tau_2)$ on variable τ_2, we conclude that $\varphi(\tau_2) < 0$ for sufficiently small $\tau_2 > 0$, and whence $\varphi(\tau_2) < 0$, which means that $\tilde{E}$ keeps the stability unchanged. Now we are interested in finding a critical value of τ_2 such that $\lambda(\tau_2)$ is a purely imaginary number. We denote it by $\tilde{\tau}_2$, i.e., $\varphi(\tilde{\tau}_2) = 0$ and $\omega(\tilde{\tau}_2) \neq 0$. In such case, the coexistence steady state loses its stability. Otherwise, if there is no such $\tilde{\tau}_2$, then the steady state will always be stable regardless of any τ_2. Substituting $\lambda(\tau_2) = \varphi(\tau_2) + i\omega(\tau_2)$ in (3.9) gives

$$\begin{cases} e^{-\phi\tau_2}\left((\sigma_3+\sigma_4\varphi+\sigma_5)\cos\omega\tau_2+\sigma_4\omega\sin\omega\tau_2\right) = -(\varphi^2-\omega^2+\sigma_1\varphi+\sigma_2), \\ e^{-\phi\tau_2}\left(\sigma_4\omega\cos\omega\tau_2-(\sigma_3+\sigma_4\varphi+\sigma_5)\sin\omega\tau_2\right) = -(2\omega\varphi+\sigma_1\omega). \end{cases}$$

Now let $\varphi(\tau_2) = 0$, then

$$\begin{cases} (\sigma_3+\sigma_5)\cos\omega\tau_2+\sigma_4\omega\sin\omega\tau_2 = \omega^2-\sigma_2, \\ \sigma_4\omega\cos\omega\tau_2-(\sigma_3+\sigma_5)\sin\omega\tau_2 = -\sigma_1\omega. \end{cases} \tag{3.10}$$

Squaring and adding the two sides of (3.10), we get a biquadratic equation of ω as follows:

$$\omega^4+(\sigma_1^2-\sigma_4^2-2\sigma_2)\omega^2+\sigma_2^2-(\sigma_3+\sigma_5)^2 = 0. \tag{3.11}$$

Denote $y = \omega^2, c_1 = \sigma_1^2-\sigma_4^2-2\sigma_2, c_2 = \sigma_2^2-(\sigma_3+\sigma_5)^2$, then (3.11) is equivalent to the following quadratic equation:

$$y^2+c_1y+c_2 = 0. \tag{3.12}$$

Next we prove the conclusions given above one by one.

Hypothesis H_1 : $c_1 > 0$ and $c_2 > 0$.

If H_1 holds, then there is no positive real root, and therefore there exists no real ω for equation (3.12). Hence, for any $\tau_2 > 0$, the coexistence equilibrium point $\tilde{E}$ is still locally asymptotically stable. Otherwise, if H_1 fails, then $\tilde{E}$ is unstable, yet H_2 keeps instability of $\tilde{E}$ unchanged. Therefore (i) of Theorem 3.3 is proved. Next we prove (ii) of Theorem 3.3.

Hypothesis H_2 : $c_2 < 0$.

Under H_2, system (3.12) has exactly one real root, denoted by $\tilde{y}$. Then $\tilde{\omega} = \pm\sqrt{\tilde{y}}$ are two real roots of (3.11). Then we can find a threshold value of τ_2, denoted by $\tilde{\tau}_2$, such that $\varphi(\tilde{\tau}_2) = 0, \omega(\tilde{\tau}_2) = \tilde{\omega}$, that is, the roots of the characteristic equation (3.9) are purely imaginary $\pm i\tilde{\omega}$. From (3.10), we obtain that the critical value of τ_2 is

$$\tau_{2_j} = \frac{1}{\tilde{\omega}}\cos^{-1}\frac{(\sigma_3+\sigma_5-\sigma_1\sigma_4)\omega^2-\sigma_2(\sigma_3+\sigma_5)}{\sigma_4^2\omega^2+(\sigma_3+\sigma_5)^2}+\frac{2\pi j}{\tilde{\omega}}, j = 0,1,2,\ldots. \tag{3.13}$$

Let

$$\tilde{\tau}_2 = \min_j \tau_{2_j}, j = 0,1,2,\ldots.$$

Then, by Lemma 3.2 and Butler's lemma, we obtain that $\tilde{E}$ is LAS if $0 < \tau_2 < \tilde{\tau}_2$ and unstable if $\tau_2 > \tilde{\tau}_2$. Whether the Hopf bifurcation occurs, we only need to verify whether the below transversality condition holds or not.

$$\frac{d}{d\tau_2}Re(\lambda)\bigg|_{\tau_2=\tilde{\tau}_2} = \frac{d\varphi}{d\tau_2}\bigg|_{\tau_2=\tilde{\tau}_2} > 0.$$

Differentiating (3.9) w.r.t τ_2 leads to

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = -\frac{2\lambda + \sigma_1}{\lambda(\lambda^2 + \sigma_1\lambda + \sigma_2)} - \frac{\tau_2}{\lambda} + \frac{\sigma_4}{\lambda(\sigma_3 + \sigma_5 + \sigma_4\lambda)}.$$

Hypothesis H_3 : $\sigma_1^2 + 2\tilde{\omega}^2 - \sigma_4^2 - 2\sigma_2 > 0$.

At $\tau_2 = \tilde{\tau}_2$, combining (3.10), under condition H_3, we get

$$\begin{aligned}\left[\frac{d}{d\tau_2}Re(\lambda)\right]^{-1}_{\tau_2=\tilde{\tau}_2} &= \left[\frac{d\varphi}{d\tau_2}\right]^{-1}_{\tau_2=\tilde{\tau}_2} \\ &= \frac{\sigma_1^2 + 2\tilde{\omega}^2 - 2\sigma_2}{\sigma_1^2\tilde{\omega}^2 + (\sigma_2 - \tilde{\omega}^2)^2} - \frac{\sigma_4^2}{\sigma_4^2\tilde{\omega}^2 + (\sigma_3 + \sigma_5)^2} \\ &= \frac{\sigma_1^2 + 2\tilde{\omega}^2 - \sigma_4^2 - 2\sigma_2}{\sigma_4^2\tilde{\omega}^2 + (\sigma_3 + \sigma_5)^2} > 0. \end{aligned} \tag{3.14}$$

That is, under H_2 and H_3, the Hopf transversality condition holds at $\tau_2 = \tilde{\tau}_2$. Then $\tilde{E}$ keeps stable when $\tau_2 < \tilde{\tau}_2$, while if $\tau_2 > \tilde{\tau}_2$, then $\tilde{E}$ changes from stable to unstable, and Hopf bifurcation appears. The case (ii) of Theorem 3.3 is verified.

Hypothesis H_4 : $c_1 < 0, c_2 > 0, c_1^2 > 4c_2$.

If H_4 holds, then equation (3.12) has two positive roots, denoted by $\tilde{y}_1$ and $\tilde{y}_2$ respectively. Consequently, $\tilde{\omega}_1 = \pm\sqrt{\tilde{y}_1}$ are two real roots and $\tilde{\omega}_2 = \pm\sqrt{\tilde{y}_2}$ are another two real roots of (3.12), respectively. From (3.10), we obtain that

$$\tau_{2_j}^{(m)} = \frac{1}{\tilde{\omega}_m}\cos^{-1}\frac{(\sigma_3 + \sigma_5 - \sigma_1\sigma_4)\tilde{\omega}_m^2 - \sigma_2(\sigma_3 + \sigma_5)}{\sigma_4\tilde{\omega}_m^2 + (\sigma_3 + \sigma_5)^2} + \frac{2\pi j}{\tilde{\omega}_m}, j = 1, 2, \dots, m = 1, 2. \tag{3.15}$$

At $\tau_2 = \tau_{2_j}^{(m)}, m = 1, 2$, combining (3.11), similarly we get

$$\frac{d\varphi}{d\tau_2}\bigg|^{-1}_{\tau_2=\tau_{2_j}^{(1)}} = \frac{2\sigma_2 - \sigma_1^2 - \sigma_4^2 - 2{\tilde{\omega}_1}^2}{\sigma_4^2\tilde{\omega}_1^2 + (\sigma_3 + \sigma_5)^2}, \qquad \frac{d\varphi}{d\tau_2}\bigg|^{-1}_{\tau_2=\tau_{2_j}^{(2)}} = \frac{2\sigma_2 - \sigma_1^2 - \sigma_4^2 - 2{\tilde{\omega}_2}^2}{\sigma_4^2\tilde{\omega}_2^2 + (\sigma_3 + \sigma_5)^2}. \tag{3.16}$$

Hypothesis H_5 : $2\sigma_2 - \sigma_1^2 - \sigma_4^2 - 2\tilde{\omega}_1^2 > 0, 2\sigma_2 - \sigma_1^2 - \sigma_4^2 - 2\tilde{\omega}_2^2 < 0$.

Under condition H_5, we have

$$\left[\frac{d}{d\tau_2}Re(\lambda)\right]^{-1}_{\tau_2=\tau_{2_j}^{(1)}} > 0, \qquad \left[\frac{d}{d\tau_2}Re(\lambda)\right]^{-1}_{\tau_2=\tau_{2_j}^{(2)}} < 0, j = 0, 1, 2, \dots.$$

Then there is a positive integer r such that the equilibrium point $\tilde{E}$ switches r times from stability to instability to stability. That is, $\tilde{E}$ is locally asymptotically stable when $\tau_2 \in [0, \tau_{2_0}^{(1)}) \cup (\tau_{2_0}^{(2)}, \tau_{2_1}^{(1)}) \cup \cdots \cup (\tau_{2_{r-1}}^{(2)}, \tau_{2_r}^{(1)})$ and unstable for all $\tau_2 \in [\tau_{2_0}^{(1)}, \tau_{2_0}^{(2)}) \cup (\tau_{2_1}^{(1)}, \tau_{2_1}^{(2)}) \cup \cdots \cup (\tau_{2_{r-1}}^{(1)}, \tau_{2_{r-1}}^{(2)}) \cup (\tau_{2_r}^{(1)}, \infty)$. That is, system (3.8) experiences Hopf bifurcation at $\tau_2 = \tau_{2_j}^{(1)}$ and $\tau_2 = \tau_{2_j}^{(2)}, j = 1, 2, \ldots, r$, respectively. Therefore, (iii) of Theorem 3.3 is verified. □

In system 1.2, let $\tau_1 > 0, \tau_2 = 0$, then (1.2) leads to

$$\begin{cases} \dfrac{dx}{dt} = x\left(\dfrac{r_1}{1 + ky(t - \tau_1)} - a_1 x - \dfrac{b_1 y}{1 + \mu_1 x + \mu_2 y}\right), \\ \dfrac{dy}{dt} = y\left(-r_2 - a_2 y + \dfrac{b_2 x(t)}{1 + \mu_1 x(t) + \mu_2 y(t)}\right) dt. \end{cases} \tag{3.17}$$

Hypothesis H_6 : $(\sigma_1 + \sigma_4)^2 + 2(\sigma_2 + \sigma_3 - \tilde{\omega}^2) > 0$.
For system (3.17), we have the following result.

Theorem 3.4 *Suppose that system (1.1) is LAS around the unique equilibrium state $\tilde{E}$, then under H_6 system (3.17) undergoes Hopf bifurcation at $\tau_1 = \tilde{\tau}_1$ with*

$$\tilde{\tau}_1 = \min_j \tau_{1_j}, \quad \tau_{1_j} = \frac{1}{\tilde{\omega}} \cos^{-1} \frac{\tilde{\omega}^2 - (\sigma_2 + \sigma_3)}{\sigma_5} + \frac{2\pi j}{\tilde{\omega}}, \quad j = 0, 1, 2, \ldots,$$

where $\tilde{\omega}$ is the root of the corresponding characteristic equation.

3.2.2 Model with two delays

In this part, we study the Hopf bifurcation of model (1.2) with $\tau_1 > 0, \tau_2 > 0$. First, we let $\tau_2 \in (0, \tilde{\tau}_2)$ and τ_1 be a parameter.

Theorem 3.5 *Suppose that system (1.1) has a unique $\tilde{E}$ that is LAS. Let $\tau_1 > 0$, $\tau_2 \in (0, \tilde{\tau}_2)$ and τ_1 is considered as a parameter. If H_7 holds, then system (1.2) is LAS around the equilibrium point $\tilde{E}$ for $\tau_1 \in (0, \check{\tau}_1)$, and experiences Hopf bifurcation at $\tau_1 = \check{\tau}_1$, where H_7 is defined in the proof.*

Proof We recall that the characteristic equation of (1.2) is as follows:

$$\lambda^2 + \sigma_1 \lambda + \sigma_2 + (\sigma_3 + \sigma_4 \lambda) e^{-\lambda \tau_2} + \sigma_5 e^{-\lambda(\tau_1 + \tau_2)} = 0. \tag{3.18}$$

First we fix $\tau_2 = \hat{\tau}_2 \in (0, \tilde{\tau}_2)$. Take τ_1 as a variable, then by the same reasoning, the root of (3.18) is denoted by $\lambda(\tau_1) = \varphi_1(\tau_1) + i w_1(\tau_1)$. Substitute $\lambda(\tau_1)$ in (3.18) and separate the real and imaginary parts, then

$$\begin{aligned} &\varphi_1^2 - \omega_1^2 + \sigma_1 \varphi_1 + \sigma_2 + (\sigma_3 + \sigma_4 \varphi_1) e^{-\hat{\tau}_2 \varphi_1} \cos \omega_1 \hat{\tau}_2 + \sigma_4 \omega_1 e^{-\hat{\tau}_2 \varphi_1} \sin \omega_1 \hat{\tau}_2 \\ &= -\sigma_5 e^{-(\tau_1 + \hat{\tau}_2)\varphi_1} \cos(\tau_1 + \hat{\tau}_2)\omega_1, \\ &2\omega_1 \varphi_1 + \sigma_1 \omega_1 - (\sigma_3 + \sigma_4 \varphi_1) e^{-\hat{\tau}_2 \varphi_1} \sin \omega_1 \hat{\tau}_2 + \sigma_4 \omega_1 e^{-\hat{\tau}_2 \varphi_1} \cos \omega_1 \hat{\tau}_2 \\ &= \sigma_5 e^{-(\tau_1 + \hat{\tau}_2)\varphi_1} \sin(\tau_1 + \hat{\tau}_2)\omega_1. \end{aligned} \tag{3.19}$$

Let $\varphi_1(\tau_1) = 0$. Now we eliminate τ_1. By squaring both sides of (3.19) and adding them leads to

$$\begin{aligned} g(w_1) &= \left(-\omega_1^2 + \sigma_2 + \sigma_3 \cos\omega_1\hat{\tau}_2 + \sigma_4\omega_1 \sin\omega_1\hat{\tau}_2\right)^2 \\ &\quad + \left(\sigma_1\omega_1 + \sigma_4\omega_1\cos\omega_1\hat{\tau}_2 - \sigma_3\sin\omega_1\hat{\tau}_2\right)^2 - \sigma_5^2 = 0. \end{aligned}$$

Since this is a transcendental equation, one cannot predict the nature of its roots. We assume that it has finite positive roots $w_1^{(1)}, w_1^{(2)}, \dots, w_1^{(n)}$. For some fixed $w_1^{(i)}$, there exists a sequence $\tau_{1_j}^{(i)}$ satisfying (3.19). Then (3.19) can be rewritten as

$$-G_1\cos(w_1^{(i)}\tau_{1_j}^{(i)}) + G_2\sin(w_1^{(i)}\tau_{1_j}^{(i)}) = G_3 \tag{3.20}$$

$$G_1\sin(w_1^{(i)}\tau_{1_j}^{(i)}) + G_2\cos(w_1^{(i)}\tau_{1_j}^{(i)}) = G_4, \tag{3.21}$$

where $G_1 = \sigma_5\cos w_1^{(i)}\hat{\tau}_2, G_2 = \sigma_5\sin w_1^{(i)}\hat{\tau}_2, G_3 = -(w_1^{(i)})^2 + \sigma_2 + \sigma_3\cos w_1^{(i)}\hat{\tau}_2 + \sigma_4 w_1^{(i)}\sin w_1^{(i)}\hat{\tau}_2$, $G_4 = \sigma_1 w_1^{(i)} + \sigma_4 w_1^{(i)}\cos w_1^{(i)}\hat{\tau}_2 - \sigma_3\sin w_1^{(i)}\hat{\tau}_2$. Then the critical value of $\tau_{1_j}^{(i)}$ for each $w_1^{(i)}$ is

$$\tau_{1_j}^{(i)} = \frac{1}{w_1^{(i)}}\sin^{-1}\left[\frac{G_2G_3 + G_2G_4}{G_1^2 + G_2^2}\right] + \frac{2\pi m}{w_1^{(i)}}, j = 0, 1, 2, \dots.$$

Assume that $\check{\tau}_1 = \min\{\tau_{1_j}^{(i)}, j = 0, 1, 2, \dots,\}$ and $\check{\omega}_1$ is the corresponding root of (3.19) with $\hat{\tau}_2 \in [0, \tilde{\tau}_2)$. For verifying the transversality condition, we differentiate both sides of (3.19) with respect to τ_1, and letting $\tau_1 = \check{\tau}_1, \omega_1(\check{\tau}_1) = \check{\omega}_1, \varphi_1(\check{\tau}_1) = 0$, then

$$\begin{aligned} H_1\left[\frac{d\varphi_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} - H_2\left[\frac{dw_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} &= H_3, \\ H_2\left[\frac{d\varphi_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} + H_1\left[\frac{dw_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} &= H_4. \end{aligned}$$

By eliminating $\frac{dw_1}{d\tau_1}$, we have

$$\left[\frac{d\varphi_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} = \frac{H_1H_3 + H_2H_4}{H_1^2 + H_2^2},$$

where

$$\begin{aligned} H_1 &= \sigma_1 + (\sigma_4 - \hat{\tau}_2\sigma_3)\cos\check{\omega}_1\hat{\tau}_2 - \sigma_4\check{\omega}_1\hat{\tau}_2\sin\check{\omega}_1\hat{\tau}_2 - \sigma_5(\check{\tau}_1 + \hat{\tau}_2)\cos(\check{\tau}_1 + \hat{\tau}_2)\check{\omega}_1, \\ H_2 &= -2\check{\omega}_1 - \sigma_3\hat{\tau}_2\sin\check{\omega}_1\hat{\tau}_2 + \sigma_4\sin\check{\omega}_1\hat{\tau}_2 + \sigma_4\check{\omega}_1\hat{\tau}_2\cos\check{\omega}_1\hat{\tau}_2 - \sigma_5\sin(\check{\tau}_1 + \hat{\tau}_2)\check{\omega}_1(\check{\tau}_1 + \hat{\tau}_2), \\ H_3 &= \sigma_5\check{\omega}_1\sin(\check{\tau}_1 + \hat{\tau}_2)\check{\omega}_1, \\ H_4 &= \sigma_5\check{\omega}_1\cos(\check{\tau}_1 + \hat{\tau}_2)\check{\omega}_1. \end{aligned}$$

Hypothesis H_7 : $H_1H_3 + H_2H_4 > 0$.

Under H_7, then $\left[\frac{d\varphi_1}{d\tau_1}\right]_{\tau_1=\check{\tau}_1,\omega=\check{\omega}_1} > 0$, which means the transversality condition is satisfied. By the theory of Hopf bifurcation, we conclude that system (1.2) is locally asymptotically stable around the equilibrium point $\tilde{E}$ for $\tau_1 \in [0, \breve{\tau}_1)$. Further, system (1.2) undergoes a Hopf bifurcation at $\tau_1 = \breve{\tau}_1$. □

Similarly, if τ_1 lies in the stable interval $(0, \tilde{\tau}_1)$ and τ_2 is considered as a parameter, we can derive the following theorem.

Theorem 3.6 *Suppose that system (1.1) is LAS around the equilibrium state $\tilde{E}$. If $\tau_2 > 0$, $\tau_1 \in (0, \tilde{\tau}_1)$ and τ_2 is taken as a parameter, then system (1.2) undergoes Hopf bifurcation at $\tau_2 = \breve{\tau}_2$ provided that $M_1M_3 + M_2M_4 > 0$ holds, where $M_1 = \sigma_2 + (\sigma_4 - \sigma_3\breve{\tau}_2)\cos\breve{\omega}_2\breve{\tau}_2 - \sigma_4\breve{\omega}_2\breve{\tau}_2\sin\breve{\omega}_2\breve{\tau}_2 - \sigma_5(\hat{\tau}_1 + \breve{\tau}_2)\cos(\hat{\tau}_1 + \breve{\tau}_2)\breve{\omega}_2$, $M_2 = -2\breve{\omega}_2 - \sigma_3\breve{\tau}_2\sin\breve{\omega}_2\breve{\tau}_2 + \sigma_4\breve{\omega}_2\breve{\tau}_2\cos\breve{\omega}_2\breve{\tau}_2 + \sigma_5(\hat{\tau}_1 + \breve{\tau}_2)\sin(\hat{\tau}_1 + \breve{\tau}_2)\breve{\omega}_2$, $M_3 = \sigma_3\breve{\omega}_2\sin\breve{\omega}_2\breve{\tau}_2 - \sigma_4\sin\breve{\omega}_2\breve{\tau}_2 - \sigma_4\breve{\omega}_2^2\cos\breve{\omega}_2\breve{\tau}_2 - \sigma_5\breve{\omega}_2\sin(\hat{\tau}_1 + \breve{\tau}_2)\breve{\omega}_2$, $M_4 = \sigma_3\breve{\omega}_2\cos\breve{\omega}_2\breve{\tau}_2 - \sigma_4\cos\breve{\omega}_2\breve{\tau}_2 - \sigma_4\breve{\omega}_2^2\sin\breve{\omega}_2\breve{\tau}_2 - \sigma_5\breve{\omega}_2\cos(\hat{\tau}_1 + \breve{\tau}_2)\breve{\omega}_2$, and $\hat{\tau}_1$ is a fixed constant within $(0, \tilde{\tau}_1)$, $\breve{\omega}_2 = \omega(\breve{\tau}_2)$ is the root of the corresponding characteristic equation.*

Remark 3.2 Theorems 3.5 and 3.6 reveal the impact of two delays on system stability, while only one fear delay is studied in reference [24]. Compared to [24], the impact of two delays is much more complex than that of one delay. Therefore, for food chain system, if three or more biological process delays are considered, how will the system stability be? It is a challenging subject that needs further research.

4 Stochastic model

Almost every population system is exposed in the open natural environment, and hence it is unavoidable to be affected by random environment disturbances [30, 35–37]. Commonly, a colored noise is used to represent the random environmental fluctuation, but if it is weakly correlated, then the Gaussian white noise is usually used to approximate the colored noise. This approach is reasonable [38]. Generally speaking, the white noise is introduced into population model by more than one way, whereas the authors [39] proposed that the growth rate of population was influenced significantly by the white noise and usually in the study of system dynamics, it was sufficient to study the impacts of environmental noise on the growth rate. Incorporating the stochastic environmental effect on the growth rate of species, we have

$$r_i \to r_i + \delta_i d\omega_i(t),$$

where r_i is the growth rate, $\omega_i(t)$ is a Gaussian white noise defined on a usual probability space, which is supposed to be standard and mutually independent as general discussion. δ_i^2 is the density of white noise, $i = 1, 2$. Then we get the stochastic model

$$\begin{cases} dx = x\left(\dfrac{r_1}{1 + ky(t-\tau_1)} - a_1x - \dfrac{b_1y}{1 + \mu_1x + \mu_2y}\right)dt + \delta_1xdw_1(t), \\ dy = y\left(-r_2 - a_2y + \dfrac{b_2x(t-\tau_2)}{1 + \mu_1x(t-\tau_2) + \mu_2y(t-\tau_2)}\right)dt - \delta_2ydw_2(t). \end{cases} \tag{4.1}$$

For stochastic system, there is no equilibrium state, so the stability [30, 36] is a crucial topic to understand the long behaviors of species. Qi and Meng [36] obtained the threshold of extinction and persistence in the mean of predator by use of stochastic analysis techniques.

Due to delay effect, the stochastic solutions of (4.1) do not possess Markov property, so we focus on the existence, boundedness, and GAS of solutions.

Let $U(t)$ be an N-dimensional stochastic process such that

$$dU(t) = F(t, U(t))dt + G(t, U(t))dW(t),$$

and $V(U(t))$ be a Lyapunov functional with respect to $U(t)$. The operator $\mathcal{L}$ of $V(U(t))$ is defined as follows (see, Refs [35, 38]):

$$\mathcal{L}V(U(t)) = V_U(U(t))F(t, U(t)) + \frac{1}{2}trace[G^T(t, U(t))V_{UU}(U(t))G(t, U(t))].$$

First, we investigate the property of solutions of stochastic system (4.1).

Theorem 4.1 *For system* (4.1) *with initial data* $\chi_i(\theta) > 0, \theta \in [-\tau, 0], i = 1, 2$, *there exists a positive stochastic process that is unique and global almost surely* (*a.s. for short*).

The proof is standard. Readers may refer to Theorem 3.1 in Refs [19] or [35], and hence we omit it.

Theorem 4.2 *For system* (4.1) *with initial value* $\chi_i(\theta) > 0 (i = 1, 2), \theta \in [-\tau, 0]$, *there are constants* $\mathcal{M}(p)$ *and* $\mathcal{N}(p)$ *large enough such that*

$$E(x^p) \le \mathcal{M}(p), \qquad E(y^p) \le \mathcal{N}(p), \quad p \ge 1,$$

where E is the mathematical expectation.

Proof Applying Itô's formula to $V(x, t) = e^t x^p$, then

$$\begin{aligned} dV(t) &= \left(e^t x^p + e^t p x^{p-1} x \left(\frac{r_1}{1 + ky(t-\tau_1)} - a_1 x - \frac{b_1 y}{1 + \mu_1 x + \mu_2 y}\right)\right. \\ &\quad \left. + \frac{p(p-1)}{2} e^t x^{p-2} \delta_1^2 x^2\right) dt + e^t p x^{p-1} \delta_1 x dw_1(t) \\ &= e^t x^p \left(1 + \frac{\delta_1^2 p(p-1)}{2} + p\left(\frac{r_1}{1 + ky(t-\tau_1)} - a_1 x - \frac{b_1 y}{1 + \mu_1 x + \mu_2 y}\right)\right) dt \\ &\quad + e^t x^p p \delta_1 dw_1(t) \\ &\le e^t x^p \left(1 + \frac{\delta_1^2 p(p-1)}{2} + p(r_1 - a_1 x)\right) dt + e^t x^p p \delta_1 dw_1(t). \end{aligned} \tag{4.2}$$

Integrating and then taking mathematical expectation of (4.2), we have

$$E(V(t)) \le x_0^p + \int_0^t e^s x^p \left(1 + \frac{\delta_1^2 p(p-1)}{2} + p(r_1 - a_1 x)\right) ds. \tag{4.3}$$

Let

$$g(x) = x^p \left(1 + \frac{\delta_1^2 p(p-1)}{2} + p(r_1 - a_1 x)\right).$$

Compute the derivative of $g(x)$ with respect to variable x and let $g'(x) = 0$, then

$$g'(x) = px^{p-1}\left(1 + \frac{\delta_1^2 p(p-1)}{2} + p(r_1 - a_1 x)\right) - a_1 p x^p = 0.$$

By the monotonicity, thus

$$x = \frac{1 + \frac{\delta_1^2 p(p-1)}{2} + pr_1}{a_1(1+p)}$$

is the maximum point of $g(x)$, and consequently, the maximum value is

$$g_{max} = a_1\left[\frac{1 + \frac{\delta_1^2 p(p-1)}{2} + pr_1}{a_1(1+p)}\right]^{p+1}.$$

Substituting it in (4.3) gives

$$E(V(t)) \leq x_0^p + \int_0^t e^s g_{max} ds \leq x_0^p + g_{max}(e^t - 1).$$

Thus

$$E(x^p) \leq (x_0^p - g_{max})e^{-t} + g_{max}.$$

So when $t = 0, E(x^p) \leq x_0^p$, and when $t \to \infty$, we have

$$E(x^p) \leq g_{max}.$$

Let

$$\mathcal{M}(p) = \max\left\{x_0^p, a_1\left[\frac{1 + \frac{\delta_1^2 p(p-1)}{2} + pr_1}{a_1(1+p)}\right]^{p+1}\right\},$$

then

$$E(x^p) \leq \mathcal{M}(p).$$

By the same manner, we obtain

$$E(y^p) \leq \mathcal{N}(p),$$

where

$$\mathcal{N}(p) = \max\left\{y_0^p, a_2\left[\frac{1 + \frac{\delta_2^2 p(p-1)}{2} + pb_2/\mu_1}{a_2(1+p)}\right]^{p+1}\right\}.$$

Therefore the solution of system (4.1) is bounded in expectation. □

Theorem 4.3 *Suppose that the following conditions hold*:

(C_1^*) $A^* := a_1 - \frac{b_1\mu_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} - \frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})} - \frac{r_1 k}{2(1+k\tilde{y})} > 0,$

(C_2^*) $B^* := a_2 - \frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})} - \frac{r_1 k}{2(1+k\tilde{y})} > 0,$

(C_3^*) $\delta := \frac{\delta_1^2\tilde{x}+\delta_2^2\tilde{y}}{2} < \min\{A^*, B^*\}.$

Then (4.1) *is globally asymptotically stable around the equilibrium state* $\tilde{E}(\tilde{x},\tilde{y})$ *almost surely, i.e., for any initial data* $\chi_i(\theta) > 0 (i=1,2), -\tau \le \theta \le 0$, *the solution of* (4.1) *satisfies* $\lim_{t\to+\infty} x(t) = \tilde{x}, \lim_{t\to+\infty} y(t) = \tilde{y}$.

Proof We recall that $\tilde{E}(\tilde{x},\tilde{y})$ meets the following equalities:

$$\frac{r_1}{1+k\tilde{y}} - a_1\tilde{x} - \frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} = 0, \qquad -r_2 - a_2\tilde{y} + \frac{b_2\tilde{x}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} = 0.$$

Define

$$W_1(x) = x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}, \quad W_2(y) = y - \tilde{y} - \tilde{y}\ln\frac{y}{\tilde{y}},$$

then $W_i(\cdot) \in C(R^+)$ is positive and continuous defined on $t \ge 0, i = 1,2$. By Itô's formula, we have

$$\begin{aligned}
\mathcal{L}W_1(x) &= (x-\tilde{x})\left(\frac{r_1}{1+ky(t-\tau_1)} - a_1 x - \frac{b_1 y}{1+\mu_1 x+\mu_2 y}\right) + \frac{\delta_1^2\tilde{x}}{2} \\
&= (x-\tilde{x})\left(\frac{r_1}{1+ky(t-\tau_1)} - a_1 x - \frac{b_1 y}{1+\mu_1 x+\mu_2 y}\right. \\
&\quad \left. - \frac{r_1}{1+k\tilde{y}} + a_1\tilde{x} + \frac{b_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right) + \frac{\delta_1^2\tilde{x}}{2} \\
&= -a_1(x-\tilde{x})^2 - \frac{r_1 k}{(1+k\tilde{y})(1+ky(t-\tau_1))}(y(t-\tau_1)-\tilde{y})(x-\tilde{x}) \\
&\quad + \frac{b_1\mu_1\tilde{y}(x-\tilde{x}) - b_1(1+\mu_1\tilde{x})(y-\tilde{y})}{(1+\mu_1\tilde{x}+\mu_2\tilde{y})(1+\mu_1 x+\mu_2 y)}(x-\tilde{x}) + \frac{\delta_1^2\tilde{x}}{2} \\
&\le -\left(a_1 - \frac{b_1\mu_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}} - \frac{r_1 k}{2(1+k\tilde{y})}\right)(x-\tilde{x})^2 \\
&\quad + \frac{r_1 k}{2(1+k\tilde{y})}(y(t-\tau_1)-\tilde{y})^2 + \frac{\delta_1^2\tilde{x}}{2}.
\end{aligned} \tag{4.4}$$

Define

$$W_2(t) = \int_t^{t+\tau_2} \frac{r_1 k}{2(1+k\tilde{y})}(y(s-\tau_1)-\tilde{y})^2 ds.$$

Differentiating $W_2(t)$ about variable t yields

$$\frac{dW_2(t)}{dt} = \frac{r_1 k}{2(1+k\tilde{y})}(y(t)-\tilde{y})^2 - \frac{r_1 k}{2(1+k\tilde{y})}(y(t-\tau_1)-\tilde{y})^2. \tag{4.5}$$

Adding (4.4) and (4.5) leads to

$$\begin{aligned}\mathcal{L}(W_1(x)+W_2(t)) \leq &-\left(a_1-\frac{b_1\mu_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{r_1k}{2(1+k\tilde{y})}\right)(x-\tilde{x})^2\\ &+\frac{r_1k}{2(1+k\tilde{y})}(y(t)-\tilde{y})^2+\frac{\delta_1^2\tilde{x}}{2}.\end{aligned}\tag{4.6}$$

Similar computation yields

$$\begin{aligned}\mathcal{L}W_3(y) &= (y-\tilde{y})\left(-r_2-a_2y+\frac{b_2x(t-\tau_2)}{1+\mu_1x(t-\tau_2)+\mu_2y(t-\tau_2)}\right)+\frac{\delta_2^2\tilde{y}}{2}\\ &\leq (y-\tilde{y})\left(-r_2-a_2y+\frac{b_2x(t-\tau_2)}{1+\mu_1x(t-\tau_2)+\mu_2y(t-\tau_2)}+r_2\right.\\ &\quad\left.+a_2\tilde{y}-\frac{b_2\tilde{x}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}\right)+\frac{\delta_2^2\tilde{y}}{2}\\ &= -a_2(y-\tilde{y})^2+(y-\tilde{y})\frac{b_2(1+\mu_2\tilde{y})(x(t-\tau_2)-\tilde{x})-b_2\mu_2\tilde{x}(y(t-\tau_2)-\tilde{y})}{(1+\mu_1x(t-\tau_2)+\mu_2y(t-\tau_2))(1+\mu_1\tilde{x}+\mu_2\tilde{y})}+\frac{\delta_2^2\tilde{y}}{2}\\ &\leq -a_2(y-\tilde{y})^2+\frac{b_2(1+\mu_2\tilde{y})}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}(x(t-\tau_2)-\tilde{x})(y-\tilde{y})+\frac{\delta_2^2\tilde{y}}{2}\\ &\leq -a_2(y-\tilde{y})^2+\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}\left((x(t-\tau_2)-\tilde{x})^2\right.\\ &\quad\left.+\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}(y-\tilde{y})^2\right)+\frac{\delta_2^2\tilde{y}}{2}.\end{aligned}\tag{4.7}$$

Define

$$W_4(t)=\int_t^{t+\tau_2}\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}((x(s-\tau_2)-\tilde{x})^2ds.$$

Then

$$\frac{dW_4(t)}{dt}=\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}((x(t)-\tilde{x})^2-\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}((x(t-\tau_2)-\tilde{x})^2.\tag{4.8}$$

Let $W(t)=W_1(x)+W_2(t)+W_3(y)+W_4(t)$, and add both sides of (4.6), (4.7), and (4.8), then

$$\begin{aligned}\mathcal{L}W(t)) &\leq -\left(a_1-\frac{b_1\mu_1\tilde{y}}{1+\mu_1\tilde{x}+\mu_2\tilde{y}}-\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}-\frac{r_1k}{2(1+k\tilde{y})}\right)(x-\tilde{x})^2\\ &\quad-\left(a_2-\frac{b_2(1+\mu_2\tilde{y})}{2(1+\mu_1\tilde{x}+\mu_2\tilde{y})}-\frac{r_1k}{2(1+k\tilde{y})}\right)(y-\tilde{y})^2+\frac{\delta_1^2\tilde{x}+\delta_2^2\tilde{y}}{2}\\ &= -A^*(x-\tilde{x})^2-B^*(y-\tilde{y})^2+\delta.\end{aligned}$$

By $A^*>0, B^*>0$, together with condition C_3^*, then the following ellipse

$$A^*(x-\tilde{x})^2+B^*(y-\tilde{y})^2=\delta$$

is situated entirely in R_+^2. We denote the domain of ellipsoid by U and take a neighborhood of U as O, that is, $U\subseteq O$, then $O\subset R_+^2$. Therefore, for any $(x,y)\in R_+^2/O$, we have $\mathcal{L}W(t)\leq 0$.

Table 1 The values of $\tilde{E}$ for different fears for (5.1)

Fear	Value of the equilibrium state $\tilde{E}(\tilde{x}, \tilde{y})$
0	$\tilde{x} = 1.3499$, $\tilde{y} = 0.5139$
1	$\tilde{x} = 1.1272$, $\tilde{y} = 0.3181$
2	$\tilde{x} = 1.0528$, $\tilde{y} = 0.2481$
5	$\tilde{x} = 0.9647$, $\tilde{y} = 0.1617$
10	$\tilde{x} = 0.9125$, $\tilde{y} = 0.1085$
50	$\tilde{x} = 0.8416$, $\tilde{y} = 0.0336$

Then, by the stochastic differential equation theory (pp. 156–160 in [35]), we conclude that system (4.1) is globally asymptotically stable. □

Remark 4.1 Having compared Theorems 4.3 and 2.2, we find that, by constructing different Lyapunov functionals, different sufficient conditions guaranteeing the stability of $\tilde{E}$ are obtained even if the white noise is absent. On the other hand, Theorem 4.3 implies that time delays also have no effect on the stability of $\tilde{E}$ if some required conditions hold.

Remark 4.2 Compared with [19], the impact of stochastic environment on birth rate of prey is considered in (4.1), which is independent of the equilibrium point. However, in [19], the impact of stochastic environment on birth rate of prey depends on the distance between the species and the equilibrium point, which is closely related to the equilibrium point. They are completely different. In addition, if the regime switching and Lévy jump are included in the stochastic factors [40], then the system dynamics deserve further research in the future.

5 Numerical analysis

Some examples are carried out by Matlab2014 to verify our theoretical results and explore the impacts of such factors as fear, delay, and random environmental perturbations on the system dynamics.

(1) *Numerical verification of our findings*

Fix a set of parameter values as follows:

$$r_1 = 0.5, k = 2, a_1 = 0.2, b_1 = 0.6, \mu_1 = 0.1, \mu_2 = 0.4, r_2 = 0.3, a_2 = 0.2, b_2 = 0.4, \tag{5.1}$$

which are sufficient to guarantee the existence of unique positive equilibrium state, and here the equilibrium is $\tilde{E}(1.0528, 0.2481)$. Now we study the impact of fear k on the equilibrium state. Take $k = 0, 1, 2, 5, 10$ and compute the corresponding values of $\tilde{E}$ respectively, which are given in Table 1.

From Table 1 we find that the values of equilibrium points of prey and predator change along with the values of fear from predator. Specifically, from a biological prospect, if the fear is larger, then the prey will produce some anti-predation acts and reduce its foraging activity, which leads to the reduction of the prey's offsprings, and vise versa. Due to the reduction of prey, the predator can produce few offsprings resulting in the reduction of predator species.

Second, we verify the stability of $\tilde{E}$. For system (1.1) with parameter values given in (5.1), numerical simulation implies that $\tilde{E}$ is GAS, which is accordant with Theorem 2.2 (see Figs. 1 (a) and (b)). For system (1.2) with above coefficients, let the delays $\tau_1 = \tau_2 = 2$.

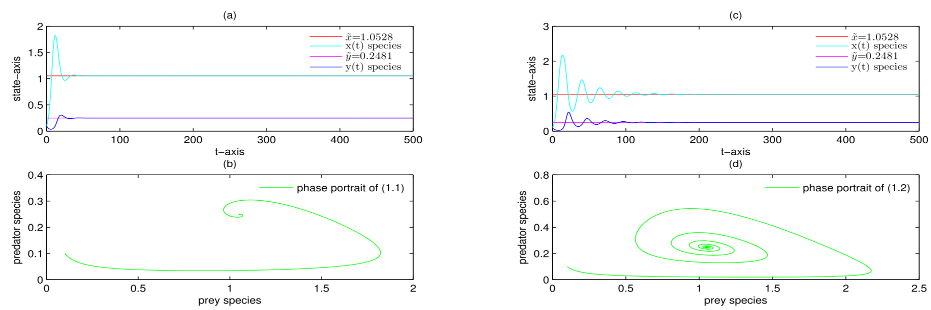
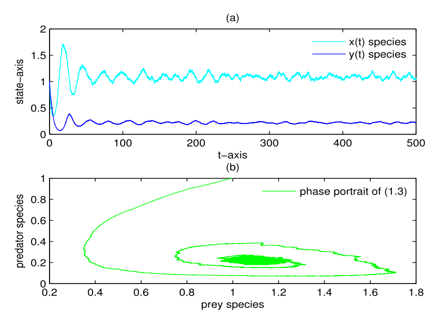


Figure 1 Numerical simulations of system (1.1) and (1.2), the parameter values are taken from (5.1). (a) is the curve of solutions and (b) is the phase graph of (1.1), (c) is the curve of solutions and (d) is the phase graph of (1.2) with $\tau_i = 2(i = 1, 2)$ respectively, which show (1.1) and (1.2) are both GAS around the $\tilde{E}$

Figure 2 The long time behaviors of stochastic system (4.1) with $\delta_i = 0.2(i = 1, 2)$, other parameter values are as (5.1). (a) is the time curves and (b) is the phase graph of solutions



Using Theorem 2.2 again, system (1.2) is GAS, which is visualized by Figs. 1 (c) and (d). For system (4.1), all parameters keep fixed as in (1.2) and $\delta_1 = 0.2, \delta_2 = 0.2$. By use of methods in [41], we simulate the long behaviors of prey and predator species; see Fig. 2, which means that (4.1) is GAS around $\tilde{E}$.

(2) *Impacts of fear, delay, and stochastic parameter*

Next we begin to explore the specific influence of fear, delay, and white noise on the equilibrium states. Take the parameter values as follows:

$$r_1 = 0.6, a_1 = 0.8, b_1 = 10, \mu_1 = 5, \mu_2 = 4, r_2 = 0.06, a_2 = 10^{-3}, b_2 = 0.73, k = 5. \tag{5.2}$$

An easy computation implies that the $\tilde{E}(0.1769, 0.0664)$ is unique.

Case 1 Take fear k as the variable with neither delay nor white noise. If $k = 5$, then the equilibrium $\tilde{E}$ is GAS, which is showed by Figs. 3 (a) and (b), while if $k = 0.5$, then the equilibrium is changed to $\tilde{E}(0.1951, 0.0985)$ and it is instable from Figs. 3 (c) and (d).

Case 2 Take τ_1 as the variable with $\tau_2 = 0$ and no noise. Other parameter values are taken as above. We take $\tau_1 = 0.01$ and $\tau_1 = 10$ respectively, and simulations imply that the former is GAS but the latter is unstable (Figs. 4(a) and (b)). Analogously, we change the values of τ_2 and keep other parameters unchanged; for example, take $\tau_2 = 0.001$ and $\tau_2 = 5$ respectively, then the system also changes from GAS to unstable emerging periodic changes and stable periodic limit circle appears, see Figs. 4(c) and Figs. 4(d). All these mean that different values of delays can make $\tilde{E}$ alter from stable to unstable and produce Hopf bifurcation, which is to be studied later.

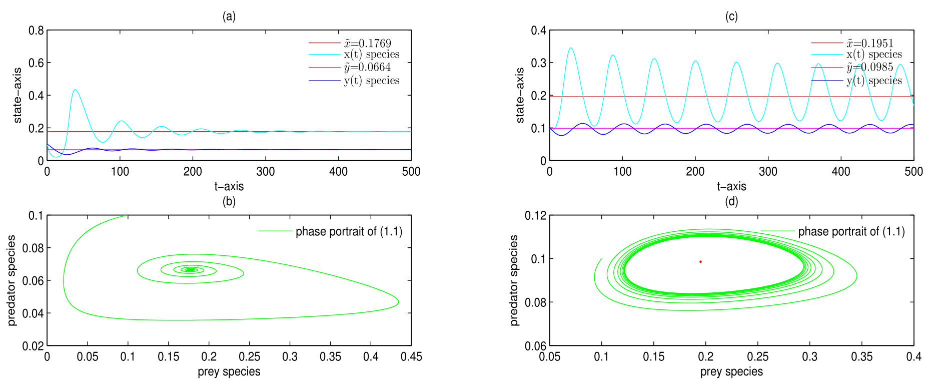


Figure 3 Numerical simulation for system (1.1) with parameter values as in (5.2). (a) is the curves of solutions and (b) is the phase graph with $k = 5$, (c) is the curves of solutions and (d) is the phase graph with $k = 0.5$ respectively. The red point (equilibrium point) in (d) is $\tilde{E}(0.1951, 0.0985)$, which shows (a) is GAS and (c) is unstable

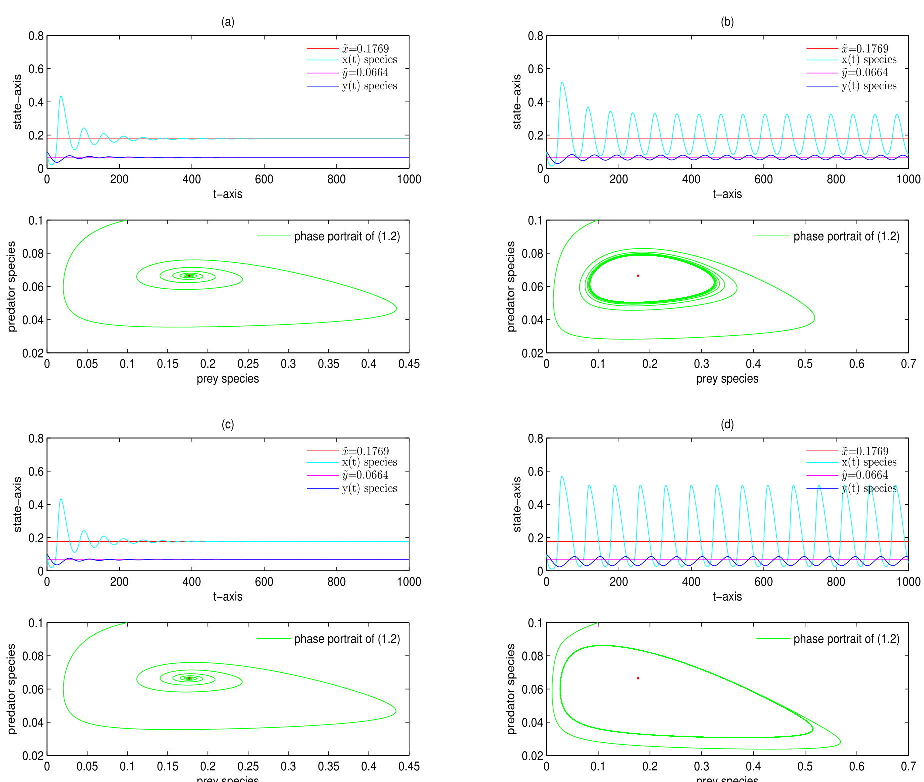


Figure 4 Numerical simulation for system (1.2) with parameter values as in (5.2). (a) is the stable case with $\tau_1 = 0.01$ and (b) is the unstable case with $\tau_1 = 10$, where $\tau_2 = 0$; (c) is the stable case with $\tau_2 = 0.001$ and (d) is the unstable case with $\tau_2 = 5$, where $\tau_1 = 0$ respectively. The red point is $\tilde{E}(0.1769, 0.0664)$

Case 3 For the study of random environmental perturbations, we take $\delta_i = 0.01$, and $\delta_i = 0.8$ and $\tau_i = 0, i = 1, 2$, respectively, then system (4.1) turns from stable (Fig. 5 (a)) to unstable (Fig. 5 (b)) around $\tilde{E}$ resulting in the extinction of some species. This

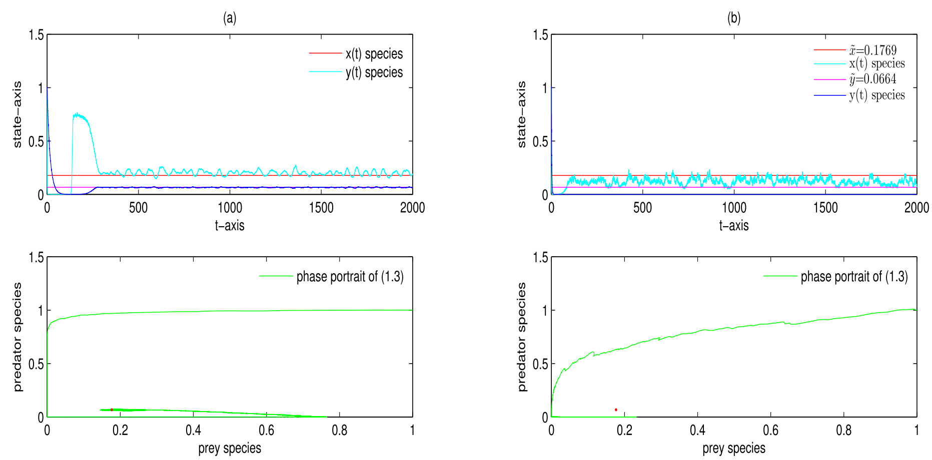


Figure 5 Numerical simulation of system (4.1) with parameter values as in (5.2). (a) Time curves and phase graph for the stable case with $\delta_i = 0.01$ and (b) the unstable case with $\delta_i = 0.8$. The red point is $\tilde{E}(0.1769, 0.0664)$

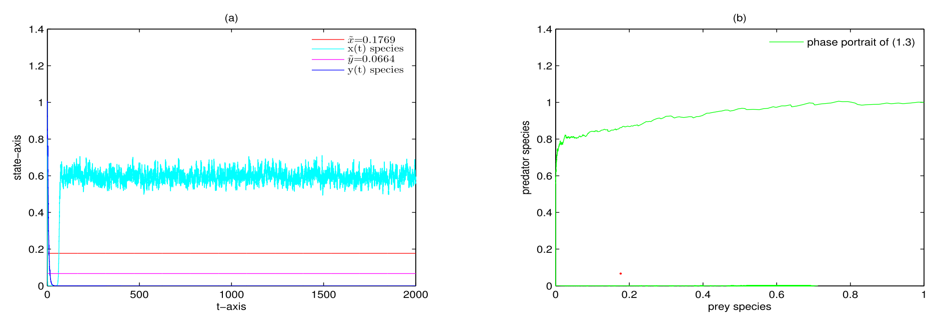


Figure 6 Numerical simulation of system (4.1) with parameter values as in (5.2). (a) and (b) are time curves and phase graph with $\delta_i = 1, \tau_i = 2 (i = 1, 2)$, respectively. The red point is $\tilde{E}(0.1769, 0.0664)$

phenomenon indicates that white noise can make (4.1) turn from stable to unstable and affect the stability of $\tilde{E}$ significantly.

Case 4 Finally, we consider the effects of delay and noise simultaneously. Take $\tau_i = 5, \delta_i = 0.5, (i = 1, 2)$, then numerical examples show that system (4.1) varies from stable status (Figs. 3 (a) and (b)) to unstable status, see Figs. 6 (a) and (b).

All above analysis demonstrates that fear, delay, and random noise are all very important for the system stability. Different values of fear and delay can make system change from GAS to unstable. The curves in the simulation show periodic changes, so Hopf bifurcation may occur. Consequently, we should choose suitable parameter values to keep the system stable for continuous development in the future.

(3) *Hopf bifurcation analysis*

The solution curves of above figures present periodic changes, then it is reasonable to guess that Hopf bifurcation may occur. Therefore, we analyze the Hopf bifurcation of the parameters k, b_1, τ_1, and τ_2 of systems (1.1) and (1.2), respectively. For a determinate system, Hopf bifurcation is an important character for people to understand the long behavior of population dynamics, which can reflect the stability and instability of species. The

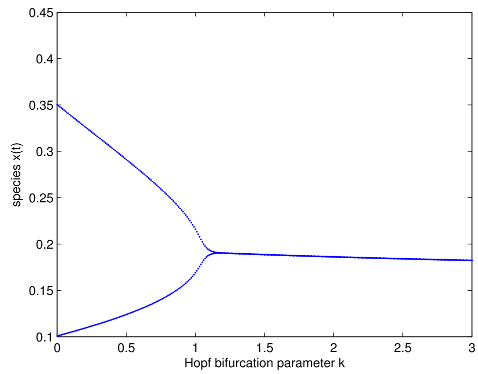


Figure 7 Hopf bifurcation diagrams of parameter k. System (1.1) becomes stable from unstable after crossing the threshold $\tilde{k} = 1.0326$

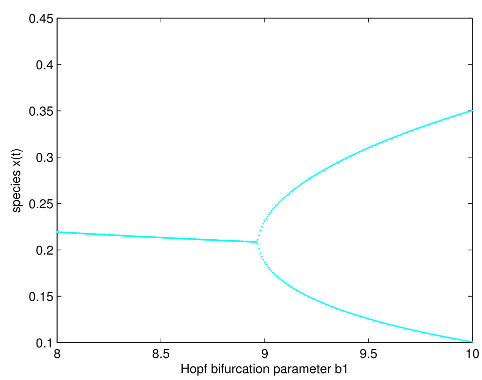


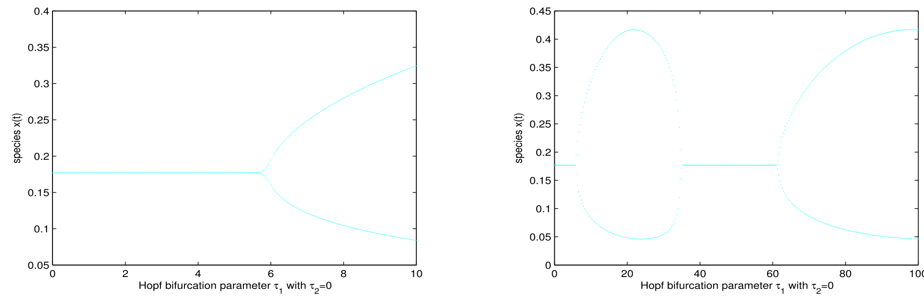
Figure 8 Hopf bifurcation diagrams of parameter b_1. System (1.1) changes the stability and becomes unstable from stable after crossing the threshold $\tilde{b}_1 = 8.9815$

Hopf threshold of parameters is a significant index for researchers to make some scientific decisions and keep the long balance of system dynamics.

In this part, we choose the parameter values given in (5.2). Select fear k as a bifurcation parameter. By Matlab2014a, we depict the drawings of species of predator varying in pace with the changes of fear k and find that the Hopf bifurcation appears at $k = 1.0326$, see Fig. 7. It exhibits that when $k < 1.326$, the prey species is unstable, whereas when $k > 1.0326$, the Hopf curves vary from unstable to stable, that is, the system gets stable and the threshold of k is $\tilde{k} = 1.0326$. Similarly, we take b_1 as a bifurcation parameter and $k = 0$, then we get the bifurcation diagram of prey (see Fig. 8). Figure 8 implies the threshold is $\tilde{b}_1 = 8.9815$.

Second, we choose τ_1 as the bifurcation parameter with $\tau_2 = 0$ and get the diagrams of prey species, see Fig. 9. From Fig. 9 (a) we find that the threshold is $\tilde{\tau}_1 = 5.8762$. Figure 9 (b) displays that prey species show multi-stabilities varying from stable to unstable, to stable and to unstable again, that is, multiple Hopf bifurcations appear. By same reasoning, when $\tau_1 = 0$, the threshold of τ_2 is $\tilde{\tau}_2 = 1.5921$. See Fig. 10.

Finally, take τ_1 as the bifurcation parameter and let $\tau_2 \in (0, 1.6921)$ and $\hat{\tau}_2 = 1$, then the threshold is $\check{\tau}_1 = 2.1645$, see Fig. 11 (a). Figure 11 (b) implies that multiple bifurcations



(a) (b)

Figure 9 Hopf bifurcation diagrams of parameter τ_1 with $\tau_2 = 0$. System (1.2) becomes unstable from stable after the threshold $\tilde{\tau}_1 = 5.8762$. (a) is with $\tau \in (0, 10)$ and (b) is with $\tau \in (0, 100)$, where multiple stabilities occur

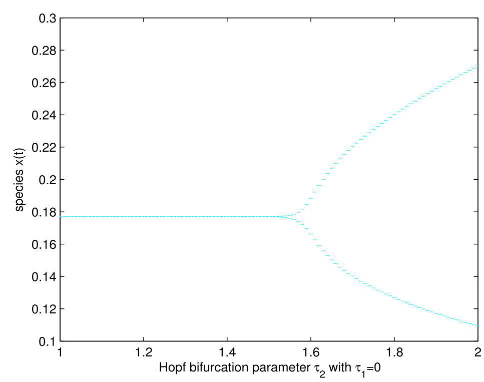
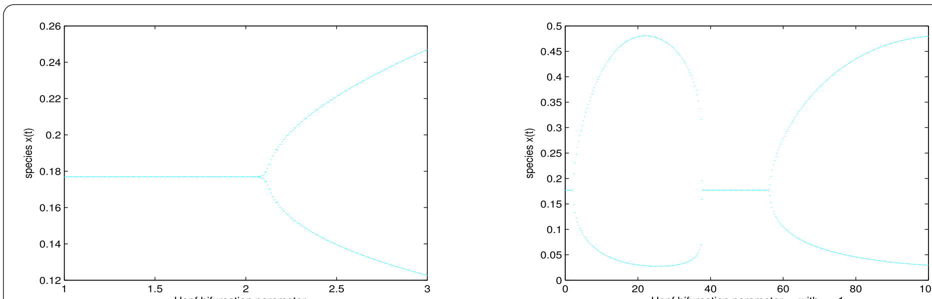


Figure 10 Hopf bifurcation diagrams of parameter τ_2 with $\tau_1 = 0$. System (1.2) changes the stability after the threshold $\tilde{\tau}_2 = 1.5921$



(a) (b)

Figure 11 Hopf bifurcation diagrams of parameter τ_1 with $\tau_2 = 1 \in (0, \tilde{\tau}_2)$. System (1.2) changes its stability after crossing the threshold $\check{\tau}_1 = 2.1645$. (a) is with $\tau \in (1, 3)$ and (b) is with $\tau \in (0, 100)$, where multiple stabilities occur

exist when $\tau_2 \in (0, 1.6921)$. Take τ_2 as the bifurcation parameter and let $\tau_1 \in (0, 5.8762)$ and $\hat{\tau}_1 = 5$, then simulations show the threshold is $\check{\tau}_2 = 0.2342$, see Fig. 12.

We summarize the thresholds of above parameters and simulation results in Table 2. In a word, Hopf bifurcations further verify that all these parameters affect the system dynamics significantly.

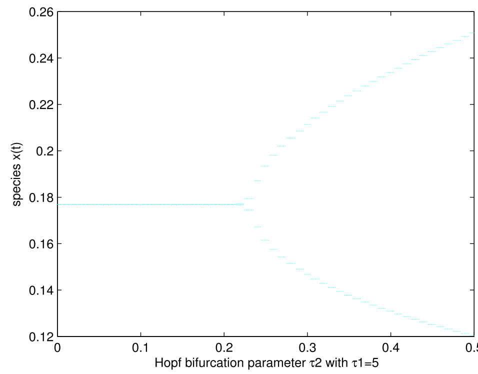


Figure 12 Hopf bifurcation diagrams of parameter τ_2 with $\tau_1 = 5 \in (0, \tilde{\tau}_1)$. System (1.2) changes the stability from stable to unstable after crossing the threshold $\breve{\tau}_2 = 0.2342$

Table 2 Threshold of Hopf bifurcation for (1.1) and (1.2) with parameters given in (5.2)

Bifurcation parameters	Threshold of parameter	Theoretical conclusion	Simulation results
Fear k	$\tilde{k} = 1.0326$	Theorem 3.1	Fig. 7
Parameter b_1	$\tilde{b}_1 = 8.9815$	Theorem 3.2	Fig. 8
Delay τ_1 with $\tau_2 = 0$	$\tilde{\tau}_1 = 5.8762$	Theorem 3.4	Fig. 9
Delay τ_2 with $\tau_1 = 0$	$\tilde{\tau}_2 = 1.5921$	Theorem 3.3	Fig. 10
Delay τ_1 with $\tau_2 = 1$	$\breve{\tau}_1 = 2.1645$	Theorem 3.5	Fig. 11
Delay τ_2 with $\tau_1 = 5$	$\breve{\tau}_2 = 0.2342$	Theorem 3.6	Fig. 12

Table 3 The main results and numerical verification.

System	GAS around $\tilde{E}$	Theoretical result	Simulation results
System (1.1)	Yes	Theorem 4.2	Figs. 1 and 3, (a) and (b)
System (1.2)	Yes	Theorem 4.2	Figs. 1 (c) and (d)
System (4.1)	Yes	Theorem 3.1	Fig. 2
System (1.2)	No	\	Figs. 4 (b) and (d)
System (4.1)	No	\	Fig. 5 (b) and Fig. 6

6 Conclusions

For a prey–predator system with superabundant predators, it is necessary to introduce a predator-dependent function response and consider the effect of fear from predator. Based on the real world, we propose a delayed prey–predator model with fear and Beddington–DeAngelis functional response and generalize it to a random environmental disturbance. We study the dynamics of above systems like the existence and global asymptotic stability of equilibrium and get their criteria, see Table 3.

Next, we analyze the Hopf bifurcation and get its thresholds of fear and delay for determinate system (1.1) and (1.2). For the main results and simulation figures, see Table 2.

Finally, we numerically verify our findings and clarify the effects of fear, delay, and random fluctuations in detail. Our study shows that the value of equilibrium is influenced by the value of fear with the same tendency. The delay and white noise bring crucial influences to the GAS of the corresponding systems. They can make system dynamics change from stable to unstable, or from unstable to stable. In particular, the changes of fear and time delays will make the solution curves vary periodically and produce Hopf bifurcations.

To summarize, fear induced by predator, the delay of prey's response to predator's fear, delay of gestation of predator and environmental white noise bring important influences on the long behavior of species. All our findings can help readers to understand well the dynamics of this kind of system and afford theoretical basis for scientific decisions and continuous developments.

Taking into account the complexity of biological processes, if there exist three or more delays or regime switching and Lévy jumps [40], the system dynamics become more complex and require further research. In addition, in recent years, fractional calculus has attracted increasing attention due to its hereditary and memory properties [42, 43], so the dynamics of fractional prey–predator system is also very interesting. All of these are left for our future work.

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Author contributions
JZ conducted all the analyses and wrote the initial draft of the manuscript. YS contributed to the conception of the study. All authors read and approved the final manuscript.

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Data availability
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests in this paper.

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