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Modeling chikungunya virus infection with Black–Karasinski process: stationary distribution, probability density function, and extinction

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Abstract

Incorporating stochastic processes into biological models is crucial for capturing the inherent variability and uncertainties within biological systems. This paper explores the benefits of introducing Black–Karasinski process into chikungunya virus infection modeling. By utilizing the Black–Karasinski process researchers can capture the inherent variability in biological processes and account for uncertainties. This paper highlights the advantages of Black–Karasinski processes in biological modeling. We investigate the dynamical behavior of a stochastic model for chikungunya virus infection incorporating a Black–Karasinski process. Firstly, we establish sufficient conditions for the existence of a stationary distribution in the model. By solving the corresponding Fokker–Planck equation we obtain the local probability density function near the quasi-endemic equilibrium, which provides insights into the statistical characteristics of the stochastic system. Additionally, we present sufficient conditions for the extinction of infected host cells and chikungunya virus particles. Finally, we supplement the analytical results with numerical simulations to investigate the impact of random noise.

Keywords: Chikungunya virus infection model; Black–Karasinski process; Stationary distribution; Probability density function; Extinction

1 Introduction

1.1 Background

Chikungunya virus (CHIKV) has emerged as a significant global health threat, causing widespread outbreaks in various regions and posing significant challenges to public health systems worldwide [1]. Since its first documented outbreak in Tanzania in 1953, CHIKV has rapidly spread to different parts of the world, affecting millions of people and causing substantial morbidity [2].

The consequences of CHIKV infection extend beyond the immediate clinical manifestations experienced by affected individuals. Outbreaks of CHIKV have severe implications for public health, including the burden on health care systems, economic losses,

and disruption of social and economic activities in affected regions [3]. Furthermore, the long-term complications and chronic forms of the disease impose a significant burden on individuals and communities, impairing their quality of life and productivity [4, 5].

Understanding the background and mode of infection of CHIKV is crucial for developing effective strategies to prevent and manage its spread. The use of mathematical models, particularly stochastic models, plays a crucial role in studying and understanding the dynamics of CHIKV transmission. These models provide valuable insights into the complex interactions between the virus and host cells (e.g., among epithelial and endothelial cells, primary fibroblasts, and macrophages), allowing researchers to assess the impact of various factors and interventions on the spread of CHIKV [6].

Biological systems exhibit intricate dynamics and inherent variability, necessitating the incorporation of stochasticity into modeling frameworks [7, 8]. This paper explores the potential benefits of introducing Black–Karasinski (BK) processes, originally utilized in finance, into biological models [9]. BK processes offer a flexible framework to capture complex biological systems by considering stochasticity and time-varying parameters. By incorporating BK processes researchers can account for the inherent variability in biological processes, address uncertainties, and enhance predictive capabilities. Stochastic models have been instrumental in investigating the impact of interventions on CHIKV transmission. Importantly, mathematical models provide a platform for exploring hypothetical scenarios and conducting virtual experiments that may not be feasible in real-world settings. By manipulating model parameters and assumptions researchers can investigate the potential consequences of specific policy decisions and explore the effects of varying factors, such as the contact rate between uninfected host cells and CHIKV particles, on CHIKV transmission dynamics.

In conclusion, the use of mathematical models, particularly stochastic models, is of paramount importance in studying CHIKV transmission dynamics. These models enable researchers to capture the complexities of CHIKV transmission, account for inherent stochasticity, and generate insights into the effectiveness of interventions and control strategies. By integrating data and simulating different scenarios mathematical models contribute to evidence-based decision-making and facilitate proactive measures to prevent and control CHIKV outbreaks.

1.2 Mathematical model

Many mathematical models have been employed to elucidate the dynamics of chikungunya virus infection [10–14]. For example, Alade et al. [14] investigated a comprehensive nonlinear CHIKV dynamics model and demonstrated the global stability of the steady states of the model by constructing appropriate Lyapunov functionals. Wang and Liu [15] studied a within-host CHIKV infection model with two delays. If the delays are not taken into account, then the model consists of four compartments: S represents the concentration of uninfected host cells, I is the concentration of infected host cells, V denotes the concentration of CHIKV particles, and B is the concentration of B cells. The specific form

Table 1 Variables and parameters in model (1.1)

Symbol	Biological meaning
S	Concentration of uninfected host cells
I	Concentration of infected host cells
V	Concentration of CHIKV particles
B	Concentration of B cells
μ	Production rate of uninfected host cells
α	Death rate of uninfected host cells
b	Contact rate between uninfected host cells and CHIKV particles
β	Death rate of infected host cells
m	Virus proliferation rate
r	Death rate of CHIKV particles
p	Elimination rate of CHIKV by B cells
η	Constant production rate of B cells
c	Production rate of B cells due to increased CHIKV
δ	Death rate of B cells

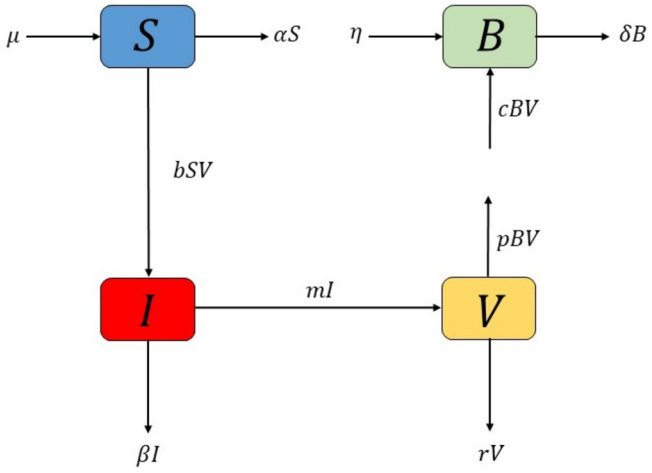

Figure 1 Infection diagram of CHIKV

is as follows:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu - \alpha S - bSV, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = bSV - \beta I, \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = mI - rV - pBV, \\ \dfrac{\mathrm{d}B}{\mathrm{d}t} = \eta + cBV - \delta B, \end{cases} \tag{1.1}$$

where the biological significance of the parameters is given in Table 1. The dynamical infection of CHIKV are presented in the flowchart in Fig. 1.

According to [15], the basic reproduction number for deterministic model (1.1) is

$$R_0 = \frac{b\mu\delta m}{\alpha\beta(r\delta + p\eta)}.$$

Moreover, the equilibria of (1.1) and their global stability are given as follows:

- If $R_0 < 1$, then system (1.1) has only the virus-free equilibrium $E_0 = (\frac{\mu}{\alpha}, 0, 0, \frac{\eta}{\delta})$, which is globally asymptotically stable (GAS);
- If $R_0 > 1$, then E_0 is unstable, and system (1.1) has a unique endemic equilibrium $E_1 = (S_1, I_1, V_1, B_1) = \left(\frac{\mu}{a+bV_1}, \frac{bS_1V_1}{\beta}, V_1, \frac{\eta}{\delta - cV_1}\right)$ is GAS, where $V_1 = -\frac{-P_2 - \sqrt{P_2^2 - 4P_1P_3}}{2P_1}$, $P_1 = r\beta bc$, $P_2 = \frac{\alpha\beta c(r\delta + p\eta)}{\delta}(1 - R_0) - rb\beta\eta - \beta p\eta b - c\alpha\beta p\eta\delta$, and $P_3 = \alpha\beta(r\delta + p\eta)(R_0 - 1)$.

Incorporating random noise or stochasticity into virus infection models is crucial for a more realistic representation of the complexities involved in the infection process [16–21]. By accounting for variations and chance events, these models capture the uncertainties at the cellular level. Random noise can affect various aspects of infection, such as the probability of successful attachment, replication, and assembly of viruses within host cells, as well as the stochastic nature of the host cell response to infection. Additionally, when considering transmission between individuals, random noise can be included in the probability and timing of transmission events. By embracing stochasticity researchers gain insights into the potential range of outcomes, the impact of chance events on infection spread, and the effectiveness of control measures. Advanced mathematical techniques are employed to analyze the behavior of these stochastic viral infection models, allowing researchers to comprehensively understand viral dynamics and develop effective strategies for disease control and prevention. Ma and Yu [22] investigated a stochastic viral infection model with two modes of transmission and immune impairment and gave a random threshold value that determines the persistence of infected cells or not. Wang et al. [23] studied a stochastic HIV infection model with immune response and distributed delay. Gokila and Sambath [24] examined a stochastic CHIKV model incorporating saturated incidence. They further elucidated the threshold condition that determines whether the disease will persist or vanish within the host.

Biological systems are subjected to inherent uncertainties, noise, and measurement errors. For stochastic mathematical models in biology, the most commonly used parameter perturbation method is linear white noise perturbation [25]. However, recent studies have shown some advantages of the Ornstein–Uhlenbeck (OU) process over linear white noise perturbation. Nevertheless, using the OU process to perturb the contact rate b may lead to negative value [26], which is not biologically meaningful.

Applying the BK process for perturbing the contact rate b can indeed resolve this problem. Introducing BK processes allows for the consideration of biological variability at various levels, ranging from cellular processes to population dynamics. By incorporating BK processes researchers can effectively model these stochastic elements and assess their impact on system behavior. This enables a more robust analysis of the system response to uncertainties, improving risk assessment and decision-making in biological research. Inspired by the aforementioned discussion, in this paper, we assume that the contact rate b is perturbed by the Black–Karasinski (BK) process

$$\mathrm{d}\ln b(t) = \theta\left(\ln\bar{b} - \ln b(t)\right)\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{B}(t), \tag{1.2}$$

where $\bar{b}$ denotes the long-run mean level of the contact rate, θ is the speed of reversion, σ is the noise intensity, and $\mathcal{B}(t)$ is a standard Brownian motion. Assuming that $b(0) = \bar{b}$,

from [27] we one can obtain that $b(t)$ approaches a stationary log-normal density with mean $\bar{b}e^{\frac{\sigma^2}{4\theta}}$ and variance $\bar{b}^2\left(e^{\frac{\sigma^2}{\theta}}-e^{\frac{\sigma^2}{2\theta}}\right)$. Furthermore, we get

(1) Topology of $b(t)$: The variable $b(t)$ represents a strictly positive stochastic process, as the logarithmic transformation $\ln b(t)$ ensures that $b(t) > 0$ for all $t \geq 0$. This is consistent with the biological interpretation of the contact rate, which must remain positive.

(2) Dynamics of $b(t)$: The process $b(t)$ reverts to its long-run mean $\bar{b}$ over time, with random fluctuations driven by the noise term $\sigma \mathrm{d}\mathcal{B}(t)$. The reversion speed θ determines how quickly $b(t)$ returns to $\bar{b}$ after deviations caused by stochastic perturbations.

(3) Stationarity and stability: The logarithmic form of the BK process ensures that $b(t)$ has a stationary distribution in the long term, and its behavior is confined to a biologically realistic positive range.

To facilitate representation, letting $x(t) = \ln b(t)$ and $\bar{x} = \ln \bar{b}$, we get the following stochastic model:

$$\begin{cases} \mathrm{d}x(t) = \theta(\bar{x} - x(t))\mathrm{d}t + \sigma \mathrm{d}\mathcal{B}(t), \\ \mathrm{d}S(t) = \left[\mu - \alpha S(t) - e^{x(t)}S(t)V(t)\right]\mathrm{d}t, \\ \mathrm{d}I(t) = \left[e^{x(t)}S(t)V(t) - \beta I(t)\right]\mathrm{d}t, \\ \mathrm{d}V(t) = \left[mI(t) - rV(t) - pB(t)V(t)\right]\mathrm{d}t, \\ \mathrm{d}B(t) = [\eta + cB(t)V(t) - \delta B(t)]\,\mathrm{d}t. \end{cases} \tag{1.3}$$

In conclusion, providing a comprehensive characterization of the dynamical properties of stochastic model (1.3) remains a challenging task. Our main contributions are as follows:

- For the first time, we attempt to incorporate the Black–Karasinski process as a stochastic fluctuation in the CHIKV infection model. Compared to the existing Ornstein–Uhlenbeck process and linear perturbation methods, this process presents a mathematically and biologically reasonable assumption of randomness.
- By constructing appropriate Lyapunov functions and utilizing the ergodicity of the BK process we establish sufficient conditions for the existence of a stationary distribution and the extinction of infected cells and CHIKV, respectively.
- To obtain the precise expression of the density function, we provide a lemma for determining the positive definiteness of a five-dimensional matrix. This lemma is particularly effective for the five-dimensional model with the BK process.
- Although our primary motivation stems from the CHIKV model, the analytical techniques employed in this study can be applied to other nonlinear virus infection models perturbed by the BK process.

The remaining sections of this paper are organized as follows. Section 2 presents essential mathematical symbols and lemmas, along with the invariant set of the stochastic model (1.3). In Sect. 3, we establish sufficient conditions for the existence of a stationary distribution in the stochastic model. In Sect. 4, by solving the Fokker–Planck equation we obtain the precise expression of the probability density function for the stochastic model. Section 5 establishes sufficient conditions for exponential extinction of infected cells and

CHIKV in the stochastic model. In Sect. 6, we conduct several numerical simulations to illustrate our theoretical findings. Lastly, in Sect. 7, we introduce biological interpretations of several theorems and discuss the limitations of our model.

2 Preliminaries and the existence and uniqueness of a global positive solution

Throughout this paper, unless otherwise specified, $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and $B(t)$ is defined on this space. If X is a vector or matrix, then by X^T we denote its transpose and by X^{-1} its inverse. Denote $\mathbb{R}^d_+ = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$. By $diag(a_1, \ldots, a_n)$ we denote the diagonal matrix with diagonal elements $a_1, \ldots, a_n$. Let $\mathbb{N}$ represent a one-dimensional normal distribution, and for an integer $k > 1$, let $\mathbb{N}_k$ represent a k-dimensional normal distribution.

2.1 Lemma of the ergodic stationary distribution

Based on Theorem 2.2 on p. 191 of Du et al. [28], Theorem 4.3 on p. 529 of Meyn et al. [29], and Theorem 2.3 on p. 98 of Dieu [30], we give the following lemma to show the existence of the ergodic stationary distribution for the stochastic system (1.3).

Lemma 2.1 *Assume that there exists a bounded closed domain $\mathbb{D} \in \mathbb{R}^d$ with regular boundary such that for any initial value $X(0) \in \mathbb{R}^d$,*

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau, X(0), \mathbb{D})\mathrm{d}\tau > 0 \ \ a.s.,$$

where $\mathbb{P}(\tau, X(0), \mathbb{D})$ is the transition probability of $X(t)$. Then system (1.3) possesses a solution that satisfies the Feller property. In addition, system (1.3) admits at least one invariant probability measure on $\mathbb{R}^d$, which means that system (1.3) has at least one ergodic stationary distribution on $\mathbb{R}^d$.

Proof 1. *Tightness*. From the condition

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau, X(0), \mathbb{D})\mathrm{d}\tau > 0$$

it follows that the stochastic process $X(t)$ has a positive probability of remaining within the bounded closed domain $\mathbb{D}$. This implies that the transition probability family $\mathbb{P}(\tau, X(0), \mathbb{D})$ is tight. The tightness is a sufficient condition for the Feller property and the existence of an invariant probability measure.

2. *Feller property*. The tightness of the process and the stochastic differential nature of system (1.3) ensure that the solution $X(t)$ is a strong Markov process satisfying the Feller property, which means that for any continuous bounded function $f \in C_b(\mathbb{R}^d)$, the mapping $t \mapsto \mathbb{E}[f(X_t)]$ is continuous and bounded.

3. *Existence of invariant probability measure*. By the Krylov–Bogoliubov theorem the tightness of the process guarantees the existence of at least one invariant probability measure μ satisfying

$$\int_{\mathbb{R}^d} \mathbb{P}(\tau, x, \mathrm{d}y)\mu(\mathrm{d}x) = \mu(\mathrm{d}y).$$

4. *Ergodic stationary distribution*. The invariant probability measure μ corresponds to at least one ergodic stationary distribution of system (1.3). Combined with the Feller property, this confirms the existence of a stationary solution.

Hence system (1.3) possesses the Feller property and admits at least one ergodic stationary distribution. □

Lemma 2.2 *Let $b(t)$ be a stochastic process satisfying the stochastic equation*

$$\mathrm{d}(\ln b(t)) = \theta(\ln \bar{b} - \ln b(t)) + \sigma \mathcal{B}(t), \tag{2.1}$$

where $\ln \bar{b}$ and σ are positive constants, and $\mathcal{B}(t)$ is a standard Brownian motion. Then

(i)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left| b(s) - \bar{b} \right| \mathrm{d}s \le \bar{b} \left(1 + e^{\frac{\sigma^2}{\theta}} - 2e^{\frac{\sigma^2}{4\theta}} \right)^{\frac{1}{2}};$$

(ii) *For $n > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t b^n(s) \mathrm{d}s = \bar{b}^n e^{\frac{n^2\sigma^2}{4\theta}}.$$

Proof (i) Denote $x(t) = \ln b(t)$ and $\bar{x} = \ln \bar{b}$. Then (2.1) becomes

$$\mathrm{d}x(t) = \theta(\bar{x} - x(t)) + \sigma \mathcal{B}(t). \tag{2.2}$$

According to the ergodicity of $x(t)$ and the strong law of large numbers, we obtain

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \int_0^t \left| e^{x(s)} - \bar{b} \right| \mathrm{d}s &= \int_{-\infty}^{+\infty} \left| e^{x(\nu)} - e^{\bar{x}} \right| \rho(\nu) \mathrm{d}\nu \\
&\le \left(\int_{-\infty}^{+\infty} \left(e^{x(\nu)} - e^{\bar{x}} \right)^2 \rho(\nu) \mathrm{d}\nu \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} 1^2 \rho(\nu) \right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{+\infty} \left(e^{x(\nu)} - e^{\bar{x}} \right)^2 \rho(\nu) \mathrm{d}\nu \right)^{\frac{1}{2}} \\
&= \left(e^{2\bar{x} + \frac{\sigma^2}{\theta}} + e^{2\bar{x}} - 2e^{2\bar{x} + \frac{\sigma^2}{4\theta}} \right)^{\frac{1}{2}} \\
&= e^{\bar{x}} \left(1 + e^{\frac{\sigma^2}{\theta}} - 2e^{\frac{\sigma^2}{4\theta}} \right)^{\frac{1}{2}},
\end{aligned}$$

where

$$\rho(\nu) = \frac{\sqrt{\theta}}{\sqrt{\pi}\sigma} e^{-\frac{\theta(\nu - \bar{x})^2}{\sigma^2}}.$$

(ii) The calculation gives

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t b^n(s) \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{nx(s)} \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{\frac{\sqrt{2\theta}(x(s) - \bar{x})}{\sigma} \frac{n\sigma}{\sqrt{2\theta}}} e^{\bar{x}n} \mathrm{d}s.$$

Let $v(t) = \frac{\sqrt{2\theta}(x(t)-\bar{x})}{\sigma}$. It is obvious that the stationary distribution of $v(t)$ obeys $\mathbb{N}(0,1)$. Therefore we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t b^n(s)\mathrm{d}s = (\bar{b})^n\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}e^{\frac{n\sigma}{\sqrt{2\theta}}v}\mathrm{d}v$$
$$= (\bar{b})^n\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(v-\frac{n\sigma}{\sqrt{2\theta}}\right)^2}{2}}e^{\frac{n^2\sigma^2}{4\theta}}\mathrm{d}v = (\bar{b})^n e^{\frac{n^2\sigma^2}{4\theta}}. \qquad \square$$

2.2 Lemma on the probability density function

Next, we give a lemma on the five-dimensional positive definite matrix from Lemma 2.4 on p. 8 of [31].

Lemma 2.3 *If a symmetric matrix Ω_0 satisfies $\Xi_0^2 + C_0\Omega_0 + \Omega_0 C_0^T = 0$, where*

$$\Xi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} -c_1 & -c_2 & -c_3 & -c_4 & -c_5\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with $c_1,\dots,c_5 > 0$ and

$$c_i c_{i+1} > c_{i-1}c_{i+2}, \quad i = 1,2,3\ (c_0 \triangleq 1),$$

then

$$\Omega_0 = \begin{pmatrix} \omega_{11} & 0 & -\omega_{22} & 0 & \omega_{33}\\ 0 & \omega_{22} & 0 & -\omega_{33} & 0\\ -\omega_{22} & 0 & \omega_{33} & 0 & -\omega_{44}\\ 0 & -\omega_{33} & 0 & \omega_{44} & 0\\ \omega_{33} & 0 & -\omega_{44} & 0 & \omega_{55} \end{pmatrix}$$

is a positive definite matrix with

$$\omega_{11} = \frac{c_2(c_3c_4 - c_2c_5) - c_4(c_1c_4 - c_5)}{2[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]},$$
$$\omega_{22} = \frac{c_3c_4 - c_2c_5}{2[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]},$$
$$\omega_{33} = \frac{c_1c_4 - c_5}{2[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]},$$
$$\omega_{44} = \frac{c_1c_2 - c_3}{2[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]},$$
$$\omega_{55} = \frac{c_3(c_1c_2 - c_3) - c_1(c_1c_4 - c_5)}{2[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2]}.$$

2.3 Existence and uniqueness of a solution

Theorem 2.1 *For any initial value $(x(0), S(0), I(0), V(0), B(0)) \in \mathbb{R}\times\mathbb{R}_+^4$, there exists a unique solution $(x(t), S(t), I(t), V(t), B(t))$ of system (1.3) on $t \geq 0$, and the solution will remain in $\mathbb{R}\times\mathbb{R}_+^4$ with probability one.*

Proof The beginning and ending of the proof are similar to that of Theorem 3.1 on p. 34 of [32] and therefore are omitted here for brevity. We focus on introducing the crucial Lyapunov function. Let us define a C^2 function U_0 on $\mathbb{R} \times \mathbb{R}^4_+ \to \mathbb{R}$ as follows:

$$U_0 = (e^x - x - 1) + (S - 1 - \ln S) + (I - 1 - \ln I) + (V - 1 - \ln V) + \frac{p}{c}(B - 1 - \ln B).$$

The nonnegativity of U_0 follows from the inequality $z - 1 - \ln z \geq 0$ for all $z > 0$.

Applying Itô's formula to U_0, we have

$$\begin{aligned}
\mathcal{L}U_0 =& \theta(e^x - 1)(\bar{x} - x) + \frac{\sigma^2 e^x}{2} + \left(1 - \frac{1}{S}\right)\left(\mu - \alpha S - e^x SV\right) \\
&+ \left(1 - \frac{1}{I}\right)\left(e^x SV - \beta I\right) + \left(1 - \frac{1}{V}\right)(mI - rV - pBV) \\
&+ \frac{p}{c}\left(1 - \frac{1}{B}\right)(\eta + cBV - \delta B) \\
\leq& \theta(e^x - 1)(\bar{x} - x) + \frac{\sigma^2 e^x}{2} + \mu + \alpha + \beta + r + \frac{p(\eta + \delta)}{c} + mI + e^x V + pB.
\end{aligned} \tag{2.3}$$

For model (1.3), we obtain that

$$\begin{aligned}
\left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)' =& \mu + \frac{p\eta\beta}{2mc} - \alpha S - \frac{\beta}{2}I - \frac{\beta}{2m}\left(rV + \frac{p\delta}{c}B\right) \\
\leq& \mu + \frac{p\eta\beta}{2mc} - \min\left\{\alpha, \frac{\beta}{2}, r, \delta\right\}\left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&S + I + \frac{\beta}{2m}V + \frac{p}{c}B \leq A \\
&\triangleq \begin{cases} \mu_{\max} & \text{if } S(0) + I(0) + V(0) + B(0) \leq \mu_{\max}, \\ S(0) + I(0) + V(0) + B(0) & \text{if } S(0) + I(0) + V(0) + B(0) > \mu_{\max}, \end{cases}
\end{aligned} \tag{2.4}$$

where $\mu_{\max} = \dfrac{\mu + \frac{p\eta\beta}{2mc}}{\min\left\{\alpha, \frac{\beta}{2}, r, \delta\right\}}$. Then substituting (2.4) into (2.3), we have

$$\begin{aligned}
\mathcal{L}U_0 \leq& \theta(e^x - 1)(\bar{x} - x) + \frac{\sigma^2 e^x}{2} + \mu + \alpha + \beta + r + \frac{p(\eta + \delta)}{c} + e^x A + (m + p)A \\
\leq& \sup_{x \in \mathbb{R}}\left\{\theta(e^x - 1)(\widehat{x} - x) + \left(A + \frac{\sigma^2}{2}\right)e^x\right\} + \mu + \alpha + \beta + r + \frac{p(\eta + \delta)}{c} + (m + p)A \\
\leq& Q_0,
\end{aligned}$$

where Q_0 is a positive constant independent of x, S, I, V, and B. The remainder of the proof is similar to [32]. This completes the proof. □

Remark 2.1 From the first three equations of the stochastic model (1.3) we have

$$\left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)' < \mu + \frac{p\eta\beta}{2mc} - \min\left\{\alpha, \frac{\beta}{2}, r, \delta\right\}\left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right),$$

which implies that

$$\begin{aligned} &S(t)+I(t)+\frac{\beta}{2m}\left(V(t)+\frac{p}{c}B(t)\right)\\ &\quad<\mu_{\max}+e^{-\min\left\{\alpha,\frac{\beta}{2},r,\delta\right\}t}\left(S(0)+I(0)+\frac{\beta}{2m}\left(V(0)+\frac{p}{c}B(0)\right)-\mu_{\max}\right). \end{aligned}$$

Similarly, $S'<\mu-\alpha S$, and $\eta-\delta B<B'$. This implies that $S(t)<\frac{\mu}{\alpha}+e^{-\alpha t}\left(S(0)-\frac{\mu}{\alpha}\right)$ and $\frac{\eta}{\delta}+e^{-\delta t}\left(B(0)-\frac{\eta}{\delta}\right)<B(t)$. Thus we obtain that

$$\Gamma=\left\{(x,S,I,V,B)\in\mathbb{R}\times\mathbb{R}_+^4:\ \frac{\eta}{\delta}<B,\ S<\frac{\mu}{\alpha},\ S+I+\frac{\beta}{2m}\left(V+\frac{p}{c}B\right)<\mu_{\max}\right\}$$

is an invariant set of the stochastic model (1.3). From now on we always assume that the initial value $(x(0),S(0),I(0),V(0),B(0))\in\Gamma$.

3 Stationary distribution

In this section, our primary objective revolves around investigating the existence of a stationary distribution for the stochastic model (1.3). This examination sheds light on the stochastic persistence of CHIKV in the mean.

Theorem 3.1 *Assume that* $R_0^s=\frac{\bar{b}\mu\delta m e^{\frac{\sigma^2}{12\theta}}}{\alpha\beta(r\delta+p\eta)}>1$. *Then the stochastic system* (1.3) *admits at least one ergodic stationary distribution on* Γ.

Proof We divide the proof of Theorem 3.1 into three steps: (i) Construct stochastic Lyapunov functions; (ii) Construct a compact set; (iii) Prove the existence of the solution of system (1.3).

Step 1. (Stochastic Lyapunov functions). Applying Itô's formula to $-\ln S$, $-\ln I$, $-\ln B$, and $\frac{p}{\delta}B$, we have, respectively,

$$\begin{aligned} \mathcal{L}(-\ln S)&=-\frac{\mu}{S}+\alpha+e^x V,\\ \mathcal{L}(-\ln I)&=-\frac{e^x SV}{I}+\beta,\\ \mathcal{L}(-\ln V)&=-\frac{mI}{V}+r+pB,\\ \mathcal{L}\left(\frac{p}{\delta}B\right)&=\frac{p\eta}{\delta}+\frac{pc}{\delta}BV-pB. \end{aligned}\tag{3.1}$$

Then define

$$U_1=-\ln I-a_1\ln S+a_2\left(-\ln V+\frac{p}{\delta}B\right),$$

where a_1 and a_2 are positive constants to be determined in (3.3). Then applying Itô's formula to U_1 and combining with (3.1), we have

$$\begin{aligned} \mathcal{L}U_1&=-\frac{e^x SV}{I}-\frac{a_1\mu}{S}-\frac{a_2 mI}{V}+\beta+a_1(\alpha+e^x V)+a_2\left(r+\frac{p\eta}{\delta}+\frac{pc}{\delta}BV\right)\\ &\le-3\sqrt[3]{a_1a_2m\mu e^x}+\beta+a_1(\alpha+e^x V)+a_2\left(r+\frac{p\eta}{\delta}+\frac{pc}{\delta}BV\right)\\ &=-3\sqrt[3]{a_1a_2m\mu\bar{b}e^{\frac{\sigma^2}{12\theta}}}+\beta+a_1(\alpha+e^x V)+a_2\left(\frac{r\delta+p\eta}{\delta}+\frac{pc}{\delta}BV\right)+f(x), \end{aligned}\tag{3.2}$$

where

$$f(x) = 3\sqrt[3]{a_1 a_2 m\mu}\left(\bar{b}^{\frac{1}{3}} e^{\frac{\sigma^2}{36\theta}} - e^{\frac{x}{3}}\right).$$

Choose

$$a_1 = \frac{m\mu\delta\bar{b}e^{\frac{\sigma^2}{12\theta}}}{\alpha^2(r\delta + p\eta)}, \quad a_2 = \frac{m\mu\delta\bar{b}e^{\frac{\sigma^2}{12\theta}}}{\alpha(r\delta + p\eta)^2}. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\begin{aligned}\mathcal{L}U_1 \leq & -\frac{m\mu\delta\bar{b}e^{\frac{\sigma^2}{12\theta}}}{\alpha(r\delta + p\eta)} + \beta + a_1 e^x V + \frac{a_2 pc}{\delta} BV + f(x) \\ = & -\beta(R_0^s - 1) + a_1 e^x V + \frac{a_2 pc}{\delta} BV + f(x),\end{aligned} \tag{3.4}$$

where

$$R_0^s = \frac{\bar{b}\mu\delta m e^{\frac{\sigma^2}{12\theta}}}{\alpha\beta(r\delta + p\eta)}.$$

Consider $e^x \leq a_3 e^{2x} + \frac{1}{4a_3}$ with a positive constant a_3 to be determined in (3.6). Then we have

$$\begin{aligned}\mathcal{L}U_1 \leq & -\beta(R_0^s - 1) + a_1\left(a_3 e^{2x} + \frac{1}{4a_3}\right)V + \frac{2a_2 mc^2\mu_{\max}}{\beta\delta} V + f(x) \\ \leq & -\beta(R_0^s - 1) + a_1 a_3 \frac{2m\mu_{\max}}{\beta} e^{2x} + \frac{a_1}{4a_3} V + \frac{2a_2 mc^2\mu_{\max}}{\beta\delta} V + f(x) \\ = & -\beta(R_0^s - 1) + a_1 a_3 \frac{2m\mu_{\max}}{\beta} \bar{b}^2 e^{\frac{\sigma^2}{\theta}} \\ & + \left(\frac{a_1}{4a_3} + \frac{2a_2 mc^2\mu_{\max}}{\beta\delta}\right) V + f(x) + g(x),\end{aligned} \tag{3.5}$$

where

$$g(x) = a_1 a_3 \frac{2m\mu_{\max}}{\beta} \bar{b}^2 \left(e^{2x} - e^{\frac{\sigma^2}{\theta}}\right).$$

Then we choose a_3 such that

$$a_1 a_3 \frac{2m\mu_{\max}}{\beta} \bar{b}^2 e^{\frac{\sigma^2}{\theta}} = \frac{\beta}{2}(R_0^s - 1). \tag{3.6}$$

Hence we obtain

$$\mathcal{L}U_1 \leq -\frac{\beta}{2}(R_0^s - 1) + \left(\frac{a_1}{4a_3} + \frac{2a_2 mc^2\mu_{\max}}{\beta\delta}\right) V + f(x) + g(x). \tag{3.7}$$

Next, we define

$$\begin{aligned} U_2 =& -\ln S - \ln I - \ln\left(B - \frac{\eta}{\delta}\right) \\ & - \ln\left(\frac{\mu}{\alpha} - S\right) - \ln\left(\mu_{\max} - \left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)\right) + e^x - x - 1. \end{aligned}$$

Applying Itô's formula to U_2, we have

$$\begin{aligned} \mathcal{L}U_2 \leq & -\frac{\mu}{S} - \frac{e^x SV}{I} + \frac{\delta B - \eta - cBV}{B - \frac{\eta}{\delta}} + \frac{\mu - \alpha S - e^x SV}{\frac{\mu}{\alpha} - S} \\ & + \frac{\mu + \frac{p\eta\beta}{2mc} - \alpha S - \frac{\beta}{2}I - \frac{\beta}{2m}\left(rV + \frac{p\delta}{c}B\right)}{\mu_{\max} - \left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)} \\ & + \theta(\bar{x} - x)(e^x - 1) + \frac{\sigma^2 e^x}{2} + \alpha + e^x V + \beta + r + pB \\ \leq & -\frac{\mu}{S} - \frac{e^x SV}{I} - \frac{\eta}{B} - \frac{cBV}{B - \frac{\eta}{\delta}} - \frac{e^x SV}{\frac{\mu}{\alpha} - S} - \frac{\left(r - \min\left\{\alpha, \frac{\beta}{2}, \delta\right\}\right)V}{\mu_{\max} - \left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)} \\ & + \theta(\bar{x} - x)(e^x - 1) + \left(\frac{2m\mu_{\max}}{\beta} + \frac{\sigma^2}{2}\right)e^x + 2\alpha + \beta + r \\ & + \frac{2mc\mu_{\max}}{\beta} + \delta + \min\left\{\alpha, \frac{\beta}{2}, r, \delta\right\}. \end{aligned} \tag{3.8}$$

Then we define

$$U_3 = M_0 U_1 + U_2,$$

where M_0 is a sufficiently large constant satisfying

$$\begin{aligned} & -\frac{M_0\beta}{2}(R_0^s - 1) + \sup_{x\in\mathbb{R}}\left\{\theta(\bar{x} - x)(e^x - 1) + \left(\frac{2m\mu_{\max}}{\beta} + \frac{\sigma^2}{2}\right)e^x\right\} \\ & + 2\alpha + \beta + r + \frac{2mc\mu_{\max}}{\beta} + \delta + \min\left\{\alpha, \frac{\beta}{2}, r, \delta\right\} \leq -2. \end{aligned} \tag{3.9}$$

Thus from (3.7) and (3.8) we have

$$\mathcal{L}U_3 \leq h(x, S, I, V, B) + M_0 f(x) + M_0 g(x),$$

where

$$\begin{aligned} h(x, S, I, V, B) = & -\frac{M_0\beta}{2}(R_0^s - 1) + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2 mc^2\mu_{\max}}{\beta\delta}\right)V \\ & - \frac{\mu}{S} - \frac{e^x SV}{I} - \frac{cBV}{B - \frac{\eta}{\delta}} - \frac{e^x SV}{\frac{\mu}{\alpha} - S} \\ & - \frac{\left(r - \min\left\{\alpha, \frac{\beta}{2}, \delta\right\}\right)V}{\mu_{\max} - \left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)} + \theta(\bar{x} - x)(e^x - 1) \end{aligned} \tag{3.10}$$

$$+\left(\frac{2m\mu_{\max}}{\beta}+\frac{\sigma^2}{2}\right)e^x$$
$$+\alpha+\beta+r+\frac{2mc\mu_{\max}}{\beta}+\delta+\min\left\{\alpha,\frac{\beta}{2},r,\delta\right\}.$$

Step 2. (A compact set). Now we construct a compact set

$$\mathbb{D}=\left\{(x,S,I,V,B)\in\Gamma:\epsilon\le e^x\le\frac{1}{\epsilon},\epsilon\le S,\epsilon^4\le I,\epsilon\le V,\frac{\eta}{\delta}+\epsilon^2\le B,\right.$$
$$\left.S<\frac{\mu}{\alpha}-\epsilon^3,S+I+\frac{\beta}{2m}\left(V+\frac{p}{c}B\right)\le\mu_{\max}-\epsilon^2\right\}$$

such that $h(x,S,I,V,B)\le -1$ for all $(x,S,I,V,B)\in\Gamma\setminus\mathbb{D}:=\mathbb{D}^c$. Then let $\mathbb{D}^c=\bigcup_{i=1}^{8}\mathbb{D}_i^c$, where

$$\mathbb{D}_1^c=\{(x,S,I,V,B)\in\Gamma:e^x<\epsilon\},\qquad \mathbb{D}_2^c=\left\{(x,S,I,V,B)\in\Gamma:e^x>\frac{1}{\epsilon}\right\},$$
$$\mathbb{D}_3^c=\{(x,S,I,V,B)\in\Gamma:0<V<\epsilon\},\qquad \mathbb{D}_4^c=\{(x,S,I,V,B)\in\Gamma:0<S<\epsilon\},$$
$$\mathbb{D}_5^c=\{(x,S,I,V,B)\in\Gamma:\epsilon\le e^x,\epsilon\le S,\epsilon\le V,0<I<\epsilon^4\},$$
$$\mathbb{D}_6^c=\left\{(x,S,I,V,B)\in\Gamma:B<\frac{\eta}{\delta}+\epsilon^2\right\},$$
$$\mathbb{D}_7^c=\left\{(x,S,I,V,B)\in\Gamma:\epsilon\le e^x,\epsilon\le V,S>\frac{\mu}{\alpha}-\epsilon^3\right\},$$
$$\mathbb{D}_8^c=\left\{(x,S,I,V,B)\in\Gamma:\epsilon\le V,S+I+\frac{\beta}{2m}\left(V+\frac{p}{c}B\right)>\mu_{\max}-\epsilon^2\right\},$$

with a small enough constant $\epsilon\in(0,1)$ satisfying the inequalities

$$\frac{\theta}{2}(1-\epsilon)(\ln\epsilon-\bar{x})+\sup_{x\in\mathbb{R}}\{\kappa_1(x)\}\le -1, \tag{3.11}$$

where

$$\kappa_1(x)=\frac{\theta}{2}(\bar{x}-x)(e^x-1)+\left(\frac{2m\mu_{\max}}{\beta}+\frac{\sigma^2}{2}\right)e^x+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}$$
$$+2\alpha+\beta+r+\frac{2mc\mu_{\max}}{\beta}+\delta+\min\left\{\alpha,\frac{\beta}{2},r,\delta\right\},$$
$$\frac{\theta}{2}\left(\frac{1}{\epsilon}-1\right)(\ln\epsilon+\bar{x})+\sup_{x\in\mathbb{R}}\{\kappa_1(x)\}\le -1, \tag{3.12}$$
$$M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\epsilon\le 1, \tag{3.13}$$
$$-\frac{\mu}{\epsilon}+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\le 1, \tag{3.14}$$
$$-\frac{1}{\epsilon}+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\le 1, \tag{3.15}$$
$$-\frac{c\eta}{\delta\epsilon}+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\le 1, \tag{3.16}$$

$$-\frac{\mu}{\alpha\epsilon}+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\le 1, \tag{3.17}$$

$$-\frac{r-\min\left\{\alpha,\frac{\beta}{2},\delta\right\}}{\epsilon}+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\le 1. \tag{3.18}$$

Case 1. If $(x,S,I,V,B)\in\mathbb{D}_1^\epsilon$, then from (3.10) and (3.11) we have

$$\begin{aligned}h(x,S,I,V,B)\le&\theta(\bar{x}-x)(e^x-1)+\left(\frac{2m\mu_{\max}}{\beta}+\frac{\sigma^2}{2}\right)e^x\\&+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\\&+2\alpha+\beta+r+\frac{2mc\mu_{\max}}{\beta}+\delta+\min\left\{\alpha,\frac{\beta}{2},r,\delta\right\}\\\le&\frac{\theta}{2}(1-\epsilon)(\ln\epsilon-\bar{x})+\sup_{x\in\mathbb{R}}\{\kappa_1(x)\}\\\le&-1.\end{aligned}$$

Case 2. If $(x,S,I,V,B)\in\mathbb{D}_2^\epsilon$, then from (3.10) and (3.12) we have

$$\begin{aligned}h(x,S,I,V,B)\le&\theta(\bar{x}-x)(e^x-1)+\left(\frac{2m\mu_{\max}}{\beta}+\frac{\sigma^2}{2}\right)e^x\\&+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\\&+2\alpha+\beta+r+\frac{2mc\mu_{\max}}{\beta}+\delta+\min\left\{\alpha,\frac{\beta}{2},r,\delta\right\}\\\le&\frac{\theta}{2}\left(\frac{1}{\epsilon}-1\right)(\ln\epsilon+\bar{x})+\sup_{x\in\mathbb{R}}\{\kappa_1(x)\}\\\le&-1.\end{aligned}$$

Case 3. If $(x,S,I,V,B)\in\mathbb{D}_3^\epsilon$, then from (3.9), (3.10), and (3.13) we have

$$\begin{aligned}h(x,S,I,V,B)\le&-2+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)V\\\le&-2+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\epsilon\le-1.\end{aligned}$$

Case 4. If $(x,S,I,V,B)\in\mathbb{D}_4^\epsilon$, then from (3.9), (3.10), and (3.14) we have

$$\begin{aligned}h(x,S,I,V,B)\le&-2+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}-\frac{\mu}{S}\\\le&-\frac{\mu}{\epsilon}-2+M_0\left(\frac{a_1}{4a_3}+\frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta}\\\le&-1.\end{aligned}$$

Case 5. If $(x,S,I,V,B) \in \mathbb{D}_5^c$, then from (3.9), (3.10), and (3.15) we have

$$\begin{aligned} h(x,S,I,V,B) &\leq -2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} - \frac{e^x SV}{I} \\ &\leq -\frac{1}{\epsilon} - 2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} \\ &\leq -1. \end{aligned}$$

Case 6. If $(x,S,I,V,B) \in \mathbb{D}_6^c$, then from (3.9), (3.10), and (3.16) we have

$$\begin{aligned} h(x,S,I,V,B) &\leq -2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} - \frac{c\eta}{\delta\epsilon} \\ &\leq -\frac{c\eta}{\delta\epsilon} - 2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} \\ &\leq -1. \end{aligned}$$

Case 7. If $(x,S,I,V,B) \in \mathbb{D}_7^c$, then from (3.9), (3.10), and (3.17) we have

$$\begin{aligned} h(x,S,I,V,B) &\leq -2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} - \frac{e^x SV}{\frac{\mu}{\alpha} - S} \\ &\leq -\frac{\mu}{\alpha\epsilon} - 2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} \\ &\leq -1. \end{aligned}$$

Case 8. If $(x,S,I,V,B) \in \mathbb{D}_8^c$, then from (3.9), (3.10), and (3.18) we have

$$\begin{aligned} h(x,S,I,V,B) &\leq -2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} \\ &\quad - \frac{\left(r - \min\left\{\alpha, \frac{\beta}{2}, \delta\right\}\right)V}{\mu_{\max} - \left(S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)\right)} \\ &\leq -\frac{r - \min\left\{\alpha, \frac{\beta}{2}, \delta\right\}}{\epsilon} - 2 + M_0\left(\frac{a_1}{4a_3} + \frac{2a_2mc^2\mu_{\max}}{\beta\delta}\right)\frac{2m\mu_{\max}}{\beta} \\ &\leq -1. \end{aligned}$$

In summary, we have $h(x,S,I,V,B) \leq -1$ for all $(x,S,I,V,B) \in \mathbb{D}^c$.

Step 3. (The existence of a stationary distribution). Since the function U_3 tends to ∞ as S, I, V, B, or $S + I + \frac{\beta}{2m}\left(V + \frac{p}{c}B\right)$ approach the boundary of Γ or as $||(x,S,I,V,B)|| \to \infty$. Thus there exists a point $(\widetilde{x},\widetilde{S},\widetilde{I},\widetilde{V},\widetilde{B})$ in the interior of Γ that makes $U_3(\widetilde{x},\widetilde{S},\widetilde{I},\widetilde{V},\widetilde{B})$ take the minimum value. Hence $U = U_3 - U_3(\widetilde{x},\widetilde{S},\widetilde{I},\widetilde{V},\widetilde{B})$ is a nonnegative C^2-function.

Then applying Itô's formula to V, we have

$$\mathcal{L}U \leq h(x,S,I,V,B) + M_0 f(x) + M_0 g(x).$$

For any initial value $(x(0), S(0), I(0), V(0), B(0)) \in \Gamma$ and a interval $[0, t]$, using the Itô integral and mathematical expectation to U, we get

$$\begin{aligned}
0 \leq & \frac{\mathbb{E}U(x(t), S(t), I(t), V(t), B(t))}{t} \\
= & \frac{\mathbb{E}U(x(0), S(0), I(0), V(0), B(0))}{t} \\
& + \frac{1}{t}\int_0^t \mathbb{E}(\mathcal{L}U(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau)))\mathrm{d}\tau \\
\leq & \frac{\mathbb{E}U(x(0), S(0), I(0), V(0), B(0))}{t} \\
& + \frac{1}{t}\int_0^t h(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau)))\mathrm{d}\tau \\
& + 3M_0\sqrt[3]{a_1 a_2 m\mu}\left(\bar{b}^{\frac{1}{3}} e^{\frac{\sigma^2}{36\theta}} - \frac{1}{t}\int_0^t e^{\frac{x(\tau)}{3}}\mathrm{d}\tau\right) \\
& + M_0 a_1 a_3 \frac{2m\mu_{\max}}{\beta}\left(\frac{1}{t}\int_0^t e^{2x(\tau)}\mathrm{d}\tau - \bar{b}^2 e^{\frac{\sigma^2}{\theta}}\right).
\end{aligned} \tag{3.19}$$

According to Lemma 2.2, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\frac{x(\tau)}{3}}\mathrm{d}\tau = \bar{b}^{\frac{1}{3}} e^{\frac{\sigma^2}{36\theta}} \text{ a.s.} \tag{3.20}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{2x(\tau)}\mathrm{d}\tau = \bar{b}^2 e^{\frac{\sigma^2}{\theta}} \text{ a.s.} \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19) and letting $t \to +\infty$, we have

$$0 \leq \liminf_{t\to+\infty}\frac{1}{t}\int_0^t h(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau)))\mathrm{d}\tau \text{ a.s.}$$

On the other hand, we obtain

$$h(x, S, I, V, B) \leq \sup_{(x,S,I,V,B)\in\Gamma}\{h(x, S, I, V, B)\} := M_1, \quad (x, S, I, V, B) \in \mathbb{R}\times\mathbb{R}^4_+.$$

Then we have

$$\begin{aligned}
& \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}[h(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau))]\mathrm{d}\tau \\
= & \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}[h(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau))]\mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}\}}\mathrm{d}\tau \\
& + \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}[h(x(\tau), S(\tau), I(\tau), V(\tau), B(\tau))]\mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}^c\}}\mathrm{d}\tau \\
\leq & M_1\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}\}}\mathrm{d}\tau - \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}^c\}}\mathrm{d}\tau \\
\leq & (M_1+1)\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}\}}\mathrm{d}\tau - 1.
\end{aligned}$$

Therefore

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau))\in\mathbb{D}\}}\mathrm{d}\tau \geq \frac{1}{M_1+1} > 0 \ \text{ a.s.} \tag{3.22}$$

Let $\mathbb{P}(t,(x(t),S(t),I(t),V(t),B(t)),\Omega)$ be the (transition) probability that $(x(t),S(t),I(t),V(t),B(t))$ belongs to the set Ω. Using Fatou's lemma [28], we have

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{P}(\tau,(x(\tau),S(\tau),I(\tau),V(\tau),B(\tau)),\mathbb{D})\mathrm{d}\tau \geq \frac{1}{M_1+1} > 0 \ \text{ a.s.} \tag{3.23}$$

According to Lemma 2.1, system (1.3) has at least one stationary distribution on Γ, and it has the Feller and ergodic properties. This completes the proof. □

4 Probability density function

The local probability density function (local PDF) is a function used in probability theory and statistics to describe the probability distribution of a random variable at different values. Unlike the global probability density function (global PDF), the local PDF is defined at specific points or regions.

The key characteristic of the local PDF is its ability to provide more detailed and accurate probability information as it describes the distribution characteristics of the random variable within specific points or regions. By utilizing the local PDF, researchers can obtain information about the probability density, distribution shape, and variations in the probability distribution of the random variable at specific values. In this section, we present the local PDF of stochastic model (1.3) near its quasi-endemic equilibrium.

Firstly, if

$$R_0^p = \frac{\bar{b}\mu\delta m}{\alpha\beta(r\delta+p\eta)} > 1, \tag{4.1}$$

then we can obtain that the stochastic model (1.3) has a unique quasi-infected equilibrium $E^\star = (\ln\bar{b}, S^\star, V^\star, I^\star, B^\star)$, where

$$S^\star = \frac{\mu}{\alpha+\bar{b}V^\star},\ I^\star = \frac{\bar{b}S^\star V^\star}{\beta},\ B^\star = \frac{\eta}{\delta-cV^\star},\ V^\star = \frac{Q_2-\sqrt{Q_2^2-4Q_1Q_3}}{2Q_1},$$

with

$$Q_1 = r\beta\bar{b}c,\ Q_2 = \frac{\alpha\beta c(r\delta+p\eta)}{\delta}(R_0^p-1)+r\bar{b}\beta\eta+\beta p\eta\bar{b}+\frac{c\alpha\beta p\eta}{\delta},\ Q_3 = \alpha\beta(r\delta+p\eta)(R_0^p-1).$$

Then let $Y = (y_1,y_2,y_3,y_4,y_5)^T = (x-\bar{x}, S-S^\star, V-V^\star, I-I^\star, B-B^\star)^T$. Applying Itô's integral, we obtain the corresponding linearized system around $E^\star$ of model (1.3):

$$\begin{cases} \mathrm{d}y_1 = -\theta y_1\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{B}(t), \\ \mathrm{d}y_2 = (-c_{21}y_1 - c_{22}y_2 - c_{23}y_3)\mathrm{d}t, \\ \mathrm{d}y_3 = (-c_{33}y_3 + c_{34}y_4 - c_{35}y_5)\mathrm{d}t, \\ \mathrm{d}y_4 = (c_{21}y_1 + c_{42}y_2 + c_{23}y_3 - c_{44}y_4)\mathrm{d}t, \\ \mathrm{d}y_5 = (c_{53}y_3 - c_{55}y_5)\mathrm{d}t, \end{cases} \tag{4.2}$$

where

$$c_{21} = \bar{b}S^\star V^\star,\ c_{22} = \alpha + \bar{b}V^\star,\ c_{23} = \bar{b}S^\star,\ \ c_{33} = r + pB^\star,\ c_{34} = m,$$
$$c_{35} = pV^\star,\ c_{42} = \bar{b}V^\star,\ c_{44} = \beta,\ c_{53} = cB^\star,\ c_{55} = \delta - cV^\star.$$

Model (4.2) can be equivalently written as

$$\mathrm{d}Y(t) = CY(t)\mathrm{d}t + \Xi \mathrm{d}\mathcal{B}(t),$$

where

$$C = \begin{pmatrix} -\theta & 0 & 0 & 0 & 0 \\ -c_{21} & -c_{22} & -c_{23} & 0 & 0 \\ 0 & 0 & -c_{33} & c_{34} & -c_{35} \\ c_{21} & c_{42} & c_{23} & -c_{44} & 0 \\ 0 & 0 & c_{53} & 0 & -c_{55} \end{pmatrix},\ \Xi = \begin{pmatrix} \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1 *If $\beta \neq \alpha$, $r + pB^\star > \alpha$ (in general) and $R_0^p > 1$, then the stationary solution $(x(t), S(t), V(t), I(t), B(t))$ of system (1.3) around $E^\star = (\ln \bar{\beta}, S^\star, V^\star, I^\star, B^\star)$ follows the normal distribution $\mathbb{N}_5(E^\star, \Sigma)$, where*

$$\Sigma = \left[c_{21}c_{34}c_{53}\sigma \left(\alpha + \delta - cV^\star + \frac{(\alpha + \bar{b}V^\star)(\delta - cV^\star)}{r + pB^\star - \alpha} \right) \right]^2 (T_5T_3T_2T_1)^{-1} \times \Omega[(T_5T_3T_2T_1)^{-1}]^T,$$

and the matrices T_1, T_2, T_3, T_5, and Ω are defined in the following proof.

Proof Based on [32–34], it has been established that system (1) exhibits a singular probability density function, denoted as $\Psi(Y(t))$, in the vicinity of the equilibrium $E^\star$. This particular density function is governed by the Fokker–Planck equation

$$-\frac{\sigma^2}{2}\frac{\partial^2\Psi}{\partial y_1^2} + \frac{\partial}{\partial y_1}\left(-\theta y_1\Psi\right) + \frac{\partial}{\partial y_2}\left[(-c_{21}y_1 - c_{22}y_2 - c_{23})\Psi\right] + \frac{\partial}{\partial y_3}\left[(-c_{33}y_3 + c_{34}y_4 - c_{35}y_5)\Psi\right] + \frac{\partial}{\partial y_4}\left[(c_{21}y_1 + c_{42}y_2 + c_{23}y_3 - c_{44}y_4)\Psi\right] + \frac{\partial}{\partial y_5}\left[(c_{53}y_3 - c_{55}y_5)\Psi\right] = 0.$$

Based on the work by Roozen [35], we can easily deduce that the matrix Ξ is constant. Consequently, the probability density function $\Psi(Y(t))$ follows a Gaussian distribution represented as

$$\Psi(Y(t)) = \psi e^{-\frac{1}{2}Y(t)\Theta Y(t)},$$

where Θ is a real symmetric matrix satisfying the algebraic equation $\Theta\Xi\Theta + C^T\Theta + \Theta C = 0$, and the constant ψ is determined by the normalization condition $\int_{R^5}\Psi(Y)\mathrm{d}Y = 1$. As-

suming that Θ is invertible, let $\Sigma = \Theta^{-1}$. Then we obtain

$$\Xi^2 + C\Sigma + \Sigma C^T = 0.$$

Let

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$C_1 = T_1 C T_1^{-1} = \begin{pmatrix} -\theta & 0 & 0 & 0 & 0 \\ -c_{21} & -c_{22} & -c_{23} & 0 & 0 \\ 0 & -c_{34} & -c_{33} & c_{34} & -c_{35} \\ 0 & c_{42}+c_{44}-c_{22} & 0 & -c_{44} & 0 \\ 0 & 0 & c_{53} & 0 & -c_{55} \end{pmatrix}.$$

Note that $c_{42} + c_{44} - c_{22} = \beta - \alpha \neq 0$. Now let

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{c_{42}+c_{44}-c_{22}}{c_{34}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$C_2 = T_2 C_1 T_2^{-1} = \begin{pmatrix} -\theta & 0 & 0 & 0 & 0 \\ -c_{21} & -c_{22} & -c_{23} & 0 & 0 \\ 0 & -c_{34} & c_{22}-c_{33}-c_{42}-c_{44} & c_{34} & -c_{35} \\ 0 & 0 & -\frac{\vartheta_1}{c_{34}} & c_{42}-c_{22} & -\frac{c_{35}}{c_{34}}(c_{42}+c_{44}-c_{22}) \\ 0 & 0 & c_{53} & 0 & -c_{55} \end{pmatrix},$$

where $\vartheta_1 = (c_{33} + c_{42} - c_{22})(c_{42} + c_{44} - c_{22})$. Since $c_{33} + c_{42} - c_{22} = r + pB^\star - \alpha > 0$, we have $\vartheta_1 = (r + pB^\star - \alpha)(\beta - \alpha) \neq 0$. Denote

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{c_{34}c_{53}}{\vartheta_1} & 1 \end{pmatrix}.$$

Then

$$C_3 = T_3 C_2 T_3^{-1} = \begin{pmatrix} -\theta & 0 & 0 & 0 & 0 \\ -c_{21} & -c_{22} & -c_{23} & 0 & 0 \\ 0 & -c_{34} & c_{22}-c_{33}-c_{42}-c_{44} & c_{35}+\frac{c_{34}c_{35}c_{53}}{\vartheta_1} & -c_{35} \\ 0 & 0 & -\frac{\vartheta_1}{c_{34}} & c_{42}-c_{22}+\frac{c_{35}c_{53}}{c_{33}+c_{42}-c_{22}} & -\frac{c_{35}}{c_{34}}(c_{42}+c_{44}-c_{22}) \\ 0 & 0 & 0 & -\frac{c_{34}c_{53}\vartheta_2}{\vartheta_1} & c_{55}\left(\frac{c_{22}}{c_{33}+c_{42}-c_{22}}-1\right) \end{pmatrix}$$

with $\vartheta_2 = c_{22}+c_{55}-c_{42}+\frac{c_{22}c_{55}}{c_{33}+c_{42}-c_{22}} = \alpha+\delta-cV^\star+\frac{(\alpha+\bar{b}V^\star)(\delta-cV^\star)}{r+pB^\star-\alpha} > 0$.

Then letting $T_4 = (0,0,0,0,1)$ and $T_5 = (T_4C_3^4, T_4C_3^3, T_4C_3^2, T_4C_3, T_4)^T$, we have

$$C_4 = T_5 C_3 T_5^{-1} = \begin{pmatrix} -c_1 & -c_2 & -c_3 & -c_4 & -c_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
c_1 =& \theta+\varrho_1,\ c_2 = \varrho_1\theta+\varrho_2,\ c_3 = \varrho_2\theta+\varrho_3,\ c_4 = \varrho_3\theta+\varrho_4,\ c_5 = \varrho_4\theta, \\
\varrho_1 =& c_{22}+c_{33}+c_{44}+c_{55}, \\
\varrho_2 =& c_{22}c_{33}+c_{22}c_{44}+c_{22}c_{55}+c_{33}c_{44}+c_{33}c_{55}+c_{35}c_{53}+c_{44}c_{55}-c_{23}c_{34} \\
=& c_{22}c_{33}+c_{22}c_{44}+c_{22}c_{55}+c_{33}c_{55}+c_{35}c_{53}+c_{44}c_{55}, \\
\varrho_3 =& c_{22}(c_{33}c_{44}-c_{23}c_{34})+c_{23}c_{34}c_{42}+c_{22}c_{33}c_{55}+c_{22}c_{35}c_{53}+c_{22}c_{44}c_{55} \\
&+c_{55}(c_{33}c_{44}-c_{23}c_{34})+c_{35}c_{44}c_{53} \\
=& c_{23}c_{34}c_{42}+c_{22}c_{33}c_{55}+c_{22}c_{35}c_{53}+c_{22}c_{44}c_{55}+c_{35}c_{44}c_{53}, \\
\varrho_4 =& c_{22}c_{55}(c_{33}c_{44}-c_{23}c_{34})+c_{22}c_{35}c_{44}c_{53}+c_{23}c_{34}c_{42}c_{55} = c_{22}c_{35}c_{44}c_{53}+c_{23}c_{34}c_{42}c_{55},
\end{aligned}$$

due to $c_{33}c_{44} = \beta(r+pB^\star) = \frac{\bar{b}S^\star V^\star}{I^\star}\frac{mI^\star}{V^\star} = m\bar{b}S^\star = c_{23}c_{34}$. After calculation, we obtain

$$\begin{aligned}
\varrho_2\varrho_3-\varrho_1\varrho_4 >& c_{22}c_{55}(c_{33}+c_{44})(c_{22}c_{33}+c_{22}c_{44}+c_{22}c_{55}+c_{33}c_{55}+c_{35}c_{53}) \\
&+c_{33}c_{42}c_{44}(c_{22}c_{33}+c_{22}c_{44}+c_{35}c_{53})+c_{35}c_{44}c_{53}(c_{33}c_{55}+c_{44}c_{55}+c_{35}c_{53}) \\
&+c_{22}c_{35}c_{53}(c_{22}c_{33}+c_{22}c_{55}+c_{33}c_{55}+c_{35}c_{53}+c_{44}c_{55})
\end{aligned}$$

and

$$\begin{aligned}
\varrho_3^2 <& (c_{33}c_{42}c_{44}+c_{35}c_{44}c_{53}+c_{22}c_{35}c_{53})(c_{22}c_{33}c_{44}+c_{22}c_{33}c_{55}+c_{22}c_{44}c_{55} \\
&+c_{35}c_{53}(c_{22}+c_{44}))+c_{22}c_{55}(c_{33}+c_{44})(c_{22}c_{33}c_{44}+c_{22}c_{33}c_{55} \\
&+c_{22}c_{44}c_{55}+c_{35}c_{53}(c_{22}+c_{44})).
\end{aligned}$$

Therefore it is not difficult to obtain $\varrho_1(\varrho_2\varrho_3 - \varrho_1\varrho_4) > \varrho_3^2$, which implies that $c_1, \ldots, c_5 > 0$ and

$$
\begin{aligned}
&c_1c_2 - c_3 > \varrho_1\varrho_2 - \varrho_3 > 0,\\
&c_2c_3 - c_1c_4 > \theta^2(\varrho_1\varrho_2 - \varrho_3) + \theta(\varrho_2^2 - \varrho_4) + (\varrho_2\varrho_3 - \varrho_1\varrho_4) > 0,\\
&c_3c_4 - c_2c_5 > \theta^2(\varrho_2\varrho_3 - \varrho_1\varrho_4) > 0.
\end{aligned}
$$

The conditions in Lemma 2.3 are satisfied. Denote

$$
\begin{aligned}
&\omega_{11} = \frac{c_2(c_3c_4 - c_2c_5) - c_4(c_1c_4 - c_5)}{2\Delta},\ \omega_{22} = \frac{c_3c_4 - c_2c_5}{2\Delta},\ \omega_{33} = \frac{c_1c_4 - c_5}{2\Delta},\\
&\omega_{44} = \frac{c_1c_2 - c_3}{2\Delta},\ \omega_{55} = \frac{c_3(c_1c_2 - c_3) - c_1(c_1c_4 - c_5)}{2\Delta},\\
&\Delta = (c_1c_2 - c_3)(c_3c_4 - c_2c_5) - (c_1c_4 - c_5)^2 > 0,
\end{aligned}
$$

and

$$
\Omega = \begin{pmatrix}
\omega_{11} & 0 & -\omega_{22} & 0 & \omega_{33}\\
0 & \omega_{22} & 0 & -\omega_{33} & 0\\
-\omega_{22} & 0 & \omega_{33} & 0 & -\omega_{44}\\
0 & -\omega_{33} & 0 & \omega_{44} & 0\\
\omega_{33} & 0 & -\omega_{44} & 0 & \omega_{55}
\end{pmatrix}.
$$

Then we obtain

$$
\begin{aligned}
&(T_5T_3T_2T_1)\Xi^2(T_5T_3T_2T_1)^T + C_4[(T_5T_3T_2T_1)\Sigma(T_5T_3T_2T_1)^T]\\
&\quad + [(T_5T_3T_2T_1)\Sigma(T_5T_3T_2T_1)^T]C_4^T = 0.
\end{aligned}
$$

From Lemma 2.3 we obtain that $(T_5T_3T_2T_1)\Sigma(T_5T_3T_2T_1)^T = (c_{21}c_{34}c_{53}\vartheta_2\sigma)^2\Omega$ is a positive definite matrix. Hence

$$
\begin{aligned}
\Sigma = &\left[c_{21}c_{34}c_{53}\sigma\left(\alpha + \delta - cV^\star + \frac{(\alpha + \bar{b}V^\star)(\delta - cV^\star)}{r + pB^\star - \alpha}\right)\right]^2 (T_5T_3T_2T_1)^{-1}\\
&\times \Omega[(T_5T_3T_2T_1)^{-1}]^T
\end{aligned}
$$

is also positive definite. □

5 Exponential extinction of CHIKV

In this section, our primary focus is on discussing the exponential extinction of infected cells and CHIKV within stochastic model (1.3). Denote

$$
R_0^e = R_0^p + \frac{\bar{b}\mu(r\delta + p\eta)\left(1 + e^{\frac{\sigma^2}{\theta}} - 2e^{\frac{\sigma^2}{4\theta}}\right)^{\frac{1}{2}}}{\alpha\beta\min\{\beta\delta, r\delta + p\eta\}},
$$

where R_0^p is defined in (4.1).

Theorem 5.1 *If* $R_0^e < 1$, *then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \left(\frac{\sqrt{\alpha\beta\delta m}}{\beta\sqrt{\bar{b}\mu(r\delta + p\eta)}} I(t) + \frac{\delta}{r\delta + p\eta} V(t) \right) < 0 \ \ a.s.,$$

which implies that the infected host cells I *and CHIKV particles* V *in model* (1.3) *will go extinct in the long term.*

Proof First, we get

$$\sqrt{R_0^p}(p,1) = (p,1) \begin{pmatrix} 0 & \frac{\bar{b}\mu}{\alpha\beta} \\ \frac{\delta m}{r\delta + p\eta} & 0 \end{pmatrix}, \tag{5.1}$$

where $p = \sqrt{\dfrac{\alpha\beta\delta m}{\bar{b}\mu(r\delta + p\eta)}}$.

Define the C^2-function

$$P = \frac{p}{\beta} I + \frac{\delta}{r\delta + p\eta} V,$$

where $\frac{1}{\beta}$ and $\frac{\delta}{r\delta+p\eta}$ are positive constants to be determined later. Then applying Itô's formula to $\ln P$, we have

$$\begin{aligned} \mathcal{L}(\ln P) =& \frac{1}{P}\left[\frac{p}{\beta}(e^x SV - \beta I) + \frac{\delta}{r\delta + p\eta}(mI - rV - pBV)\right] \\ \leq& \frac{1}{P}\left[\frac{p}{\beta}\left(\frac{\bar{b}\mu}{\alpha} V - \beta I\right) + \frac{\delta}{r\delta + p\eta}\left(mI - rV - \frac{p\eta}{\delta} V\right)\right] \\ &+ \frac{\frac{\mu}{\alpha\beta} V}{\frac{pI}{\beta} + \frac{\delta V}{r\delta+p\eta}} \left(e^x - \bar{b}\right) \\ \leq& \frac{1}{P}\left[\left(\frac{\delta m}{r\delta + p\eta} - p\right) I + \left(\frac{p\bar{b}\mu}{\alpha\beta} - 1\right) V\right] + \frac{\mu(r\delta + p\eta)}{\alpha\beta\delta} |e^x - \bar{b}| \\ =& \frac{1}{P}(p,1)\left[\begin{pmatrix} 0 & \frac{\bar{b}\mu}{\alpha\beta} \\ \frac{\delta m}{r\delta+p\eta} & 0 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix} - \begin{pmatrix} I \\ V \end{pmatrix}\right] + \frac{\mu(r\delta + p\eta)}{\alpha\beta\delta} |e^x - \bar{b}|. \end{aligned} \tag{5.2}$$

Substituting (5.1) into (5.2), if $R_0^p < 1$, then we have

$$\begin{aligned} \mathcal{L}(\ln P) \leq& \frac{1}{P}\left(\sqrt{R_0^p} - 1\right)(pI + V) + \frac{\mu(r\delta + p\eta)}{\alpha\beta\delta} |e^x - \bar{b}| \\ \leq& -\min\left\{\beta, \frac{r\delta + p\eta}{\delta}\right\}\left(1 - \sqrt{R_0^p}\right) + \frac{\mu(r\delta + p\eta)}{\alpha\beta\delta} |e^x - \bar{b}|. \end{aligned} \tag{5.3}$$

Integrating (5.3) from 0 to t and dividing both sides by t, we obtain

$$\begin{aligned}\frac{\ln P(t)-\ln P(0)}{t} \leq & -\min\left\{\beta, \frac{r\delta+p\eta}{\delta}\right\}\left(1-\sqrt{R_0^p}\right) \\ & +\frac{\mu(r\delta+p\eta)}{\alpha\beta\delta}\left(\frac{1}{t}\int_0^t \left|e^{x(\tau)}-\bar{b}\right| \mathrm{d}\tau\right).\end{aligned} \tag{5.4}$$

From Lemma 2.2 we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left|e^{x(\tau)}-\bar{b}\right| \mathrm{d}\tau \leq \bar{b}\left(1+e^{\frac{\sigma^2}{\theta}}-2e^{\frac{\sigma^2}{4\theta}}\right)^{\frac{1}{2}}. \tag{5.5}$$

Taking the superior limit of t on both sides of (5.4) and combining with (5.5), we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln P(t)}{t} \leq & -\min\left\{\beta, \frac{r\delta+p\eta}{\delta}\right\}\left(1-\sqrt{R_0^p}\right)+\frac{\bar{b}\mu(r\delta+p\eta)}{\alpha\beta\delta}\left(1+e^{\frac{\sigma^2}{\theta}}-2e^{\frac{\sigma^2}{4\theta}}\right)^{\frac{1}{2}} \\ = & -\min\left\{\beta, \frac{r\delta+p\eta}{\delta}\right\}(1-R_0^e).\end{aligned}$$

This completes the proof. □

6 Numerical simulations

In this section, we provide several numerical examples to validate our theoretical results. Based on [36, 37], we adopt the parameter values for model (1.3) as shown in Table 2. Taking into account the influence of environmental fluctuations in practice, we obtain the discrete model over the time interval $[0, T]$ using Milstein's higher-order method [38] as follows:

$$\begin{cases} x_{i+1} = x_i + \theta(\ln\bar{b} - x_i)\Delta t + \sigma\eta_i\sqrt{\Delta t}, \\ S_{i+1} = S_i + \left(\mu - \alpha S_i - e^{x_i}S_iV_i\right)\Delta t, \\ I_{i+1} = I_i + \left(e^{x_i}S_iV_i - \beta I_i\right)\Delta t, \\ V_{i+1} = V_i + \left(mI_i - rV_i - pB_iV_i\right)\Delta t, \\ B_{i+1} = B_i + \left(\eta + cB_iV_i - \delta B_i\right)\Delta t, \end{cases} \tag{6.1}$$

where $(x_i, S_i, I_i, V_i, B_i)$ denotes the corresponding value of the ith iteration of the discretization equation, the time increment $\Delta t > 0$, and η_i are Gaussian random variables with distribution $\mathbb{N}(0,1)$ for $i = 1, 2, \ldots, n$. We choose the initial value $(x(0), S(0), I(0), V(0), B(0)) = (\ln 0.5269, 2, 0.2, 0.1, 2)$ in the invariant set Γ.

Example 1 (Stationary distribution) First, to investigate the existence of the stationary distribution, we chose the parameters shown in Table 2 and obtain

$$R_0^s = \frac{\bar{b}\mu\delta m e^{\frac{\sigma^2}{12\theta}}}{\alpha\beta(r\delta+p\eta)} = 5.0746 > 1.$$

According to Theorem 3.1, the stochastic system (1.3) admits at least one ergodic stationary distribution. The phase diagrams of $S(t)$, $I(t)$, $V(t)$, and $B(t)$ are given in the left-hand

Table 2 Value of parameters in (6.1)

Parameters	Description	Value	Source
θ	Reversion speed	0.3	[36]
$\bar{b}$	Long-run mean level of contact rate between uninfected host cells and CHIKV particles	0.5269	[37]
ξ	Noise intensity	0.1	[36]
μ	Production rate of uninfected host cells	1.826	[37]
α	Death rate of uninfected host cells	0.7979	[37]
β	Death rate of infected host cells	0.4441	[37]
m	Virus proliferation rate	2.02	[37]
r	Death rate of CHIKV particles	0.4418	[37]
p	Elimination rate of CHIKV by B cells	0.5946	[37]
η	Constant production rate of B cells	1.402	[37]
c	Production rate of B cells due to increased CHIKV	1.2129	[37]
δ	Death rate of B cells	1.251	[37]

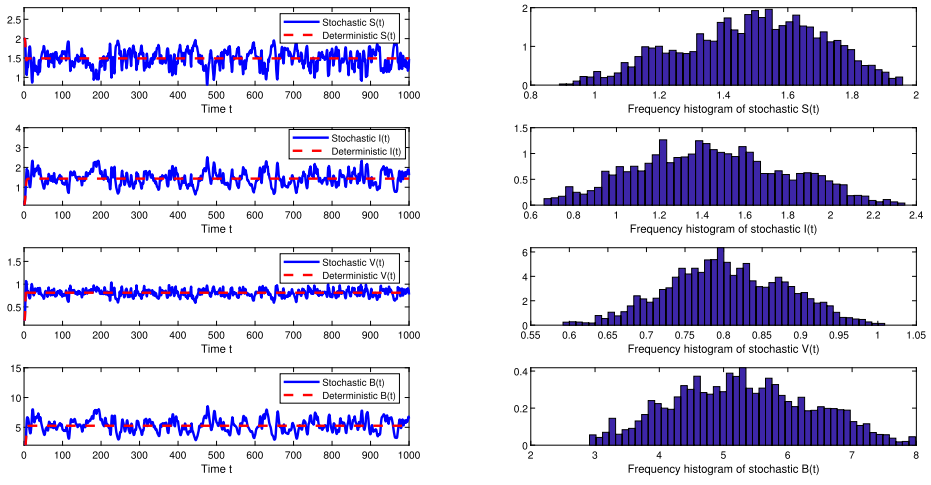

Figure 2 Computer simulations for (i) the temporal evolution of the concentrations of uninfected host cells (S), infected host cells (I), CHIKV particles (V), and B cells (B) in system (1.3) and (ii) the frequency histograms of S, I, V, and B in system (1.3), illustrating the stationary distribution. The parameters used for the simulations are provided in Table 2

column of Fig. 2, and the frequency histograms are presented in the right-hand column of Fig. 2.

To facilitate comparison with the deterministic model (1.1), we choose the contact rate 0.5269 and the other parameters as in Table 2, which results in

$$R_0 = \frac{b\mu\delta m}{\alpha\beta(r\delta + p\eta)} = 4.9493 > 1.$$

Then by [15] the disease of the deterministic system (1.1) will persist in a long term; see the left-hand column of Fig. 2. Figure 2 illustrates the dynamics and statistical behavior of the system variables under the stochastic framework of system (1.3). Panel (i) depicts the temporal evolution of the concentrations of uninfected host cells (S), infected host cells (I), CHIKV particles (V), and B cells (B). It highlights the transient dynamics and eventual stabilization of these variables, showing how they fluctuate and reach a steady state under stochastic perturbations. Panel (ii) presents frequency histograms for S, I, V, and B,

demonstrating the stationary distribution derived from the long-term simulations. These histograms reflect the probabilistic nature of the system equilibrium under the influence of noise.

Example 2 (Probability density function) Letting the parameters of model (1.3) be as in Table 2, we obtain that there exists a quasi-endemic equilibrium for the model (1.3), $(\bar{x}, S^\star, I^\star, V^\star, B^\star) = (\ln 0.5269, 1.4899, 1.4349, 0.81175, 5.2621)$. In addition, we can calculate that $R_0^p = 4.9493 > 1$ and $r + pB^\star = 3.5707 > \alpha$. By Theorem 4.1 the solution $(x(t), S(t), V(t), I(t), B(t))$ of system (1.3) has a normal probability density function $\Psi(x, S, V, I, B) \sim \mathbb{N}_5((\ln 0.5269, 1.4899, 0.8116, 1.4349, 5.2621)^T, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 0.15 & -0.0750 & 0.02402 & 0.1107 & 0.2706 \\ -0.0750 & 0.04868 & -0.015116 & -0.07561 & -0.2067 \\ 0.02402 & -0.01511 & 0.005358 & 0.0215074 & 0.0504 \\ 0.1107 & -0.07561 & 0.0215074 & 0.1240 & 0.3671 \\ 0.2706 & -0.2067 & 0.0504 & 0.3671 & 1.2067 \end{pmatrix}.$$

In addition, we can calculate the following marginal density functions:

$$\begin{aligned} \frac{\partial \Psi(x, S, V, I, B)}{\partial S} &= 1.8082 e^{-10.2714(S-1.4899)^2}, \\ \frac{\partial \Psi(x, S, V, I, B)}{\partial I} &= 1.1329 e^{-4.0321(I-1.4349)^2}, \\ \frac{\partial \Psi(x, S, V, I, B)}{\partial V} &= 5.4503 e^{-93.3224(V-0.8116)^2}, \\ \frac{\partial \Psi(x, S, V, I, B)}{\partial B} &= 0.3632 e^{-0.4143(B-5.2621)^2}. \end{aligned}$$

From the right-hand columns of Figs. 2 and 3, we can see that the marginal density function of $\Psi(x, S, V, I, B)$ given by Theorem 4.1 is highly consistent with the corresponding

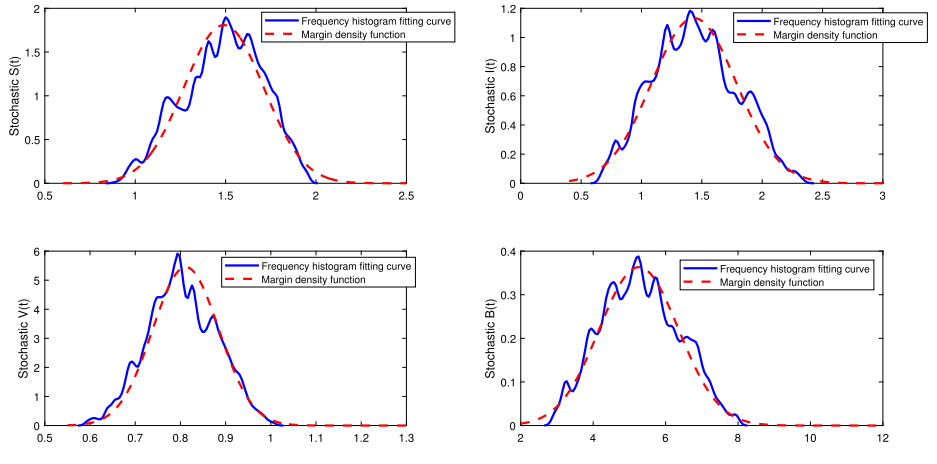

Figure 3 Computer simulations for the frequency fitting density functions and the marginal density functions of S (uninfected host cells), I (infected host cells), V (CHIKV particles), and B (B cells) in system (1.3). The results are based on 500,000 iteration points, highlighting the probabilistic distribution and alignment between simulated and theoretical densities. The parameters are given in Table 2

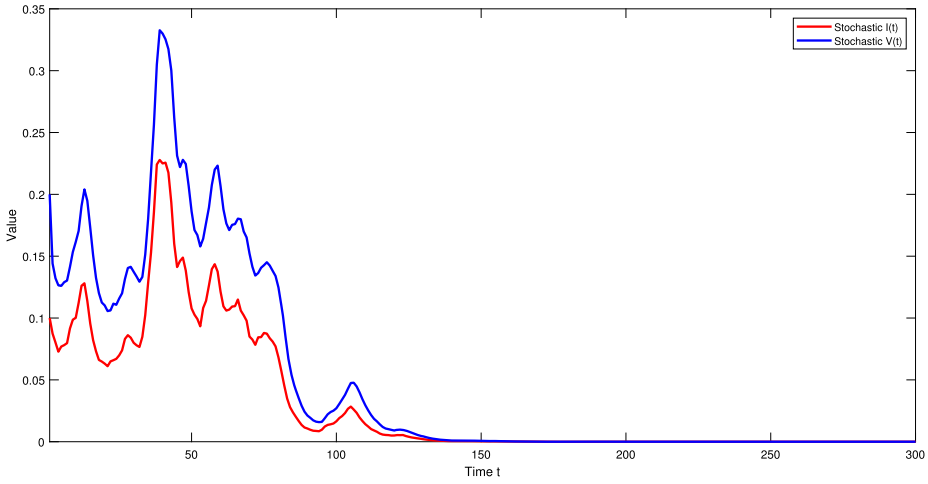

Figure 4 Computer simulations showing the dynamic variation trends of infected host cells (I) and CHIKV particles (V) in system (1.3) under the mean contact rate $\bar{b} = 0.08$. These simulations highlight the interaction between the two compartments over time, demonstrating how changes in one variable influence the other. The other parameters are given in Table 2

frequency histogram. Figure 3 illustrates the statistical properties of the system variables S, I, V, and B by comparing their simulated marginal density functions (derived from long-term simulations) with theoretical probability density functions. The fitting curves demonstrate good agreement, validating the stochastic dynamics of system (1.3). The large number of iterations ensures the robustness of the results, capturing the equilibrium behavior of the variables under noise.

Example 3 (Extinction) Choosing $\bar{b} = 0.08$ and the other parameters as in Table 2, we get

$$R_0^e = R_0^p + \frac{\bar{b}\mu(r\delta + p\eta)\left(1 + e^{\frac{\sigma^2}{\theta}} - 2e^{\frac{\sigma^2}{4\theta}}\right)^{\frac{1}{2}}}{\alpha\beta \min\{\beta\delta, r\delta + p\eta\}} = 0.9511 < 1.$$

Thus by Theorem 5.1 the infected host cells I and CHIKV particles V of the stochastic system (1.3) will be extinct exponentially in a long term, which is supported by Fig. 4. Figure 4 presents the time-series trends of I (infected host cells) and V (CHIKV particles) when the mean contact rate is set to $\bar{b} = 0.08$. The figure emphasizes the close coupling between these two compartments, with fluctuations in I driving corresponding changes in V, and vice versa. The results capture the feedback dynamics inherent in the infection process and demonstrate the system capacity to stabilize under stochastic perturbations.

7 Conclusions

The primary objective of this study is to establish and analyze the dynamical behavior of a stochastic model for CHIKV infection incorporating the Black–Karasinski process. Drawing inspiration from the work of [15], we introduce a stochastic CHIKV infection model by perturbing the contact rate b using the BK process.

Following the establishment of existence and uniqueness of solutions for the stochastic system, as well as the identification of invariant sets, we obtain sufficient conditions

for the stationary distribution and extinction of the stochastic model (1.3). Subsequently, we derive the sufficient conditions for both endemic persistence and extinction by solving the corresponding five-dimensional Fokker–Planck equation. Moreover, we obtain explicit expressions for the local density function of the stochastic model. Remarkably, in this study, we observe that as the noise intensity tends to zero (i.e., $\sigma \to 0$), the conditions R_0^s converge to R_0^p, and R_0^e approaches R_0^p. This finding implies that the dynamic behavior of the stochastic model encompasses that of the corresponding deterministic model.

Lastly, we propose and discuss several remaining issues to be addressed. Notably, due to the current limitations of our mathematical methods, a disparity exists in the conditions for R_0^s and R_0^e, particularly when the noise intensity is significant. Consequently, our future work will focus on establishing a threshold condition to determine the persistence of CHIKV in the model, which is considered a key research direction. Additionally, another intriguing avenue for investigation involves considering CHIKV models driven by alternative types of stochastic noise, such as colored noise [39] or Lévy jumps [40]. Ongoing research efforts are dedicated to exploring these areas of interest. Furthermore, a critical extension involves incorporating the exponential expansion phase of CHIKV into the model to better capture the dynamics of early epidemic outbreaks. This phase is particularly relevant during the initial rapid growth of infection cases and requires a hybrid deterministic–stochastic framework or the integration of explicit exponential growth terms [41, 42]. Such advancements will enhance the applicability of the model to both early and long-term epidemic dynamics, complementing the insights gained in this study.

Appendix

Itô's process is a general stochastic process described by

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}\mathcal{B}_t,$$

where X_t is the state variable, $\mu(X_t, t)$ is the drift term, $\sigma(X_t, t)$ is the diffusion term, and $\mathcal{B}_t$ is a standard Brownian motion.

The Black–Karasinski process is a particular case of Itô's process, with a logarithmic transformation to ensure the positivity and mean-reverting dynamics, which is described by the following stochastic differential equation:

$$\mathrm{d}\ln X_t = \theta\left(\ln\bar{X} - \ln X_t\right)\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{B}_t,$$

where X_t is the variable of interest (e.g., contact rate), $\bar{X}$ is the long-run mean level, $\theta > 0$ is the rate of mean reversion, $\sigma > 0$ is the noise intensity, and $\mathcal{B}_t$ is a standard Brownian motion.

Author contributions

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Data availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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References
1. Moulay, D., Aziz-Alaoui, M.A., Cadivel, M.: The chikungunya disease: modeling, vector and transmission global dynamics. Math. Biosci. **229**(1), 50–63 (2011). https://doi.org/10.1016/j.mbs.2010.10.008
2. Pialoux, G., Gaüzère, B.-A., Jauréguiberry, S., Strobel, M.: Chikungunya, an epidemic arbovirosis. Lancet Infect. Dis. **7**(5), 319–327 (2007). https://doi.org/10.1016/S1473-3099(07)70107-X
3. Simon, F., Javelle, E., Oliver, M., Leparc-Goffart, I., Marimoutou, C.: Chikungunya virus infection. Curr. Infect. Dis. Rep. **13**(3), 218–228 (2011). https://doi.org/10.1007/s11908-011-0180-1
4. Weaver, S.C., Osorio, J.E., Livengood, J.A., Chen, R., Stinchcomb, D.T.: Chikungunya virus and prospects for a vaccine. Expert Rev. Vaccines **11**(9), 1087–1101 (2012). https://doi.org/10.1586/erv.12.84
5. Wu, Z., Cai, Y., Wang, Z., Wang, W.: Global stability of a fractional order SIS epidemic model. J. Differ. Equ. **352**, 221–248 (2023). https://doi.org/10.1016/j.jde.2022.12.045
6. Dumont, Y., Chiroleu, F., Domerg, C.: On a temporal model for the Chikungunya disease: modeling, theory and numerics. Math. Biosci. **213**(1), 80–91 (2008). https://doi.org/10.1016/j.mbs.2008.02.008
7. Cai, Y., Kang, Y., Banerjee, M., Wang, W.: A stochastic SIRS epidemic model with infectious force under intervention strategies. J. Differ. Equ. **259**(12), 7463–7502 (2015). https://doi.org/10.1016/j.jde.2015.08.024
8. Cai, Y., Kang, Y., Banerjee, M., Wang, W.: A stochastic epidemic model incorporating media coverage. Commun. Math. Sci. **14**(4), 893–910 (2016). https://doi.org/10.4310/CMS.2016.v14.n4.a1
9. Black, F., Karasinski, P.: Bond and option pricing when short rates are lognormal. Financ. Anal. J. **47**(4), 52–59 (1991). https://doi.org/10.2469/faj.v47.n4.52
10. Ruiz-Moreno, D., Vargas, I.S., Olson, K.E., Harrington, L.C.: Modeling dynamic introduction of chikungunya virus in the United States. PLoS Negl. Trop. Dis. **6**(11), e1918 (2012). https://doi.org/10.1371/journal.pntd.0001918
11. Yakob, L., Clements, A.C.A.: A mathematical model of chikungunya dynamics and control: the major epidemic on Réunion Island. PLoS ONE **8**(3), e57448 (2013). https://doi.org/10.1371/journal.pone.0057448
12. Liu, X., Stechlinski, P.: Application of control strategies to a seasonal model of chikungunya disease. Appl. Math. Model. **39**(12), 3194–3220 (2015). https://doi.org/10.1016/j.apm.2014.10.035
13. Feng, X., Huo, X., Tang, B., Tang, S., Wang, K., Wu, J.: Modelling and analyzing virus mutation dynamics of chikungunya outbreaks. Sci. Rep. **9**(1), 2860 (2019). https://doi.org/10.1038/s41598-019-38792-4
14. Alade, T.O., Elaiw, A.M., Alsulami, S.M.: Stability dynamics of a delayed generalized Chikungunya virus infection model. J. Appl. Math. Comput. **65**(1), 575–595 (2021). https://doi.org/10.1007/s12190-020-01405-9
15. Wang, Y., Liu, X.: Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays. Math. Comput. Simul. **138**, 31–48 (2017). https://doi.org/10.1016/j.matcom.2016.12.011
16. Shi, Z., Jiang, D., Shi, N., Alsaedi, A.: Virus infection model under nonlinear perturbation: ergodic stationary distribution and extinction. J. Franklin Inst. **359**, 11039–11067 (2022). https://doi.org/10.1016/j.jfranklin.2022.03.035
17. Zhao, S., Yuan, S., Zhang, T.: The impact of environmental fluctuations on a plankton model with toxin-producing phytoplankton and patchy agglomeration. Chaos Solitons Fractals **162**, 112426 (2022). https://doi.org/10.1016/j.chaos.2022.112426
18. Lan, G., Song, B., Yuan, S.: Epidemic threshold and ergodicity of an SEIR model with vertical transmission under the telegraph noise. Chaos Solitons Fractals **167**, 113017 (2023). https://doi.org/10.1016/j.chaos.2022.113017
19. Shi, Z., Cao, Z., Jiang, D.: Host vector dynamics of a nonlinear pine wilt disease model in deterministic and stochastic environments. J. Franklin Inst. **360**, 5171–5210 (2023). https://doi.org/10.1016/j.jfranklin.2023.03.020
20. Wang, W., Cai, Y., Ding, Z., Gui, Z.: A stochastic differential equation SIS epidemic model incorporating Ornstein–Uhlenbeck process. Phys. A, Stat. Mech. Appl. **509**, 921–936 (2018). https://doi.org/10.1016/j.physa.2018.06.099
21. Tan, Y., Cai, Y., Wang, X., Peng, Z., Wang, K., Yao, R., Wang, W.: Stochastic dynamics of an SIS epidemiological model with media coverage. Math. Comput. Simul. **204**, 1–27 (2023). https://doi.org/10.1016/j.matcom.2022.08.001
22. Ma, Y., Yu, X.: The effect of environmental noise on threshold dynamics for a stochastic viral infection model with two modes of transmission and immune impairment. Chaos Solitons Fractals **134**, 109699 (2020). https://doi.org/10.1016/j.chaos.2020.109699
23. Wang, Y., Liu, J., Zhang, X., Heffernan, J.M.: An HIV stochastic model with cell-to-cell infection, B-cell immune response and distributed delay. J. Math. Biol. **86**(3), 35 (2023). https://doi.org/10.1007/s00285-022-01863-8
24. Gokila, C., Sambath, M.: The threshold for a stochastic within-host CHIKV virus model with saturated incidence rate. Int. J. Biomath. **14**(06), 2150042 (2021). https://doi.org/10.1142/S179352452150042X
25. Zhao, X., Liu, L., Wang, H., Fan, M.: Ecological effects of predator harvesting and environmental noises on Oceanic coral reefs. Bull. Math. Biol. **85**(7), 59 (2023)
26. Shi, Z., Jiang, D.: Dynamical behaviors of a stochastic HTLV-I infection model with general infection form and Ornstein–Uhlenbeck process. Chaos Solitons Fractals **165**, 112789 (2022). https://doi.org/10.1016/j.chaos.2022.112789
27. Allen, E.: Environmental variability and mean-reverting processes. Discrete Contin. Dyn. Syst., Ser. B **21**(7), 2073–2089 (2016). https://doi.org/10.3934/dcdsb.2016037

28. Du, N.H., Nguyen, D.H., Yin, G.G.: Conditions for permanence and ergodicity of certain stochastic predator–prey models. J. Appl. Probab. **53**(1), 187–202 (2016). https://doi.org/10.1017/jpr.2015.18
29. Meyn, S.P., Tweedie, R.L.: Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. Adv. Appl. Probab. **25**(3), 518–548 (1993). https://doi.org/10.2307/1427522
30. Dieu, N.T.: Asymptotic properties of a stochastic SIR epidemic model with Beddington–DeAngelis incidence rate. J. Dyn. Differ. Equ. **30**(1), 93–106 (2018). https://doi.org/10.1007/s10884-016-9532-8
31. Zhou, B., Jiang, D., Dai, Y., Hayat, T.: Threshold dynamics and probability density function of a stochastic avian influenza epidemic model with nonlinear incidence rate and psychological effect. J. Nonlinear Sci. **33**(2), 29 (2023). https://doi.org/10.1007/s00332-022-09885-8
32. Yang, Q., Zhang, X., Jiang, D.: Dynamical behaviors of a stochastic food chain system with Ornstein–Uhlenbeck process. J. Nonlinear Sci. **32**(3), 34 (2022). https://doi.org/10.1007/s00332-022-09796-8
33. Zhou, B., Han, B., Jiang, D., Hayat, T., Alsaedi, A.: Stationary distribution, extinction and probability density function of a stochastic vegetation–water model in arid ecosystems. J. Nonlinear Sci. **32**(3), 30 (2022). https://doi.org/10.1007/s00332-022-09789-7
34. Shi, Z., Jiang, D.: Environmental variability in a stochastic HIV infection model. Commun. Nonlinear Sci. Numer. Simul. **120**, 107201 (2023). https://doi.org/10.1016/j.cnsns.2023.107201
35. Roozen, H.: An asymptotic solution to a two-dimensional exit problem arising in population dynamics. SIAM J. Appl. Math. **49**(6), 1793–1810 (1989). https://doi.org/10.1137/0149110
36. Cai, Y., Jiao, J., Gui, Z., Liu, Y., Wang, W.: Environmental variability in a stochastic epidemic model. Appl. Math. Comput. **329**, 210–226 (2018). https://doi.org/10.1016/j.amc.2018.02.009
37. Tilahun, G.T., Alemneh, H.T.: Mathematical modeling and optimal control analysis of COVID-19 in Ethiopia. J. Interdiscip. Math. **24**(8), 2101–2120 (2021). https://doi.org/10.1080/09720502.2021.1874086
38. Higham, D.J.: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**(3), 525–546 (2001). https://doi.org/10.1137/S0036144500378302
39. Chen, Z., Zhang, R., Li, J., Zhang, S., Wei, C.: A stochastic nutrient-phytoplankton model with viral infection and Markov switching. Chaos Solitons Fractals **140**, 110109 (2020). https://doi.org/10.1016/j.chaos.2020.110109
40. Berrhazi, B.-E., El Fatini, M., Laaribi, A., Pettersson, R.: A stochastic viral infection model driven by Lévy noise. Chaos Solitons Fractals **114**, 446–452 (2018). https://doi.org/10.1016/j.chaos.2018.08.002
41. Borrow, P.: Innate immunity in acute HIV-1 infection. Curr. Opin. HIV AIDS **6**(5), 353–363 (2011)
42. Schmid, H., Dobrovolny, H.M.: An approximate solution of the interferon-dependent viral kinetics model of influenza. J. Theor. Biol. **498**, 110266 (2020)

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