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Dynamic analysis of differential advertising model based on single parameter sales promotion strategy

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Abstract

Advertising and promotion strategies are crucial marketing tools for increasing sales. In this paper, we primarily investigate the mathematical mechanisms for increasing sales by inducing a single-parameter sales promotion strategy into a differential advertising model. Based on the continuous model, we derive the discrete governing system for sales. By leveraging the existence, stability, and bifurcation and chaotic behavior of fixed points of the discrete system, we have elucidated the dynamic behavior of sales in the continuous model. The specific parameter ranges for the existence of a T-period solution and its stability conditions are given. Furthermore, we perform a flip bifurcation analysis of the positive fixed point. This analysis helps us to obtain the existence and stability conditions for nT-period solutions. Interestingly, for the same model, when we take different parameter values, flip bifurcation and inverse flip bifurcation can coexist. The bifurcation provides a route to chaos. In the simulations, we find that in some situations, there exists a pathway for the system to enter into chaos from a stable state through flip bifurcation, and then enter into a stable state through inverse flip bifurcation, while in other situations, there exists no such pathway. We propose an effective control strategy that serves to suppress flip bifurcation and promote inverse flip bifurcation to eliminate chaos. These findings have significant theoretical implications and practical applications in relevant markets.

Keywords: Single parameter sales promotion strategy; Flip bifurcation; Inverse flip bifurcation; Chaos; Control strategy

1 Introduction

With the rapid development of the commodity economy and the proliferation of mass media, advertising has become more prevalent than ever [1]. Advertising is an indispensable tool in the competitive strategies of enterprises. It plays an important role in market expansion and the generation of economic benefits. Numerous studies on consumer behavior have shown that consumers' acceptance of advertising information significantly influences their responses to products. One of the primary challenges any company must address when implementing its advertising strategy is determining the optimal allocation of advertising expenditures over the planning horizon within a constrained budget [2].

Over the past few decades, research based on advertising models has been steadily growing and evolving. Scholars have continuously developed numerous models to examine the influences of advertising on product sales from various perspectives.

For instance, Vidale and Wolfe [3] conducted numerous controlled experiments. They employed a linear differential equation to analyze the influence of advertising on product sales. In [4], the author extended the V-W model to a duopoly. Wang et al. [5] introduced a bipolar model of dynamic competitive advertising, which outlined the differential game model for competitive advertising decision-making concerning non-durable products. Bass [6] categorized purchasers into innovators (the initial group of purchasers) and imitators (those who follow suit due to certain factors), and developed a quadratic differential equation based on the infectious disease model in their mathematical framework. In [7], scholars utilized the diffusion model to investigate optimal advertising strategies following the introduction and diffusion of a new product. The advertising capital model [8] was used to explore the interconnected effects of three advertising oscillator models. The SEM-ANN model [9] was used to explore the impact of advertising on consumer purchasing behavior by analyzing consumers' trust in advertising. In [10], Chenavaz et al. investigated the interplay between price, advertising, and quality in an optimal control model. For more advertising models, one can refer to the references [11–13] and the references therein.

Though numerous scholars have developed many advertising-based models, a majority of these models rely on linear differential equations. This is mainly because linear differential equations are easy to analyze to some extent. There is a scarcity of studies focusing on the creation of more intricate differential equations to comprehensively depict and analyze the influences of advertising and promotion on product sales, as well as their practical applications.

As an essential component of a company's marketing communication strategy, promotional activities have the potential to promptly increase sales [14, 15]. Currently, research on promotion strategies primarily focuses on their applications within supply chain-related cooperation models [16–20]. These models optimize advertising and promotional strategies throughout the entire sales process, from manufacturers and sellers to consumers. Competitive models related to promotion strategies are also derived within this cooperative framework. Furthermore, a linear differential model for export commodities has been developed. This model, with promotion cost as the control variable, identified the optimal promotion strategy across various stages of the export commodity life cycle and established a predictive demand model for export commodities at specific time [21].

Advertising and promotion represent two distinct marketing communication tools whose dissemination significantly influences product sales, thereby enabling companies to achieve their target market share [22]. In [23], Bandyopadhyay et al. investigated whether various types of sales promotions together with hedonic shopping motivation (value shopping) and positive effects drive impulse buying. Although sales promotion strategy is very important and popular in the real market [24, 25], there have been limited studies on the immediate impact of promotions on sales levels. Jiang et al. [26] established a linear advertising competition model incorporating a two-parameter promotion strategy. However, this two-parameter promotion strategy fails to guarantee consistent positive changes in sales.

It is worth noting that linear models have some shortcomings, the most significant of which is their inability to describe the complexity of advertising's impacts on sales. The relationship between advertising and sales is very complex. Initial advertising investments can lead to a significant increase in sales. However, as market saturation increases, the incremental growth in sales gradually decreases. Moreover, consumers' responses to advertising are influenced by a variety of factors, such as product characteristics, advertisement content, and consumer preferences. There may be nonlinear relationships among these factors. Consumers might become weary of repeated advertisements, thereby reducing their willingness to purchase the product. All of these factors indicate that the impact of advertising on sales is not a simple linear relationship and cannot be adequately described by a straightforward linear model. Nonlinear models, on the other hand, are more capable of uncovering the complexity of advertising's impact on sales. At present, some nonlinear models are available. However, the existing models rarely study periodic solutions and complex bifurcation phenomena. In fact, when advertising expenditure surpasses a certain threshold, there may be a significant increase in sales levels. The sales of some products may also exhibit periodic fluctuations, which can be explained by the bifurcation behavior of nonlinear models. Moreover, when the market is subject to minor disturbances, the sales of products can become unpredictable, a phenomenon that may be explained by the chaotic behavior of the system. In [27], Ma et al. introduced a quadratic term to describe the advertising model within a promotional strategy framework. However, due to the complexity of changes in actual sales levels, it is necessary to introduce a higher-order term to accurately describe the advertising model.

From the literature mentioned above, we find that there is a scarcity of discussions regarding the periodic solution and its bifurcation within the extended nonlinear V-W model, particularly concerning promotions within a differential model. Consequently, this study aims to address this gap by integrating the promotion strategy and cubic nonlinear term into the differential advertising model. We focus on exploring the complex dynamics of the model and understanding the influence mechanism of the promotion strategy on sales.

The structure of this paper is outlined as follows. In the second section, we introduce the cubic term and incorporate the promotion strategy into the V-M model to derive the differential advertising model based on a single-parameter promotion strategy. In the third section, we delve into the examination of the existence and stability of periodic solutions of this model. Via utilizing numerical simulations, we illustrate periodic solutions and bifurcation diagrams generated by the system. Furthermore, we explore the stability of the solutions under the other four different parameter sets and propose a control strategy by utilizing the promotion coefficient. In the forth section, we propose a promotional control strategy that realizes the elimination of chaos by suppressing flip bifurcation and promoting inverse flip bifurcation. In the fifth section, we provide a summary and discussion of our findings.

2 Model description

Vidale and Wolfe established a linear differential model [3]:

$$\dot{S(t)} = \frac{r}{M}u(t)(M - S(t)) - \lambda S(t), \tag{1}$$

to analyze the impact of advertising on the company's product sales. Here, $S(t)$ represents the current sales level at time t, M denotes the potential market size and market saturation level, $u(t)$ stands for the advertising expenditure at the moment t, r means the advertising response rate, and λ represents the sales decay constant. Bass established a growth model for the initial purchase time of new products [6]:

$$\dot{S(t)} = (p + qS(t))(M - S(t)). \tag{2}$$

Among them, $p > 0$ and $q > 0$ are innovation and imitation parameters. M is the fixed market potential of the product.

The model (1) only considers the linear term; given the complexity of the impact of advertising on sales, it becomes necessary to employ nonlinear equations to describe the changes in product sales. Motivated by the nonlinear model (2), we use $\frac{r}{M}u(t)S(t)^2(M - S(t))$ to replace the effect of advertising $\frac{r}{M}u(t)(M - S(t))$, so the model (1) can be written as follows:

$$\dot{S(t)} = \frac{r}{M}u(t)S(t)^2(M - S(t)) - \rho S(t). \tag{3}$$

For some products, estimating their potential market size and market saturation level M proves challenging, as they are influenced by various factors. Hence, we do not consider the potential market size and market saturation level M [27]. Since $\frac{M-S(t)}{M} = 1 - \frac{S(t)}{M}$ represents the proportion of remaining market demand in the total demand, which is denoted by b. This means that the market's remaining demand generated per unit of sales is b. Now the sales level is $S(t)$, and therefore, the remaining market size is $bS(t)$. Additionally, we assume that the company's advertising expenditure remains constant, namely, $u(t) = U$. Therefore, the advertising effect $\frac{r}{M}u(t)S(t)^2(M - S(t))$ in (3) can be replaced with $rUbS(t)^3$. Let: $m = rUb$, model (3) can be written as follows:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\rho S(t) + mS(t)^3, \tag{4}$$

where $S(t)$ is the sales level at the moment t, ρ is the sales decay constant, m is the response rate to advertising.

Model (4) also can be written as

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = mS(t)^3[1 - \frac{\rho}{mS(t)^2}]. \tag{5}$$

If $S(t)^2 > \frac{\rho}{m}$, then $\frac{\mathrm{d}S(t)}{\mathrm{d}t} > 0$, $S(t)$ is an increasing function of time t. Conversely, if $S(t)^2 < \frac{\rho}{m}$, then $\frac{\mathrm{d}S(t)}{\mathrm{d}t} < 0$, $S(t)$ is a decreasing function of time t. Therefore, it can be inferred that $\frac{\rho}{m}$ represents a critical threshold. When the sales level deceases over time, it becomes necessary to introduce a promotion strategy to rapidly enhance the sales level. Thus, in this paper, we mainly discuss strategies to increase product sales when $S(t)^2 < \frac{\rho}{m}$.

There are many strategies for promotion. In [26], the authors used the impulsive promotion strategy $\Delta S(t) = S(t^+) - S(t) = (b - cS(t))S(t)$. The shortcoming is that it can not guarantee the value of $\Delta S(t)$ is always greater than 0. To avoid the emergence of negative values of $\Delta S(t)$, in our paper, the selection of promotion strategies primarily draws on reference [27]:

$$\Delta S(t) = S(t^+) - S(t) = \frac{\alpha}{S(t)},\ t = nT,\ n = 1, 2, 3, \ldots, \tag{6}$$

where $\alpha > 0$ represents the promotion coefficient, and $S(t^+) = \lim_{\varepsilon \to 0^+} S(t+\varepsilon)$. Equation (6) is called a single-parameter sales promotion strategy, which indicates that the promotion strategies are introduced at $t = nT$. At this time, sales level $S(t)$ increases from $S(nT)$ to $S(nT^+)$, where $S(nT^+) = S(nT) + \frac{\alpha}{S(nT)}$, and $\frac{\alpha}{S(nT)}$ is the increment of sales. Unlike literature [26], the promotion strategy (6) ensures that the increment of sales $\Delta S(t)$ is always greater than zero.

According to the advertising model (4) and promotion strategy (6), the following differential advertising model based on single parameter sales promotion strategy can be established:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\rho S(t) + mS(t)^3,\ t \neq nT, \tag{7a}$$

$$\Delta S(t) = \frac{\alpha}{S(t)},\ t = nT, \tag{7b}$$

where $n = 1, 2, 3, \ldots$.

Further, we analyze the related dynamical behavior of model (7a)–(7b). The dynamical behavior of the model (7a)–(7b) consists of two stages: a dynamical evolution over time (govern by (7a)) and the influence of the promotion strategy (govern by (7b)). In Fig. 1, we present a sketch to explain the mechanism of the promotion strategy on a solution of model (7a)–(7b). The solution starting from the initial point $(0, S(0))$ reaches the point $(T, S(T))$ at $t = T$ under the control of (7a), then jumps to the point $(T, S(T^+))$ under the promotion strategy (7b). The point $(T, S(T^+))$ serves as the new initial point, and the trajectory starting from it reaches the point $(2T, S(2T))$ under the control of (7a) at $t = 2T$. Under the promotion strategy (7b), the trajectory then jumps to the point $(2T, S(2T^+))$ and so on. According to (7a) and the initial condition $(0, S(0))$, we obtain the solution of

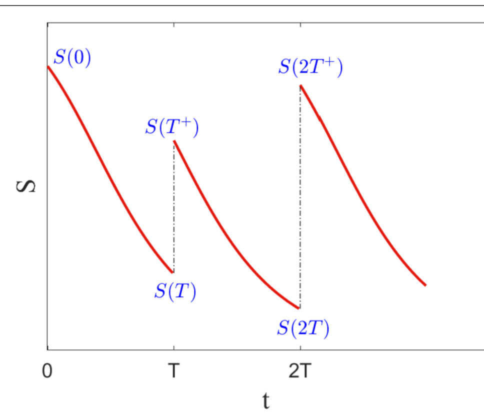

Figure 1 The mechanism of the promotion strategy on a solution of model (7a)–(7b)

(7a):

$$S(t)=\sqrt{\frac{\rho S(0)^2}{mS(0)^2+[\rho-mS(0)^2]e^{2\rho t}}},\ 0\le t\le T.$$

Obviously, with the increase in t, the sales level $S(t)$ decreases. When $t = T$, the sales level decreases to $S(T)$. After introducing the promotion strategy (7b), the sales level $S(t)$ rapidly increases to $(T, S(T^+))$, where $S(T^+)$ satisfies:

$$S(T^+)=S(T)+\frac{\alpha}{S(T)}.$$

When $t\in(T,2T]$, the solution $S(t)$ of the system (7a)–(7b) satisfies:

$$\begin{cases} S(t)=\sqrt{\dfrac{\rho S(T^+)^2}{mS(T^+)^2+[\rho-mS(T^+)^2]e^{2\rho(t-T)}}}, & T<t\le 2T,\\ S(2T^+)=S(2T)+\dfrac{\alpha}{S(2T)}. \end{cases}$$

Similarly, for any $t\in(kT,(k+1)T]$, the solution of (7a)–(7b) can be described as follows:

$$\begin{cases} S(t)=\sqrt{\dfrac{\rho S((k-1)T^+)^2}{mS((k-1)T^+)^2+[\rho-mS((k-1)T^+)^2]e^{2\rho(t-(k-1)T)}}}, & (k-1)T<t\le kT,\\ S(kT^+)=S(kT)+\dfrac{\alpha}{S(kT)}, \end{cases}$$

where $k = 0, 1, 2, \ldots$.

3 Dynamical analysis of system (7a)–(7b)

Product sales fluctuate over time, and certain product sales may exhibit periodic behavior. To better grasp the periodic behavior of product sales, we mainly discuss the dynamical behavior of the system (7a)–(7b) in this section.

3.1 Existence of *T*-period solution

Suppose the solution of the system (7a)–(7b) arrives at the point (kT, S_k) at the moment $t = kT$, where

$$S_k = S(kT). \tag{8}$$

Due to the effect of sales promotion, the solution jumps to the point (kT, S_k^+) at the moment $t = kT^+$, where

$$S_k^+ = S(kT^+) = S_k + \frac{\alpha}{S_k}, \tag{9}$$

please see Fig. 2. At the moment $t = (k+1)T$, the solution $S(t)$ reaches to the point $((k+1)T, S_{k+1})$, where

$$S_{k+1}=S((k+1)T)=\sqrt{\frac{\rho S(kT^+)^2}{\rho e^{2\rho T}+mS(kT^+)^2(1-e^{2\rho T})}}. \tag{10}$$

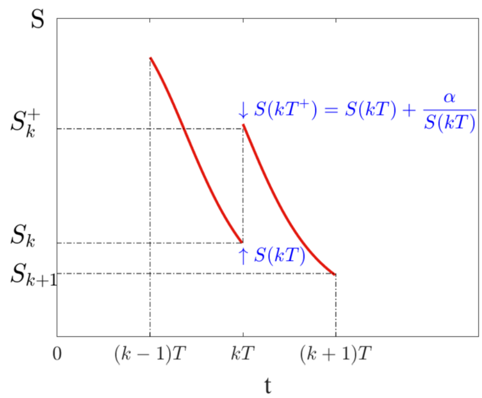

Figure 2 Promotion strategy leads to a sudden increase in sales

Substituting (9) into (10), the solution of (7a)–(7b) satisfies the following discrete system:

$$S_{k+1} = \sqrt{\frac{\rho(S_k + \frac{\alpha}{S_k})^2}{\rho e^{2\rho T} + m(S_k + \frac{\alpha}{S_k})^2(1 - e^{2\rho T})}}. \tag{11}$$

This means that the fixed point of the discrete system (11) is a T-period solution of (7a)–(7b). For the convenience of calculations, we set

$$T = \frac{1}{2\rho} \ln(1 + \frac{\rho}{m}), \tag{12}$$

namely,

$$e^{2\rho T} = 1 + \frac{\rho}{m}. \tag{13}$$

Substituting (12) into (11), we arrive at

$$S_{k+1} = \sqrt{\frac{(S_k + \frac{\alpha}{S_k})^2}{1 + \frac{\rho}{m} - (S_k + \frac{\alpha}{S_k})^2}} = f(\alpha, \rho, m, S_k). \tag{14}$$

The fixed point of (14) corresponds to a T-period solution of the system (7a)–(7b), and they exhibit the same stability. To explore the existence and stability of T-period solutions of the system (7a)–(7b), we investigate the existence and stability of fixed points of the system (14).

The fixed point S of the system (14) can be obtained by solving $S_k = S_{k+1}$, which indicates that S satisfies

$$S = \sqrt{\frac{(S + \frac{\alpha}{S})^2}{1 + \frac{\rho}{m} - (S + \frac{\alpha}{S})^2}},$$

namely,

$$S^6 + (2\alpha - \frac{\rho}{m})S^4 + (\alpha^2 + 2\alpha)S^2 + \alpha^2 = 0. \tag{15}$$

Since we do not want the sales to be zero, we only consider positive fixed points, and therefore, $S > 0$. Let $S^2 = y$, then we have

$$y^3 + (2\alpha - \frac{\rho}{m})y^2 + (\alpha^2 + 2\alpha)y + \alpha^2 = 0.$$

Let

$$h(y) = y^3 + (2\alpha - \frac{\rho}{m})y^2 + (\alpha^2 + 2\alpha)y + \alpha^2, \; y \in [0, +\infty). \tag{16}$$

Then, the existence of a positive zero point of $h(y)$ means the existence of a positive fixed point S. In the Appendix, we provide detailed information about the parameter conditions for the existence of positive zero points. Therefore, we have the following conclusion about the existence of fixed points of the system (14):

Lemma 1 *If* $0 < \rho < \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the system* (14) *has no positive fixed point. If* $\rho = \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the system* (14) *has only one positive fixed point. If* $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the system* (14) *has two different positive fixed points* S_1 *and* S_2, *which are given by* (A.4) *and* (A.5) *in the Appendix.*

From the relationship between the solution of the system (7a)–(7b) and the fixed point of the system (14), the following conclusions can be obtained:

Theorem 1 *If* $0 < \rho < \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *there is no T-period solution for the system* (7a)–(7b). *If* $\rho = \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the system* (7a)–(7b) *has only one unique T-period solution. If* $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the system* (7a)–(7b) *has the following two T-period solutions*:

$$S_1(t) = \sqrt{\frac{\rho(S_1 + \frac{\alpha}{S_1})^2}{m(S_1 + \frac{\alpha}{S_1})^2 + [\rho - m(S_1 + \frac{\alpha}{S_1})^2]e^{2\rho(t-kT)}}}, \; kT < t \le (k+1)T, \tag{17}$$

and

$$S_2(t) = \sqrt{\frac{\rho(S_2 + \frac{\alpha}{S_2})^2}{m(S_2 + \frac{\alpha}{S_2})^2 + [\rho - m(S_2 + \frac{\alpha}{S_2})^2]e^{2\rho(t-kT)}}}, \; kT < t \le (k+1)T, \tag{18}$$

corresponding to the two fixed points S_1 *and* S_2 *of the system* (14), *respectively.*

In this paper, we only consider the case when the system (14) has two fixed points. In this situation, we suppose that $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$ holds. Next, we discuss the stability of the two positive fixed points to obtain the stability conditions of the two T-period solutions $S_1(t)$ and $S_2(t)$ of system (7a)–(7b).

3.2 Stability of the two *T*-period solutions

The eigenvalues of the fixed point $S_i (i = 1, 2)$ is

$$\lambda_i = \frac{\mathrm{d}f(\alpha,\rho,m,S_i)}{\mathrm{d}S_i} = \frac{(1+\frac{\rho}{m})S_i(S_i^4-\alpha^2)}{(S_i^4+2\alpha S_i^2+\alpha^2-S_i^2-\frac{\rho}{m}S_i^2)^2\sqrt{\frac{(S_i^2+\alpha^2)^2}{-(S_i^4+2\alpha S_i^2+\alpha^2-S_i^2-\frac{\rho}{m}S_i^2)}}}.$$

From equation (15), we can get

$$\lambda_i = \frac{(1+\frac{\rho}{m})y_i^2(y_i-\alpha)}{(y_i+\alpha)^3}. \tag{19}$$

According to [28], we can get the following lemma for determining the stability of the fixed points.

Lemma 2 *Suppose that* $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$ *holds, the fixed point* S_i *of the system* (14) *is stable if the corresponding eigenvalue* $|\lambda_i| < 1$, *and it is unstable if* $|\lambda_i| > 1$, *where* $i = 1, 2$.

The stability of the fixed point $S_i (i = 1, 2)$ of the system (14) can be used to determine the stability of the solution $S_i(t)(i = 1, 2)$ of the system (7a)–(7b), so we arrive at the following conclusions.

Theorem 2 *Suppose that* $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, *the T-period solution* $S_i(t)(i = 1, 2)$ *of the system* (7a)–(7b) *is stable if the corresponding eigenvalue* $|\lambda_i| < 1$; *and it is unstable if* $|\lambda_i| > 1$, *where* $i = 1, 2$.

It is easy to verify that $\frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$ is an increasing function with $\alpha > 0$. Let $m = 0.1$, for $\alpha \in (0, 9.3326]$, the maximum of $\frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$ is 3.99997. Let $\rho = 4.0$, so $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$ always holds. At this time, we can get $T = 0.4642$. From Lemma 1, the system (14) has two positive fixed points S_1 and S_2, as shown in Fig. 3. Correspondingly, the system (7a)–(7b) has two T-period solutions $S_1(t)$ and $S_2(t)$.

For the stability of the two T-period solutions $S_i(t)$, $(i = 1, 2)$, we consider the stability of the two fixed points S_i. When $m = 0.1$, $\rho = 4.0$, we plot the images of the eigenvalues λ_1 and λ_2 of the positive fixed points S_1 and S_2 in Fig. 4. From Fig. 4(a), we find that when $\alpha \in (2.6804, 8.7862)$, we have $|\lambda_1| > 1$, and thus the fixed point S_1 is unstable, so the corresponding T-period solution $S_1(t)$ is unstable. When $\alpha \in [2, 2.6804) \cup (8.7862, 9.3326]$, we have $|\lambda_1| < 1$, thus S_1 is stable, so the corresponding T-periodic solution $S_1(t)$ is stable. From Fig. 4(b), we find that $|\lambda_2| > 1$ always holds for $\alpha \in [2, 9.3326]$, and therefore, the fixed point S_2 is always unstable, thus the T-period solution $S_2(t)$ of the corresponding system (7a)–(7b) is always unstable.

Let $m = 0.1$, $\rho = 4.0$, $\alpha = 2.0$, then $T = 0.4642$. Using equations (A.2)–(A.5), we have $S_1 \approx 0.6830$, $S_2 \approx 5.9811$, so $S_1 + \frac{\alpha}{S_1} \approx 3.6113$, $S_2 + \frac{\alpha}{S_2} \approx 6.3155$. According to (17) and (18), when $\alpha = 2.0$, the system (7a)–(7b) has two T-period solutions:

$$S_1(t) = \sqrt{\frac{40 \times 3.6113^2}{3.6113^2 + (40 - 3.6113^2)e^{8.0\times(t-kT)}}},\ kT < t \le (k+1)T, \tag{20}$$

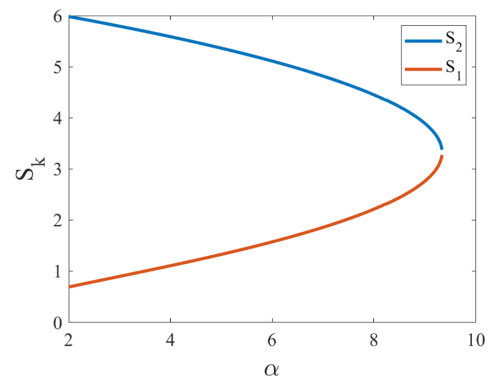

Figure 3 Two fixed points S_1 and S_2 of the system (14) when $m = 0.1$, $\rho = 4.0$

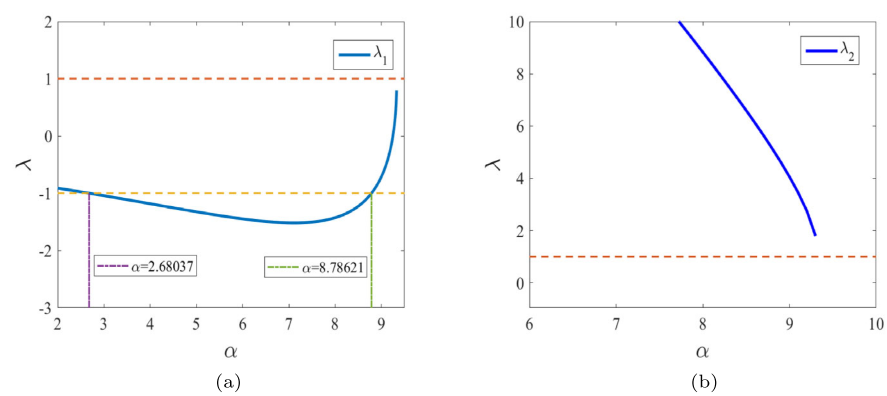

Figure 4 The images of the two eigenvalues of the two fixed points with $m = 0.1$, $\rho = 4.0$. (a) The eigenvalue λ_1 of the fixed point S_1; (b) the eigenvalue λ_2 of the fixed point S_2

and

$$S_2(t) = \sqrt{\frac{40 \times 6.3155^2}{6.3155^2 + (40 - 6.3155^2)e^{8.0\times(t-kT)}}}, \; kT < t \leq (k+1)T. \tag{21}$$

The stability of the two T-period solutions $S_1(t)$ and $S_2(t)$ are shown in Fig. 5 and Fig. 6, respectively.

In Fig. 5, the solution $S(t)$ (red line) starting from the initial point $(0, 3.9)$ converges to $S_1(t)$ (blue line) as time increases. It indicates that the T-period solution $S_1(t)$ is stable.

In Fig. 6, the solution $S(t)$ (red line) starting from the initial point $(0, 6.3)$ goes far away from the T-period solution $S_2(t)$ (blue line). Therefore, the T-period solution $S_2(t)$ is unstable.

3.3 Flip bifurcation analysis

Let $m = 0.1$, $\rho = 4.0$, so it can be seen from Fig. 4 that when $\alpha = 2.6804$ and $\alpha = 8.7862$, the eigenvalue λ_1 of the system (14) at the fixed point S_1 is -1. According to [28], the system (14) may undergo flip bifurcation or inverse flip bifurcation. At this time, a period-2 orbit can be bifurcated from the fixed point S_1. Corresponding to the period-2 orbit, there exists a 2T-period solution of the system (7a)–(7b).

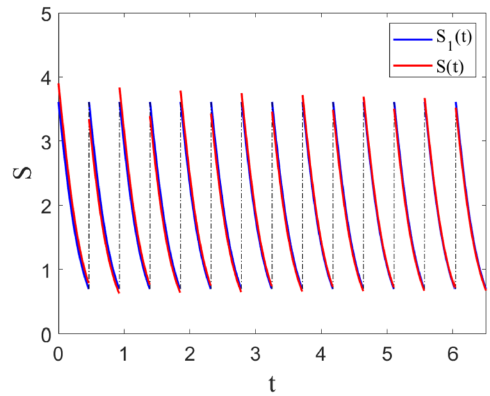

Figure 5 The stable T-period solution $S_1(t)$

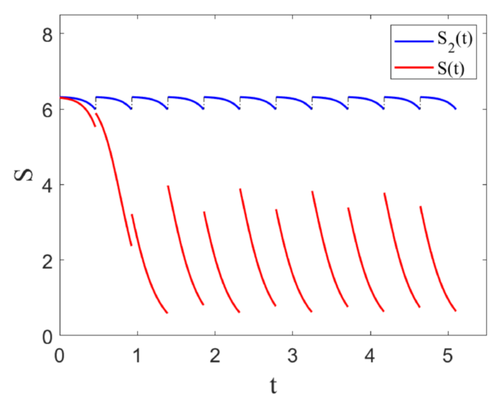

Figure 6 The unstable T-period solution $S_2(t)$

In the next, we will investigate the existence and stability of the 2T-period solution of the system (7a)–(7b) by carrying out a bifurcation analysis of the fixed point S_1 for the system (14) in the case of $m = 0.1$, $\rho = 4.0$, and α is around 2.6804.

Lemma 3 in [29] is instrumental in determining the stability and bifurcation direction of the period-2 orbit of the system (14).

Lemma 3 *Let $f_\mu : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of map such that f_{μ_0} has a fixed point x_0 with eigenvalue -1. Assume the following conditions*:

$(C1)$ $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial^2 x} + 2\frac{\partial^2 f}{\partial x \partial \mu} \neq 0$, *at* (x_0, μ_0);

$(C2)$ $g(x, \mu) = \frac{1}{2}(\frac{\partial^2 f}{\partial^2 x})^2 + \frac{1}{3}\frac{\partial^3 f}{\partial^3 x} \neq 0$, *at* (x_0, μ_0).

Then, there is a smooth curve of fixed points of f_μ passing through (x_0, μ_0), the stability of which changes at (x_0, μ_0). There is also a smooth curve γ passing through (x_0, μ_0) so that $\gamma \setminus (x_0, \mu_0)$ is a union of hyperbolic period-2 orbits.

When $m = 0.1$, $\rho = 4.0$, $\alpha = 2.6804$, then $S_1 \approx 0.8248$, and $\frac{\partial f(\alpha,S_k)}{\partial \alpha}\frac{\partial^2 f(\alpha,S_k)}{\partial^2 S_k} + 2\frac{\partial^2 f(\alpha,S_k)}{\partial S_k \partial \alpha}|_{(0.8248,2.6804)} = -0.2669 \neq 0$, so condition $(C1)$ is satisfied. In condition $(C2)$, the sign of $g(x_0, \mu_0)$ determines the stability and bifurcation direction of the period-2 orbit. If $g(x_0, \mu_0)$ is positive, the period-2 orbit is stable. If $g(x_0, \mu_0)$ is negative, the period-2 orbit

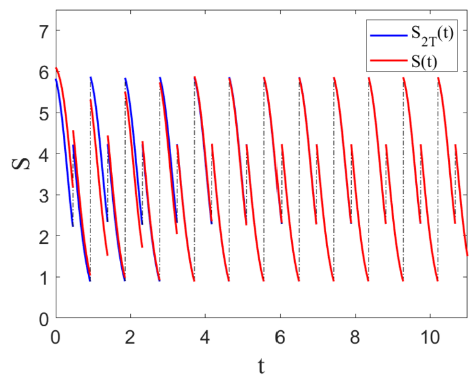

Figure 7 The stable 2T-period solution $S_{2T}(t)$ of the system (7a)–(7b)

is unstable. In our example, $g(S_1,\alpha) = 0.8656 > 0$. Therefore, the system (14) will experience flip bifurcation at the fixed point S_1 with $\alpha = 2.6804$, and the bifurcated period-2 orbit is stable. Furthermore, the system (7a)–(7b) has a stable 2T-period solution for $\alpha \in (2.6804, 2.6804 + \varepsilon)$, $\varepsilon > 0$.

To obtain the expression of the 2T-period solution $S_{2T}(t)$ of model (7a)–(7b), we consider the following quadratic iterative map:

$$S_{k+1} = \frac{\frac{S_k+\frac{\alpha}{S_k}}{\sqrt{1+\frac{\rho}{m}-(S_k+\frac{\alpha}{S_k})^2}} + \frac{\alpha\sqrt{1+\frac{\rho}{m}-(S_k+\frac{\alpha}{S_k})^2}}{S_k+\frac{\alpha}{S_k}}}{\sqrt{1+\frac{\rho}{m}-\left[\frac{S_k+\frac{\alpha}{S_k}}{\sqrt{1+\frac{\rho}{m}-(S_k+\frac{\alpha}{S_k})^2}} + \frac{\alpha\sqrt{1+\frac{\rho}{m}-(S_k+\frac{\alpha}{S_k})^2}}{S_k+\frac{\alpha}{S_k}}\right]^2}} = f^2(\alpha,\rho,m,S_k). \tag{22}$$

The system (22) has four positive fixed points. Two of them satisfy $f(\alpha,\rho,m,S_i) = S_j$, $f(\alpha,\rho,m,S_j) = S_i$ and $S_i \neq S_j$. These two fixed points are denoted as S_{11} and S_{12}. They are actually the stable period-2 points of the system (14) for $\alpha \in (2.6804, 2.6804+\varepsilon)$. Therefore, the system (7a)–(7b) has a stable 2T-period solution $S_{2T}(t)$:

$$S_{2T}(t) = \begin{cases} \frac{\rho(S_{11}+\frac{\alpha}{S_{11}})}{\sqrt{m(S_{11}+\frac{\alpha}{S_{11}})^2+[\rho-m(S_{11}+\frac{\alpha}{S_{11}})^2]e^{2\rho(t-kT)}}}, & t \in (kT,(k+1)T], \\ \frac{\rho(S_{12}+\frac{\alpha}{S_{12}})}{\sqrt{m(S_{12}+\frac{\alpha}{S_{12}})^2+[\rho-m(S_{12}+\frac{\alpha}{S_{12}})^2]e^{2\rho(t-(k+1)T)}}}, & t \in ((k+1)T,(k+2)T]. \end{cases} \tag{23}$$

Let $\alpha = 4.4382$, then $S_{11} = 0.8849$, $S_{12} = 2.3728$. The blue line in Fig. 7 is the 2T-period solution $S_{2T}(t)$. The red line in Fig. 7 shows the solution $S(t)$ starting from the initial point $(0, 6.1)$. It can be seen that with the change of time, the solution $S(t)$ of the system (7a)–(7b) gradually tends to the 2T-period solution $S_{2T}(t)$. It indicates that $S_{2T}(t)$ is stable in this case.

Figure 8 shows the flip bifurcation of the fixed point S_1 of the system (14) when $\rho = 4.0$, $m = 0.1$, $\alpha \in [2, 9.3326]$. According to Fig. 8, the flip bifurcation occurs at $\alpha = 2.6804$, and the 2T-period solution is generated. When $\alpha \in (2.6804, 4.4198)$, the 2T-period solution $S_{2T}(t)$ is stable. The stable 2T-period solution $S_{2T}(t)$ can also undergo flip bifurcation when $\alpha = 4.4198$, and the 4T-period solution can be bifurcated. As the increase of α, the

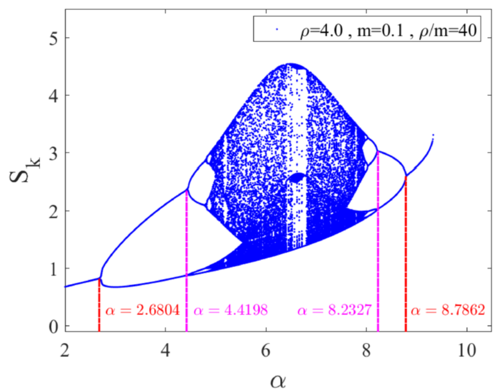

Figure 8 The bifurcation diagram of the system (14) when $\rho = 4.0$, $m = 0.1$

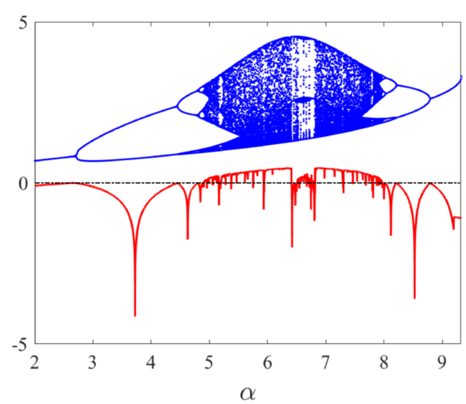

Figure 9 The diagram of the maximum Lyapunov exponent corresponding to the bifurcation diagram in Fig. 8

solution may enter into chaos. While for enough large α, we observe that an inverse flip bifurcation occurs when $\alpha = 8.7862$. There also exists a stable 2T-period solution $S_{2T}(t)$ for $\alpha \in (8.2327, 8.7862)$.

To distinguish the chaos and regular dynamical behavior, we draw the maximum Lyapunov exponent corresponding to the bifurcation diagram, see Fig. 9. It can be seen that there are some cases where the maximum Lyapunov exponent is greater than 0, so the system (14) enters into chaos for some α. From Fig. 8 and Fig. 9, we find that there exists a pathway for the solution to move from a stable state into chaos and then back to a stable state.

3.4 Numerical simulations under different parameters values

In Sect. 3.3, we selected a set of values for m and ρ, and used α as the main bifurcation parameter to numerically explain the existence of flip bifurcation and inverse flip bifurcation. In this part, we aim to investigate numerically whether the system's dynamic behavior shows significant differences with varying values of m and ρ. Through extensive numerical simulations, we found that the system exhibits five distinct dynamic behaviors under different parameter values. One of these scenarios has already been presented in Sect. 3.3. Next, we will present the other four scenarios. The four sets of values for m and ρ and the corresponding ranges of α are shown in Table 1. We will discuss numerically the existence,

Table 1 Four different cases with different ρ and m

Case	ρ	m	$\frac{\rho}{m}$	the range of α	$(\frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3}))_{max}$
Case I	9.0	0.36	25	[0.5, 5.6]	8.9697
Case II	5.0	0.175	28.5714	[2.5,6.5]	4.9988
Case III	18.0	0.5	36	[2,8]	17.3137
Case IV	23.71	0.58	40.8793	[2,9.3]	23.1237

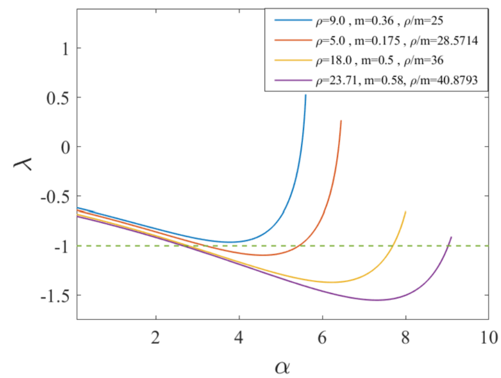

Figure 10 The graphs of eigenvalues λ_1 under the four cases

stability, and bifurcation of the solution of the system (7a)–(7b). This paper mainly uses Matlab 2021 for numerical simulations.

From Table 1, we know that $\rho > (\frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3}))_{max}$ always holds. According to Lemma 1, the system (14) has two different positive fixed points S_1 and S_2 in the four cases, which are given by (A.4) and (A.5), respectively.

In Fig. 10, we draw the eigenvalues λ_1 for the four cases. It can be seen from Fig. 10 that $|\lambda_1| < 1$ of the system (14) always holds under Case I (blue line). It indicates that the T-period solution $S_1(t)$ of the system (7a)–(7b) is stable. While in the other three cases, the modulus of eigenvalue λ_1 is larger than 1 for some α (red, orange and purple lines), which means that the T-period solution $S_1(t)$ of the system (7a)–(7b) has different stability as α changes.

In Fig. 11, we draw the eigenvalues λ_2 for the four cases. It can be seen from Fig. 11 that $|\lambda_2| > 1$ of the system (14) is always valid under all four cases. It means that the fixed point S_2 is always unstable, thus the T-period solution $S_2(t)$ of the corresponding system (7a)–(7b) is always unstable. So for the rest of this section, we will not discuss S_2 anymore.

For $\rho = 9.0$, $m = 0.36$, then we can get $\frac{\rho}{m} = 25$. When $\alpha \in [0.5, 5.6]$, the eigenvalue of the system (14) has $|\lambda_1| < 1$, see the blue line in Fig. 10. Therefore, for $\alpha \in [0.5, 5.6]$, the fixed point S_1 is always stable, which means that system (7a)–(7b) has only one stable T-period solution under the Case I, as shown in Fig. 12. In this case, no bifurcation will occur. The dynamical behavior of the system is relatively simple in this case.

For $\rho = 5.0$, $m = 0.175$, then we get $\frac{\rho}{m} = 28.5714$ and $T = 0.3387$. For $\alpha \in [2.5, 3.2007) \cup (5.4346, 6.5]$, the modulus of eigenvalue λ_1 is smaller than 1. For $\alpha \in (3.2007, 5.4346)$, the modulus of eigenvalue λ_1 is larger than 1, see the red line in Fig. 10. The bifurcation diagram of the fixed point S_1 under this case is shown in Fig. 13. The system (7a)–(7b) has a stable T-period solution for $\alpha \in [2.5, 3.2007) \cup (5.4346, 6.5]$, and a stable 2T-period solution for $\alpha \in (3.2007, 5.4346)$. Let $\alpha = 4$, the trajectory $S(t)$ starting from the initial value

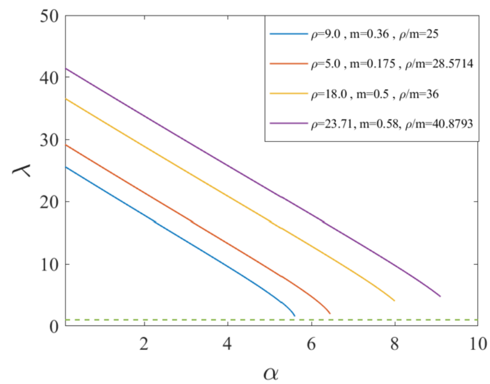

Figure 11 The graphs of eigenvalues λ_2 under the four cases

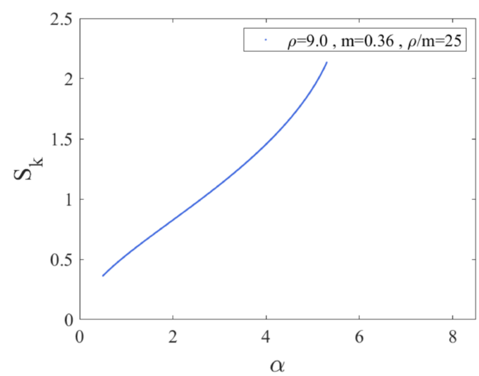

Figure 12 The stable fixed point S_1 under Case I

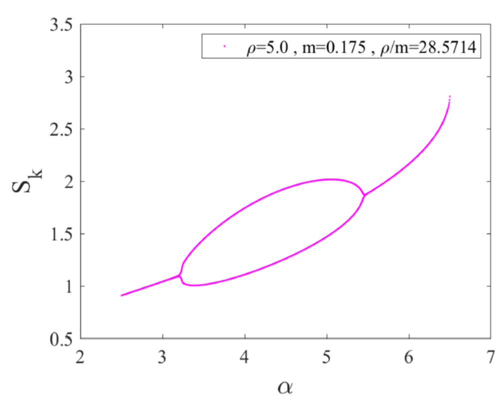

Figure 13 The bifurcation diagram of the fixed point S_1 under Case II

$(0, 4.39)$ tends to the stable 2T-period solution $S_{2T}(t)$, see Fig. 14. Except for the stable T-period and 2T-period solutions, there are no other types of period solutions in the system (7a)–(7b) under Case II.

For $\rho = 18.0, m = 0.5$, then we get $\frac{\rho}{m} = 36$ and $T = 0.1003$. For $\alpha \in [2, 2.7824) \cup (7.6921, 8]$, the modulus of the eigenvalue λ_1 of the system (14) is smaller than 1, and for $\alpha \in$

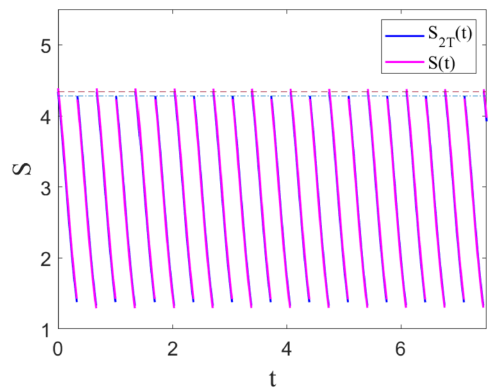

Figure 14 The stable 2T-period solution $S_{2T}(t)$ under Case II

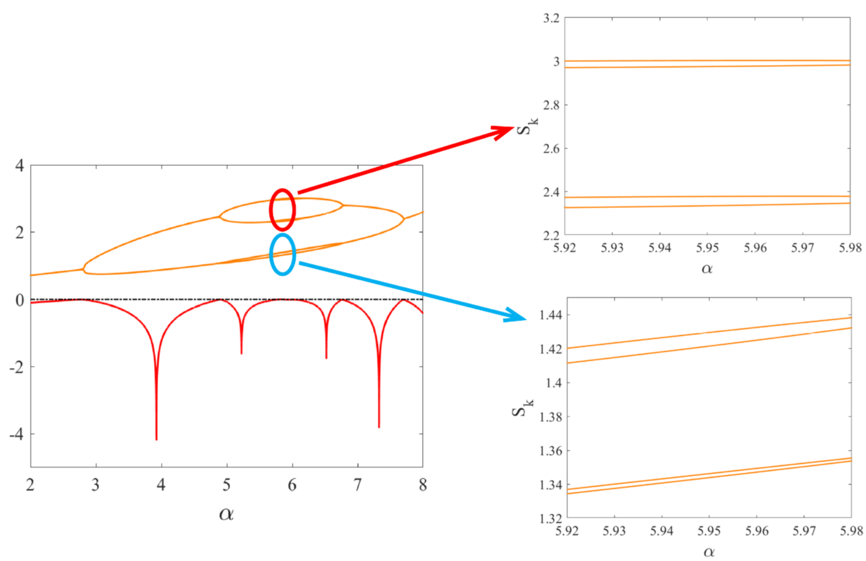

Figure 15 The stability and corresponding Lyapunov exponents of fixed point S_1 under Case III. In the diagram on the left, the horizontal axis is the promotion coefficient α, and the vertical axis is the sales level and the maximum Lyapunov value

(2.7824, 7.6921) the modulus of the eigenvalue λ_1 of the system (14) is larger than 1, please see the orange line in Fig. 10. The bifurcation diagram of the fixed point S_1 and the corresponding maximum Lyapunov exponent under Case III are shown in Fig. 15. It can be seen from Fig. 15 that all the exponent values are less than 0, which indicates that there is no chaos in the system (7a)–(7b). On the right of the Fig. 15, we show its local enlargement of the bifurcation diagram.

When $\alpha \in (2.7824, 4.9041) \cup (6.7623, 7.6921)$, the system (7a)–(7b) has a stable 2T-period solution. For $\alpha \in (4.9041, 5.8425) \cup (6.0102, 6.7623)$, system (7a)–(7b) has a stable 4T-period solution. Let $\alpha = 5.8$, the trajectory $S(t)$ starting from the initial value (0, 5) gradually tends to the stable 4T-period solution $S_{4T}(t)$ under Case III, please see Fig. 16. From the localized enlargement of Fig. 15, we can see that there exists an 8T-period solution when $\alpha \in (5.8425, 6.0102)$. In Fig. 17, we show the stable 8T-period solution. It is also

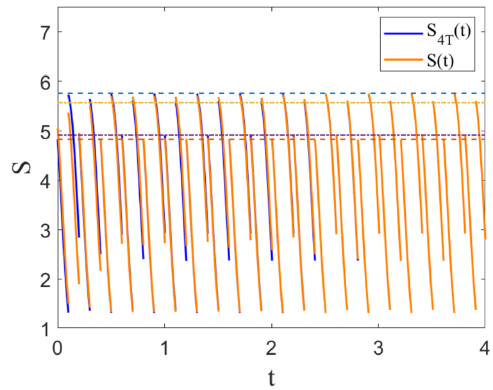

Figure 16 The stable 4T-period solution under Case III

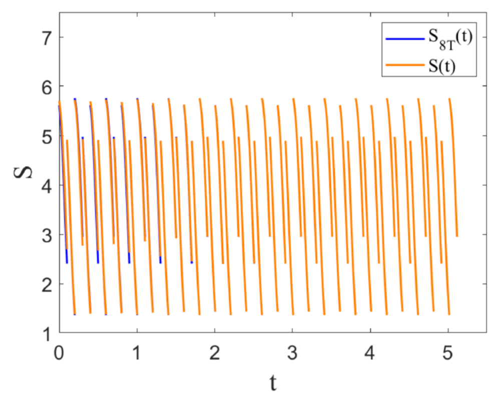

Figure 17 The stable 8T-period solution under Case III

shown that there are no other nT-period solutions in the system (7a)–(7b) under the Case III, except for T-period solution, 2T-period solution, 4T-period solution, and 8T-period solution.

For $\rho = 23.71$, $m = 0.58$, then we can get $\frac{\rho}{m} = 40.8793$ and $T = 0.0788$. When $\alpha \in [2, 2.6625) \cup (9.0233, 9.3]$, the fixed point S_1 of the system (14) has the eigenvalue $|\lambda_1| < 1$. When $\alpha \in (2.6625, 9.0233)$, the fixed point S_1 of system (14) has the eigenvalue $|\lambda_1| > 1$. Please see the purple line in Fig. 10. From the bifurcation diagram, shown in Fig. 18(a), we find that the system (7a)–(7b) can undergo both flip bifurcation and inverse flip bifurcation at $\alpha = 2.6625$ and $\alpha = 9.0233$ respectively. In Fig. 18(b), we plot the corresponding maximum Lyapunov exponents. It indicates that chaos is generated by flip bifurcation or inverse flip bifurcation. Though the system (7a)–(7b) can experience flip and inverse flip bifurcations, it is clearly different from the one presented in Fig. 8 in Sect. 3.3. There is no pathway in the Case IV. This implies that once sales enter a chaotic state, they will eventually become disorganized and uncontrollable.

4 Bifurcation control

In Sect. 3, we have carried out a detailed analysis of the dynamical behaviors of the system (7a)–(7b). We find that the system can experience flip and inverse flip bifurcation if proper parameter values are given. What is more, the system (7a)–(7b) can also enter into chaos.

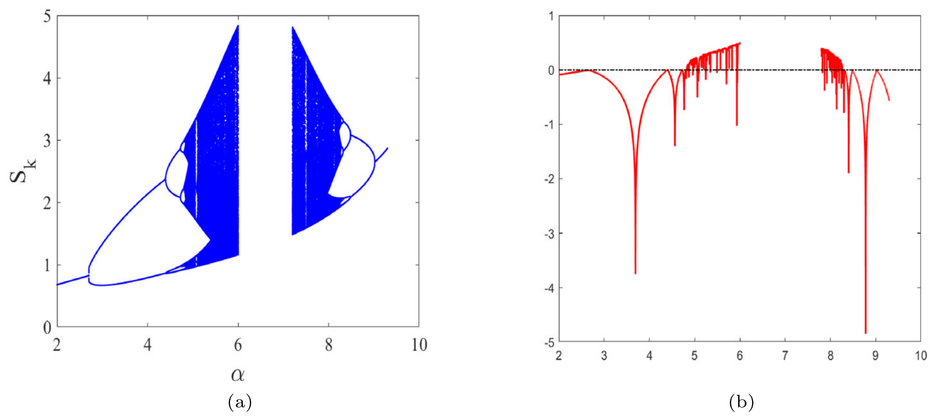

Figure 18 (a) The bifurcation diagram of the fixed point S_1 under Case IV; (b) The corresponding maximum Lyapunov exponents under Case IV

For example, when $\rho = 4.0$, $m = 0.1$ and $\alpha = 6$, the maximum Lyapunov exponent is 0.4034. It means that system (7a)–(7b) has a chaotic solution.

If product sales are chaotic, they will fluctuate erratically and uncontrollably. We would prefer that this situation does not occur. Therefore, we must adopt measures to control the bifurcation and eliminate chaos to reduce unpredictability. For our system, we introduce a constant β into promotion strategy, namely,

$$\Delta S(t) = \frac{\alpha}{S(t)} + \beta,\ t = nT, \tag{24}$$

where β is a small increase in the sales level at the promotion time $t = nT$. Thus, we obtain the following bifurcation control system:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\rho S(t) + mS(t)^3,\ t \neq nT, \tag{25a}$$

$$\Delta S(t) = \frac{\alpha}{S(t)} + \beta,\ t = nT, \tag{25b}$$

where $n = 1, 2, 3, \ldots$.

Similar to Sect. 3, let $T = \frac{1}{2\rho}\ln(1 + \frac{\rho}{m})$, then we get the following map:

$$S_{k+1} = \sqrt{\frac{(S_k + \frac{\alpha}{S_k} + \beta)^2}{1 + \frac{\rho}{m} - (S_k + \frac{\alpha}{S_k} + \beta)^2}}. \tag{26}$$

Denote the fixed point of the map (26) with $S_i{}^*$, $i = 1, 2$. Its stability can be determined by the length of the corresponding eigenvalue:

$$\lambda_i^* = \frac{\sqrt{1 + \frac{\rho}{m} - (S_i{}^* + \frac{\alpha}{S_i{}^*} + \beta)^2}(1 + \frac{\rho}{m})S_i{}^*(-\alpha + S_i{}^{*2})(\alpha + S_i{}^*(\beta + S_i{}^*))}{(S_i{}^* + \frac{\alpha}{S_i{}^*} + \beta)(\alpha^2 + 2\alpha S_i{}^*(\beta + S_i{}^*) + S_i{}^{*2}(-1 - \frac{\rho}{m} + \beta^2 + 2\beta S_i{}^* + S_i{}^{*2}))^2}, i = 1, 2. \tag{27}$$

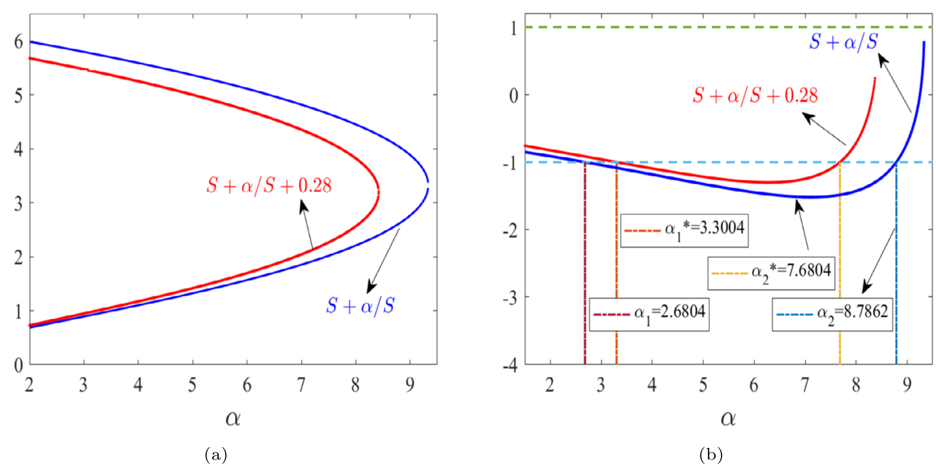

Figure 19 The parameters are $\beta = 0.28$, $\rho = 4.0$ and $m = 0.1$. The blue line represents the system (13), and the red line represents the bifurcation control system (26), (a) the fixed point lines for the system (13) and (26); (b) the eigenvalues corresponding to the fixed point lines of (13) and (26)

From the numerical simulations of the previous section, we note that when chaos happens, there are two cases: one is the existence of a pathway, and the other is not. Therefore, in this section, we mainly focus on these two cases to explain the bifurcation control.

First, let $\beta = 0.28$, $\rho = 4.0$, $m = 0.1$, it is easy to verify that the fixed point S_2^* is always unstable since the length of the eigenvalue ${\lambda_2}^*$ is larger than 1. Therefore, we only consider the stability of the fixed point ${S_1}^*$ in the latter part. The two fixed point lines S_1 and ${S_1}^*$ of the original map (13) and the bifurcation control map (26) are indicated by the blue ($\alpha \in [2, 9.3326]$) and red ($\alpha \in [2, 8.4]$) respectively in Fig. 19(a). For the convenience of the reader, the eigenvalue corresponding to the fixed point S_1 (denoted by the blue with $\alpha \in [2, 9.3326]$) and the eigenvalue corresponding to the fixed point ${S_1}^*$ (denoted by the red with $\alpha \in [2, 8.4]$) are shown in Fig. 19(b), from which we have $\lambda_1^* = -1$ when ${\alpha_1}^* = 3.3004$ and ${\alpha_2}^* = 7.6804$. The flip bifurcation is inhibited from $\alpha_1 = 2.6804$ to ${\alpha_1}^* = 3.3004$, while the inverse flip bifurcation occurs in advanced from $\alpha_2 = 8.7862$ to ${\alpha_2}^* = 7.6804$.

Then, let $\beta = 0.28$, $\rho = 23.71$ and $m = 0.58$, the fixed point ${S_2}^*$ is always unstable since the length of the eigenvalue λ_2^* is larger than 1. Therefore, only the fixed point ${S_1}^*$ is considered. The two fixed point lines of the original map (13) and the bifurcation control map (26) are indicated by the blue ($\alpha \in [2, 9.3]$) and red ($\alpha \in [2, 8.2]$) lines, respectively in Fig. 20(a). In Fig. 20(b), we draw that eigenvalues corresponding to S_1 and ${S_1}^*$. From Fig. 20(b), we see that ${\lambda_1}^* = -1$ when ${\alpha_1}^* = 3.3524$ and ${\alpha_2}^* = 7.9296$. The flip bifurcation is inhibited from $\alpha_1 = 2.6503$ to ${\alpha_1}^* = 3.3524$, while the inverse flip bifurcation occurs from $\alpha_2 = 9.0754$ to ${\alpha_2}^* = 7.9296$ in advance.

Obviously, from Fig. 19(b) and Fig. 20(b), we find that the flip bifurcation can be suppressed while the inverse flip bifurcation can occur in advance when the effect of the bifurcation control strategy (24) is considered. This means the chaos may be eliminated.

In Sect. 3.3 and Sect. 3.4, we have plotted the bifurcation diagrams when $\rho = 4.0$, $m = 0.1$, and $\rho = 23.71$, $m = 0.58$. For convenience, we show them in blue in Fig. 21(a) and 21(b), respectively. The effects of bifurcation control on system (13) are also shown in Fig. 21, see the red lines. From Fig. 21(a), we observe that the chaos in system (13) is controlled into

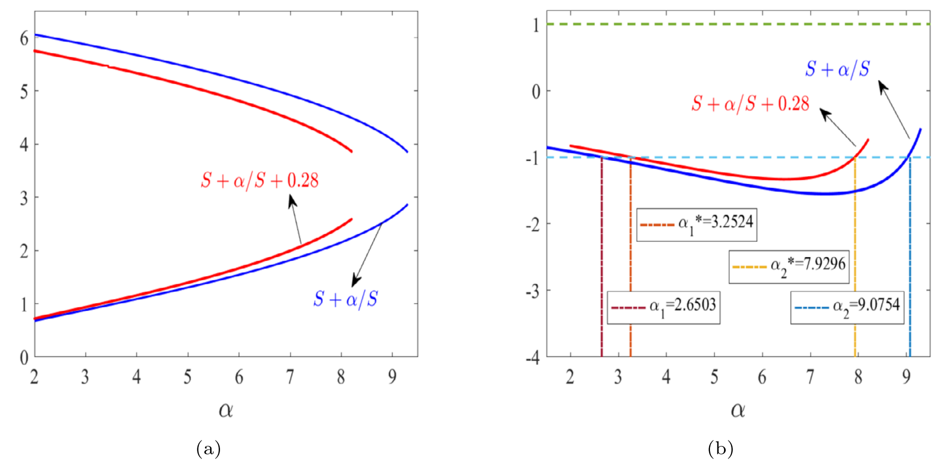

Figure 20 The parameters are $\beta = 0.28$, $\rho = 23.71$ and $m = 0.58$. The blue line represents the system (13), and the red one represents the bifurcation control system (26), (a) the fixed point lines for the system (13) and (26); (b) the eigenvalues corresponding to the fixed point lines of (13) and (26)

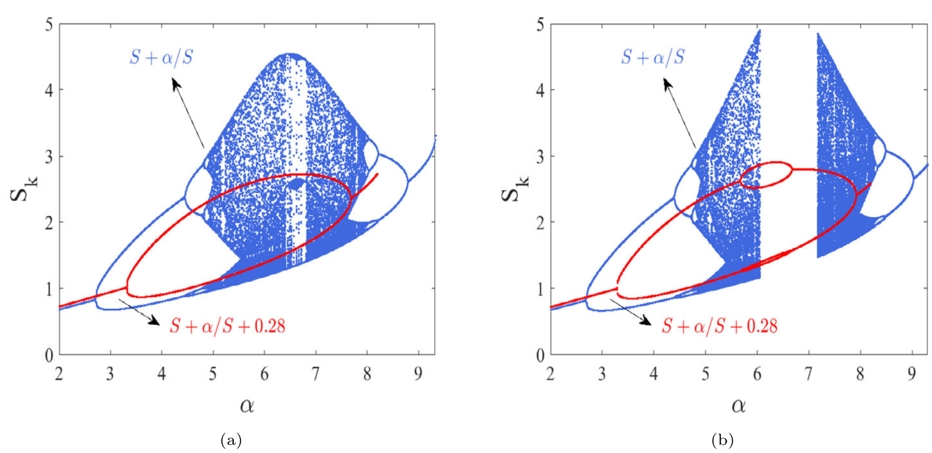

Figure 21 The effects of bifurcation control on system (13), (a) the parameters are $\rho = 4.0$, $m = 0.1$; (b) the parameters are $\rho = 23.71$, $m = 0.58$

T-period and 2T-period solutions. From Fig. 21(b), we observe that the chaos in system (13) is controlled into T-period, 2T-period, and 4T-period solutions.

To better understand the chaos control on the sales, we plot the dynamics of sales before and after the implementation of control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.28$ in Fig. 22. In the Fig. 22(a), it can be seen that the map (13) with parameters $\rho = 4$, $m = 0.1$ and $\alpha = 6$ is in a chaotic state without bifurcation control (denoted by the blue points), whereas when the control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.28$ are added at $t = kT$, ($k > 150$, $T = 0.4642$), it proceeds to the stable period-2 points (denoted as the red points). Similarly, for system (7a)–(7b), the sale $S(t)$ is in a chaotic state before bifurcation control is applied, while the sale $S(t)$ converses to a stable 2T-period solution (denoted by red) after bifurcation control is applied. Please see Fig. 22(b).

For the case with $\rho = 23.71$, $m = 0.58$ and $\alpha = 8.1$, we can also eliminate the chaos with the control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.28$. As shown in Fig. 23, the blue points are the disor-

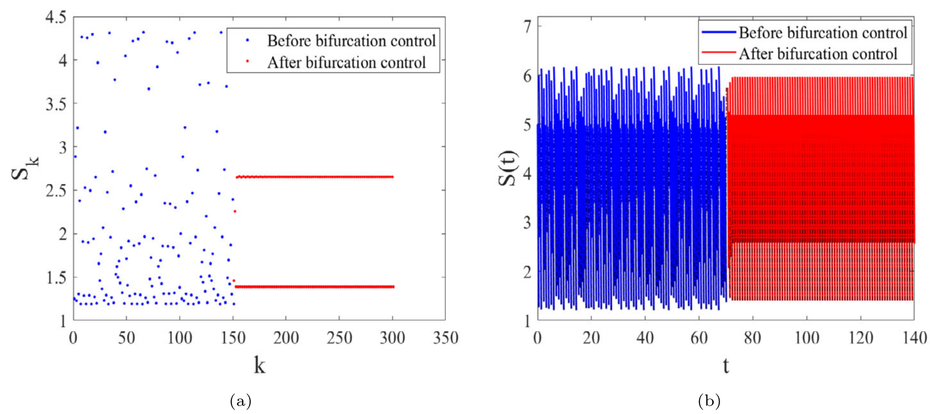

Figure 22 The effects of bifurcation control on the S_k and $S(t)$ with $\rho = 4.0$, $m = 0.1$ and $\alpha = 6$, (a) a chaotic state of S_k is controlled to a stable period-2 orbit; (b) a chaotic state of sale $S(t)$ is controlled to a stable 2T-period solution

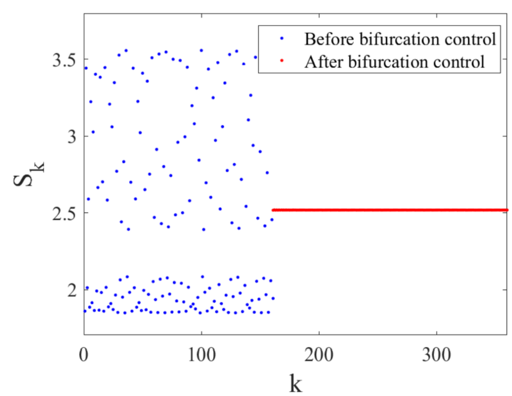

Figure 23 A chaotic state of S_k is controlled to a stable fixed point with $\rho = 23.71$, $m = 0.58$ and $\alpha = 8.1$

dered ones produced by the map (13). Before $k = 160$, we do not add the bifurcation control strategy, the orbits of the system (13) are in a chaotic state. After the implementation of the bifurcation control; the orbits are governed by a stable state (red line). Correspondingly, the effects of the bifurcation control of the sale $S(t)$ of system (7a)–(7b) is shown in the Fig. 24(a). The chaotic behavior (blue) is taken over by a stable T-period solution (red) after $t = kT(k > 160, T = 0.0788)$. By appropriately choosing other values of α, the chaotic behavior can be taken over by 2T-period and 4T-period solutions.

5 Conclusion and discussion

In this research, a nonlinear differential advertising model with a single parameter sales promotion strategy is developed. To find the mathematical mechanisms of how the promotion strategy affects sales, we conduct a series of theoretical analyses and numerical simulations. We identify the threshold for adopting a promotion strategy: $S(t)^2 < \frac{\rho}{m}$. Under this condition, we further explore the important roles of the parameters α, ρ and m in the complex and rich dynamical behaviors. An important and interesting phenomenon is that both flip bifurcation and inverse flip bifurcation can occur in this system, giving rise to the occurrence of chaos. What is more interesting is that under some values of

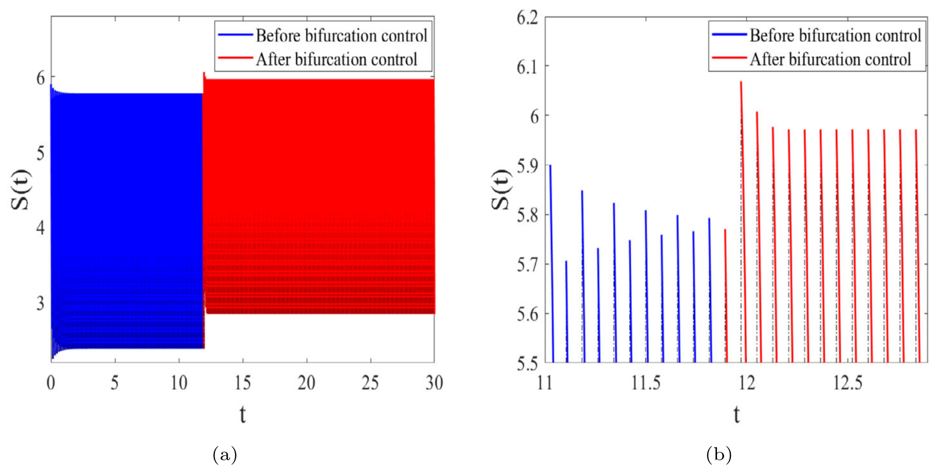

Figure 24 The effects of bifurcation control on the $S(t)$ with $\rho = 23.71$, $m = 0.58$ and $\alpha = 8.1$, (a) a chaotic state of sale $S(t)$ is controlled to a stable T-period solution; (b) Localized enlarged view of Fig. 24(a) at $\alpha \in [11, 12.9]$

the parameters, there exists a pathway in the parameter space for the system (7a)–(7b) to enter into chaos from a stable state via flip bifurcation and exit chaos to a stable state via inverse flip bifurcation. While in other values of the parameters, though flip bifurcation and inverse flip bifurcation coexist, in the parameter space, there is no such pathway between them. The promotion coefficient α plays an important role in the rich and complex dynamical behaviors. According to the theoretical analysis and the actual situation, firms can make appropriate adjustments to the promotion coefficient α to develop a sales promotion strategy and ensure the sales volume is steered towards greater stability and predictability.

The instability and unpredictability of chaos spell a lot of difficulties for practical applications. Therefore, we propose an effective bifurcation control strategy to eliminate chaos by inhibiting flip bifurcation and facilitating the occurrence of inverse flip bifurcation. After a company formulates and implements certain promotional strategies, the sales volume of the products will increase significantly. Through the analysis of the system (7a)–(7b), we found that under certain promotional strategies, sales may enter a state of chaos. To avoid this situation, the company can make fine adjustments to the existing promotional strategies. By controlling the parameter β, the company can thus control the increment of sales volume, thereby suppressing the chaotic phenomena and transforming it into a controllable sales pattern. This can help the company avoid unpredictable and uncontrolled fluctuations in sales volume.

It is worth noting that this research is primarily a theoretical analysis. The conditions which are responsible for the dynamic behaviors of the sales are obtained. A firm can refer to the actual data to get the sales decay constant ρ, the response rate to advertising m. Before conducting a promotional event, it is important to conduct research, analysis, and testing in advance to obtain the optimal promotion coefficient α and promotional control parameter β to improve sales level and maximize firm's profit.

Appendix: Detailed information about the parameter conditions for the existence of positive zero points of $h(y)$

It is easy to get that

$$h'(y) = 3y^2 + 2(2\alpha - \frac{\rho}{m})y + (\alpha^2 + 2\alpha). \tag{A.1}$$

Then when $\frac{\rho}{m} - 2\alpha \le 0$, i.e., $0 < \rho \le 2\alpha m$, for any $y \ge 0$, we have $h'(y) \ge 0$, so $h(y)$ is an increasing function on $[0, +\infty)$. Since $h(0) = \alpha^2 > 0$, we have $h(y) > 0$, $\forall y \ge 0$. The sketch of the function $h(y)$ is shown in Fig. 25. So, when $0 < \rho \le 2\alpha m$, $h(y)$ has no positive zero point. Thus, the system (14) has no positive fixed point in this case.

When $\frac{\rho}{m} - 2\alpha > 0$ and $4(2\alpha - \frac{\rho}{m})^2 - 12(\alpha^2 + 2\alpha) \le 0$, i.e. $2\alpha m < \rho \le (2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m$, we have $h'(y) \ge 0$ for any $y \ge 0$, so $h(y)$ is an increasing function on $[0, +\infty)$. In addition, $h(0) = \alpha^2 > 0$, so $h(y)$ also has no positive zero point. Therefore, the system (14) does not have a positive fixed point when $2\alpha m < \rho \le (2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m$.

When $\frac{\rho}{m} - 2\alpha > 0$ and $4(2\alpha - \frac{\rho}{m})^2 - 12(\alpha^2 + 2\alpha) > 0$, that is to say $\rho > (2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m$. At this situation, on $[0, +\infty)$, $h'(y)$ has two positive real roots:

$$y_{01} = \frac{\frac{\rho}{m} - 2\alpha - \sqrt{\alpha^2 + 4\alpha\frac{\rho}{m} + (\frac{\rho}{m})^2 - 6\alpha}}{3},$$
$$y_{02} = \frac{\frac{\rho}{m} - 2\alpha + \sqrt{\alpha^2 + 4\alpha\frac{\rho}{m} + (\frac{\rho}{m})^2 - 6\alpha}}{3}.$$

The sketch of function $h'(y)$ is shown in Fig. 26.

From Fig. 26, when $y \in (0, y_{01})$, $h(y)$ is an increasing function. Because $h(0) = \alpha^2 > 0$, so $h(y_{01}) > 0$. When $y \in (y_{01}, y_{02})$, $h(y)$ is a decreasing function, so $h(y_{01})$ is the local maximum value. When $y \in (y_{02}, +\infty)$, $h(y)$ is an increasing function, so $h(y_{02})$ is a local minimum value. There are three possible values for $h(y_{02})$: $h(y_{02}) > 0$, $h(y_{02}) = 0$ and $h(y_{02}) < 0$. These three cases result in three different images of the function $h(y)$. In Fig. 27, we show the three different sketches of $h(y)$.

In Fig. 27(a), we have $h(y_{02}) > 0$, this can be satisfied if given $(2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m < \rho < \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8 + \alpha)^3})$. It shows that $h(y)$ has no positive zero point for any $y > 0$. Together with the cases that when $0 < \rho < 2\alpha m$ and $2\alpha m < \rho \le (2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m$, $h(y)$

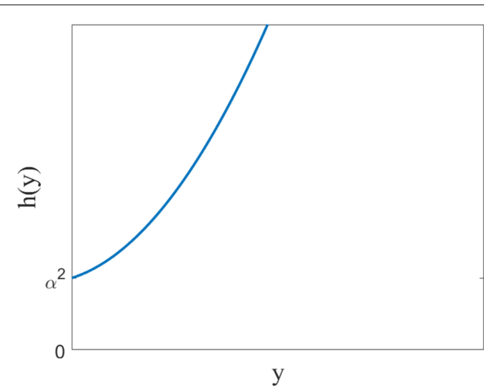

Figure 25 A sketch of $h(y)$ when $\frac{\rho}{m} - 2\alpha \le 0$

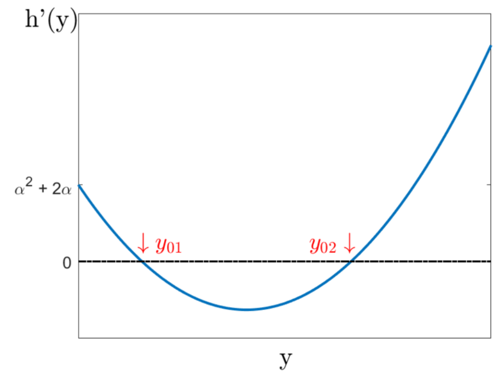

Figure 26 A sketch of $h'(y)$ when $\rho > (2\alpha + \sqrt{3(\alpha^2 + 2\alpha)})m$

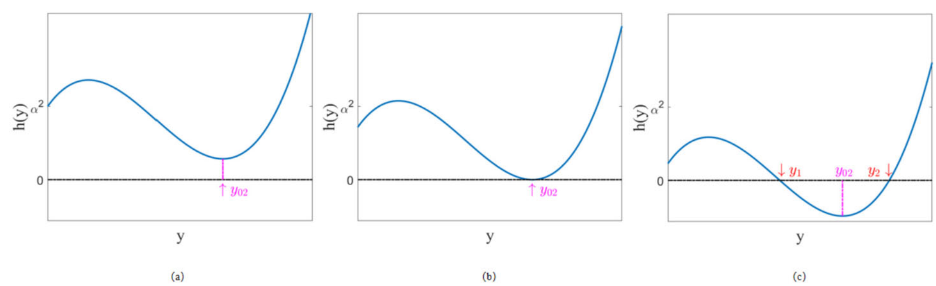

Figure 27 Three different sketches of $h(y)$ corresponding to three different values of $h(y_{02})$: (a) $h(y_{02}) > 0$; (b) $h(y_{02}) = 0$; (c) $h(y_{02}) < 0$

also has no positive zero point, so when $0 < \rho < \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, the system (14) has no positive fixed point.

In Fig. 27(b), $h(y_{02}) = 0$, this can be satisfied if given $\rho = \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$. Therefore, when $\rho = \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$, the system (14) has only one positive fixed point.

In Fig. 27(c), $h(y_{02}) < 0$, this can be satisfied if given $\rho > \frac{1}{8}m(20\alpha - \alpha^2 + \sqrt{\alpha(8+\alpha)^3})$. It is shown that $h(y)$ has two positive zero points. We denote them as y_1 and y_2. According to the root formula, the two positive zeros y_1 and y_2 are

$$y_1 = \frac{\frac{\rho}{m} - 2\alpha + \sqrt{(2\alpha - \frac{\rho}{m})^2 - 3(\alpha^2 + 2\alpha)}(\cos(\frac{\theta}{3}) - \sqrt{3}\sin(\frac{\theta}{3}))}{3}, \tag{A.2}$$

and

$$y_2 = \frac{\frac{\rho}{m} - 2\alpha + \sqrt{(2\alpha - \frac{\rho}{m})^2 - 3(\alpha^2 + 2\alpha)}(\cos(\frac{\theta}{3}) + \sqrt{3}\sin(\frac{\theta}{3}))}{3}, \tag{A.3}$$

where

$$R = \frac{2((2\alpha - \frac{\rho}{m})^2 - 3(\alpha^2 + 2\alpha))(2\alpha - \frac{\rho}{m}) - 3((2\alpha - \frac{\rho}{m})(\alpha^2 + 2\alpha) - 9\alpha^2)}{2\sqrt{((2\alpha - \frac{\rho}{m})^2 - 3(\alpha^2 + 2\alpha))^3}}, 0 < R < 1,$$

$$\theta = \arccos(R).$$

Therefore, the system (14) has two positive fixed points

$$S_1 = \sqrt{y_1}, \tag{A.4}$$

and

$$S_2 = \sqrt{y_2}. \tag{A.5}$$

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Author contributions
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