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RESEARCH

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Numerical analysis of a second-order finite difference scheme for Riesz space-fractional Allen–Cahn equations

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Abstract

The goal of this paper is to present a second-order finite difference scheme for Allen–Cahn equations with Riesz fractional derivative. The discrete maximum bound principle, the maximum-norm error estimates, and the discrete energy stability of the proposed scheme are discussed. It is shown that the proposed scheme is unconditionally energy-stable for any nonnegative stabilization parameter. Two numerical experiments are performed to verify the theoretical results.

Keywords: Riesz space-fractional Allen–Cahn equation; Finite difference method; Maximum bound principle; Error estimate; Energy stability

1 Introduction

In this paper, we consider the finite difference approximation for the following 1D space-fractional Allen–Cahn equation:

$$\frac{\partial u(x,t)}{\partial t} = \epsilon^2 \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} - f(u), \quad x \in (a,b),\ t \in (0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{2}$$

$$u(a,t) = u(b,t) = 0, \quad t \in (0,T], \tag{3}$$

where the parameter $\epsilon > 0$, $\alpha \in (1,2)$, and the nonlinear term $f(u) = u^3 - u$ presents the polynomial double well potential.

Let $\frac{\partial^\alpha}{\partial |x|^\alpha}$ be the Riesz fractional derivative operator defined by

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos\frac{\alpha\pi}{2}\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_{-\infty}^{\infty} |x-\xi|^{1-\alpha} u(\xi,t)\,\mathrm{d}\xi,$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t, \quad z > 0.$$

In recent years, there has been significant interest in using the diffusive-interface phase-field approach for modeling the mesoscale morphological pattern formation and interface motion. One of very effective mathematical models describing these physical phenomena is the Allen–Cahn equation introduced in 1979 [1].

Similar to the classical Allen–Cahn equation, the space-fractional Allen–Cahn equation (1)–(3) also has the following two intrinsic properties. One is the maximum bound principle, i.e., if $|u_0(x)| \leq 1$ for all $x \in [a,b]$ then $|u(x,t)| \leq 1$ for all $x \in [a,b]$ and $t \geq 0$. The other is that the energy function $E(u)$ is decreasing with time: $E(u(t_n)) \leq E(u(t_m))$, $\forall t_n > t_m$, where $E(u) = \int_a^b \left(F(u) - \frac{1}{2}\epsilon^2 u \frac{\partial^\alpha u}{\partial |x|^\alpha}\right) \mathrm{d}x$ and $F(u) = \frac{1}{4}(u^2-1)^2$. Such two properties are important in the study of the stability of the solution to the Allen–Cahn equation, and whether they could be inherited in the discrete level is a significant issue in numerical simulations. Many scholars have attempted to discuss the discrete maximum bound principle and the discrete energy stability of the finite difference approximations and derived some meaningful results. Tang and Yang [9] discussed the discrete maximum bound principle and the discrete energy stability of first-order linear implicit–explicit scheme for the Allen–Cahn equation. By adding a stabilizing term, the proposed scheme is unconditionally energy-stable when the parameter is greater than 2. Shen et al. [8] analyzed the discrete maximum bound principle of a first-order linear implicit–explicit scheme for the generalized Allen-Cahn equation. For temporal discretization, the standard semiimplicit scheme and the stabilized semiimplicit scheme were adopted, while for space discretization, the central finite difference was used for approximating the diffusion term and the upwind scheme was employed for the advection term. Hou et al. [5] considered the discrete maximum bound principle, the discrete energy stability, and the error estimates of the second-order Crank–Nicolson finite difference scheme for fractional-in-space Allen–Cahn equations. In that paper, the left and right Riemann–Liouville fractional derivatives were considered. Hou and Leng [4] considered a stabilized second-order Crank–Nicolson/Adams–Bashforth scheme of the Allen–Cahn equation and obtained the discrete maximum bound principle and the discrete energy stability. Liao et al. [7] presented a second-order and nonuniform time-stepping maximum bound principle preserving scheme for time-fractional Allen–Cahn equations. Du et al. [3] proposed the first- and second-order maximum bound principle preserving exponential time differencing schemes for the nonlocal Allen–Cahn equation. To obtain a higher-order scheme for solving the Allen–Cahn equation, Li et al. [6] proposed a new class of maximum principle preserving numerical schemes, which consists of a kth-order multistep exponential integrator in time, and a lumped mass finite-element method in space with piecewise rth-order polynomials and Gauss–Lobatto quadrature. Zhang et al. [12] proposed high-order (up to fourth) strong stability-preserving implicit–explicit Runge–Kutta schemes for the time integration of the space-fractional Allen–Cahn equation and discussed the discrete maximum bound principle and energy stability. Zhang et al. [13] considered the second-order finite difference method in space and the pth-order Runge–Kutta integration in time to design a class of maximum principle preserving integrators for the Allen–Cahn equation. Next, they proposed and analyzed a class of temporal up to fourth-order unconditionally structure-preserving single-step methods for Allen–Cahn-type semilinear parabolic equations in [14]. Zhang et al. [11] presented a systematic two-step approach to derive temporal up to the eighth-order, unconditionally maximum-principle-preserving schemes for a semilinear parabolic Sine–Gordon equation and its conservative modification. Zhang et al. [10] developed a class of high-order, large time-stepping and delay-free integrators for the Cahn–Hilliard-type equation with both viscous regularization and Oono-type nonlocal interaction that models microphase separation in diblock copolymer melts.

In this work, we shall present an unconditionally energy-stable second-order finite difference scheme for the Allen–Cahn equation with Riesz fractional derivative. We can prove that our scheme is maximum bound principle preserving and the discrete energy is unconditionally decreasing. Moreover, our theoretical results can be applied to 2D and 3D problems.

The rest of the paper is organized as follows. In Sect. 2, a second-order finite difference scheme will be presented. The discrete maximum bound principle, the maximum-norm error estimate and the discrete energy stability of the proposed scheme will be discussed in Sects. 3–5, respectively. Finally, two numerical examples are given in the last section to verify the theoretical results.

2 Finite difference approximation

We partition the interval (a,b) into a uniform mesh with the space step $h = (b-a)/(N+1)$ and $\tau = T/M$, where N, M are two positive integers. The set of grid points are denoted by $x_i = a + (i-1)h$ and $t_n = n\tau$ for $1 \le i \le N+2$ and $0 \le n \le M$, and we use the notations $u^n = u(x,t_n)$ and $u_i^n = u(x_i,t_n)$. Define

$$V_h = \left\{ v : v = \{v_i\} \text{ is a grid function in } \{x_i = a + ih\}_{i=1}^{N} \right\}.$$

For any $v = \{v_i\} \in V_h$, we define its pointwise maximum norm

$$\|v\|_\infty = \max_{1\le i\le N} |v_i|.$$

We adopt a second-order finite difference approach as in [2] to discretize the fractional operator $\frac{\partial^\alpha}{\partial|x|^\alpha}$. Hereinafter, we denote by D_h the discretization matrix of the fractional operator. It is given by

$$D_h = -\frac{1}{h^\alpha}\begin{bmatrix} g_0 & g_{-1} & g_{-2} & \cdots & g_{-N+1} \\ g_1 & g_0 & g_{-1} & \cdots & g_{-N+2} \\ g_2 & g_1 & g_0 & \cdots & g_{-N+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & g_{N-3} & \cdots & g_0 \end{bmatrix}_{N\times N} =: -\frac{1}{h^\alpha}A, \tag{4}$$

where

$$g_0 = \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}+1\right)^2} > 0, \quad g_{-k} = g_k < 0, \quad \sum_{k=-\infty}^{+\infty} g_k = 0, \tag{5}$$

and

$$g_{k+1} = \left(1 - \frac{\alpha+1}{\frac{\alpha}{2}+k+1}\right) g_k < 0, \quad 1 < \alpha < 2. \tag{6}$$

Using (4)–(6), one can prove that D_h satisfies the following properties. We omit the proof of the following lemma.

Lemma 1 *The matrix D_h satisfies the following properties*:

- *D_h is symmetric*;
- *D_h is negative definite, i.e., $U^T D_h U < 0$, for any nonzero $U \in \mathbf{R}^N$*;
- *The elements of $D_h = (b_{ij})$ satisfy*

$$b_{ii} = -d < 0, \quad d \geq \max_i \sum_{j \neq i} |b_{ij}|. \tag{7}$$

Next, we present the following finite difference scheme to solve equation (1), namely:

$$\begin{aligned} &\frac{U^{n+1} - U^n}{\tau} + \frac{(U^{n+1})^{.3} + (U^{n+1})^{.2} U^n + U^{n+1}\left(U^n\right)^{.2} + \left(U^n\right)^{.3}}{4} - \frac{U^{n+1} + U^n}{2} \\ &\qquad + \frac{\left(U^{n+1} - U^n\right)^{.3}}{12} + \beta\tau\left(U^{n+1} - U^n\right) = \frac{\epsilon^2 D_h \left(U^{n+1} + U^n\right)}{2}, \end{aligned} \tag{8}$$

where $0 \leq n \leq M-1$, $\beta \geq 0$ and U^n represents the vector of numerical solution at nth level. Hereinafter we define

$$\begin{aligned} U^n &:= (U_1^n, U_2^n, \ldots, U_N^n)^T, \\ (U^n)^{.2} &:= ((U_1^n)^2, (U_2^n)^2, \ldots, (U_N^n)^2)^T, \\ (U^n)^{.3} &:= ((U_1^n)^3, (U_2^n)^3, \ldots, (U_N^n)^3)^T \end{aligned}$$

and

$$U^n V^n := (U_1^n V_1^n, U_2^n V_2^n, \ldots, U_N^n V_N^n)^T.$$

3 The discrete maximum bound principle

In this section, we will discuss the discrete maximum bound principle for the scheme (8).

Theorem 1 *Assume the initial value satisfies $\max_{x\in[a,b]} |u_0(x)| \leq 1$. Then the fully discrete scheme (8) preserves the discrete maximum bound principle provided the time step size satisfies*

$$\tau \leq \min\left\{\frac{1}{2}, \frac{h^\alpha \Gamma\left(\frac{\alpha}{2}+1\right)^2}{\epsilon^2 \Gamma(\alpha+1)}\right\}.$$

Proof We prove this theorem by induction. First, it follows from the assumption on $u_0(x)$ that $||U^0||_\infty \leq 1$. We now assume that the result holds for $n = m$, i.e., $||U^m||_\infty \leq 1$. Below we will check that this upper bound is also true for $n = m+1$.

We rewrite (8) as

$$\left(1 - \frac{\tau}{2} + \beta\tau^2\right) U^{m+1} + \frac{\tau}{3}\left(U^{m+1}\right)^{.3} + \frac{\tau}{2} U^{m+1}\left(U^m\right)^{.2} - \frac{\tau}{2}\epsilon^2 D_h U^{m+1}$$

$$= \left(1+\frac{\tau}{2}+\beta\tau^2\right)U^m - \frac{\tau}{6}\left(U^m\right)^{.3} + \frac{\tau}{2}\epsilon^2 D_h U^m. \quad (9)$$

Suppose $\|U^{m+1}\|_\infty = |U_p^{m+1}|$, then $|U_p^{m+1}| \ge |U_j^{m+1}|$ for all $1 \le j \le N$. The pth component of (9) is

$$\left[1-\frac{\tau}{2}+\beta\tau^2+\frac{\tau}{2}\left(U_p^m\right)^2\right]U_p^{m+1} + \frac{\tau}{3}\left(U_p^{m+1}\right)^3 - \frac{\tau}{2}\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^{m+1}\right)$$
$$= \left(1+\frac{\tau}{2}+\beta\tau^2\right)U_p^m - \frac{\tau}{6}\left(U_p^m\right)^3 + \frac{\tau}{2}\epsilon^2\left(\sum_{j=1}^N b_{pj}U_j^m\right). \quad (10)$$

We see from $\tau \le \frac{1}{2}$ and (7) that

$$\left[1-\frac{\tau}{2}+\beta\tau^2+\frac{\tau}{2}\left(U_p^m\right)^2\right]U_p^{m+1}\left(-\frac{\tau}{2}\epsilon^2\sum_{j=1}^N b_{pj}U_j^{m+1}\right)$$
$$= \frac{\tau\epsilon^2}{2}\left[1-\frac{\tau}{2}+\beta\tau^2+\frac{\tau}{2}\left(U_p^m\right)^2\right]\left(d\left(U_p^{m+1}\right)^2 - \sum_{j\ne p} b_{pj}U_p^{m+1}U_j^{m+1}\right)$$
$$\ge \frac{\tau\epsilon^2}{2}\left[1-\frac{\tau}{2}+\beta\tau^2+\frac{\tau}{2}\left(U_p^m\right)^2\right]\left[d\left|U_p^{m+1}\right|^2 - \sum_{j\ne p}|b_{pj}||U_p^{m+1}|^2\right] \ge 0.$$

So, we find that $(1-\frac{\tau}{2}+\beta\tau^2)U_p^{m+1} + \frac{\tau}{2}\left(U_p^m\right)^2 U_p^{m+1}$, $\frac{\tau}{3}\left(U_p^{m+1}\right)^3$, and $-\frac{\tau}{2}\epsilon^2\sum b_{pj}U_j^{m+1}$ are nonpositive or nonnegative simultaneously. Then, we have

$$\left|\left(1-\frac{\tau}{2}+\beta\tau^2\right)U_p^{m+1} + \frac{\tau}{2}\left(U_p^m\right)^2 U_p^{m+1} - \frac{\tau}{2}\epsilon^2\sum_{j=1}^N b_{pj}U_j^{m+1} + \frac{\tau}{3}\left(U_p^{m+1}\right)^3\right|$$
$$= \left|\left(1-\frac{\tau}{2}+\beta\tau^2\right)U_p^{m+1} + \frac{\tau}{2}\left(U_p^m\right)^2 U_p^{m+1}\right| + \left|\frac{\tau}{2}\epsilon^2\sum_{j=1}^N b_{pj}U_j^{m+1}\right| + \left|\frac{\tau}{3}\left(U_p^{m+1}\right)^3\right|$$
$$\ge \left(1-\frac{\tau}{2}+\beta\tau^2+\frac{\tau}{2}\left(U_p^m\right)^2\right)\left|U_p^{m+1}\right| + \frac{\tau}{3}\left|U_p^{m+1}\right|^3. \quad (11)$$

Taking the absolute value of (10) and using (11), we easily obtain

$$\left(1-\frac{\tau}{2}+\beta\tau^2\right)\left|U_p^{m+1}\right| + \frac{\tau}{2}\left(U_p^m\right)^2\left|U_p^{m+1}\right| + \frac{\tau}{3}\left|U_p^{m+1}\right|^3$$
$$\le \left|\frac{1}{2}U_p^m + \frac{\tau}{2}\epsilon^2\sum_{j=1}^N b_{pj}U_j^m\right| + \left|\frac{\tau}{2}U_p^m - \frac{\tau}{6}\left(U_p^m\right)^3\right| + \beta\tau^2\left|U_p^m\right| + \frac{1}{2}\left|U_p^m\right|. \quad (12)$$

Notice that

$$\left|\frac{1}{2}U_p^m + \frac{\tau}{2}\epsilon^2\sum_{j=1}^N b_{pj}U_j^m\right| = \left|\left(\frac{1}{2}+\frac{\tau}{2}\epsilon^2 b_{pp}\right)U_p^m + \frac{\tau}{2}\epsilon^2\sum_{j\ne p} b_{pj}U_j^m\right|.$$

If

$$\frac{1}{2}+\frac{\tau}{2}\epsilon^2 b_{pp}=\frac{1}{2}-\frac{\tau\epsilon^2\Gamma(\alpha+1)}{2h^\alpha\Gamma\left(\frac{\alpha}{2}+1\right)^2}\geq 0,$$

namely

$$\tau\leq\frac{h^\alpha\Gamma\left(\frac{\alpha}{2}+1\right)^2}{\epsilon^2\Gamma(\alpha+1)},$$

then we have

$$\begin{aligned}\left|\left(\frac{1}{2}+\frac{\tau}{2}\epsilon^2 b_{pp}\right)U_p^m+\frac{\tau}{2}\epsilon^2\sum_{j\neq p}b_{pj}U_j^m\right| &\leq\left(\frac{1}{2}+\frac{\tau}{2}\epsilon^2 b_{pp}\right)\left|U_p^m\right|+\frac{\tau}{2}\epsilon^2\sum_{j\neq p}\left|b_{pj}\right|\left|U_j^m\right|\\ &\leq\frac{1}{2}+\frac{\tau}{2}\epsilon^2 b_{pp}+\frac{\tau}{2}\epsilon^2\sum_{j\neq p}\left|b_{pj}\right|\\ &=\frac{1}{2}+\frac{\tau}{2}\epsilon^2\left(\sum_{j\neq p}\left|b_{pj}\right|-d\right)\\ &\leq\frac{1}{2},\end{aligned}\tag{13}$$

where we used (7) and $\|U^m\|_\infty\leq 1$.

Let $g(x)=\frac{\tau}{2}x-\frac{\tau}{6}x^3, x\in[-1,1]$. It is easy to see that $g'(x)=\frac{\tau}{2}-\frac{\tau}{2}x^2$ and $g'(x)\geq 0$ for $x\in[-1,1]$. As $g(1)=-g(-1)=\frac{\tau}{3}$, we have $\left|g(x)\right|\leq\frac{\tau}{3}$, so

$$\left|\frac{\tau}{2}U_p^m-\frac{\tau}{6}\left(U_p^m\right)^3\right|\leq\frac{\tau}{3}.\tag{14}$$

Using (12)–(14) and $\|U^m\|_\infty\leq 1$, we get

$$\begin{aligned}&\left(1-\frac{\tau}{2}+\beta\tau^2\right)\left|U_p^{m+1}\right|+\frac{\tau}{2}\left(U_p^m\right)^2\left|U_p^{m+1}\right|+\frac{\tau}{3}\left|U^{m+1}\right|^3\\ &\quad\leq\frac{1}{2}+\frac{\tau}{3}+\beta\tau^2+\frac{1}{2}\left|U_p^m\right|.\end{aligned}\tag{15}$$

Suppose $\left\|U^{m+1}\right\|_\infty>1$, then (15) becomes

$$1-\frac{\tau}{2}+\frac{\tau}{2}\left|U_p^m\right|^2<\frac{1}{2}+\frac{1}{2}\left|U_p^m\right|,$$

namely

$$-\tau\left|U_p^m\right|^2+\left|U_p^m\right|+\tau-1>0,$$

which is in contradiction with $\left|U_p^m\right|\leq\|U^m\|_\infty\leq 1$ provided that $\tau\leq\frac{1}{2}$. Thus, we have $\left\|U^{m+1}\right\|_\infty\leq 1$. Then the proof of this theorem is completed. □

Remark 1 If $\beta < 0$, the discrete maximum bound principle also holds for $\tau \leq \min\left\{\frac{1}{2}, \frac{h^{\alpha}\Gamma(\frac{\alpha}{2}+1)^2}{\epsilon^2\Gamma(\alpha+1)}\right\}$ and $1 - \frac{\tau}{2} + \beta\tau^2 \geq 0$, so the constraint on the time step should be $\tau \leq \min\left\{\frac{1}{2}, \frac{1-\sqrt{1-16\beta}}{4\beta}, \frac{h^{\alpha}\Gamma(\frac{\alpha}{2}+1)^2}{\epsilon^2\Gamma(\alpha+1)}\right\}$.

4 The maximum-norm error estimate

In this section, we shall discuss the maximum-norm error estimate based on the discrete maximum bound principle obtained in Theorem 1. Let $C(\epsilon,\beta,T)$ be a constant which depends on ϵ, β, T, and the regularity of the exact solution, but is independent of h and τ. Other constants C, $C(\epsilon)$, and $C(\epsilon,\beta)$ can be defined similarly.

Theorem 2 *Suppose the exact solution $u(x,t)$ is smooth, and the initial value is smooth and bounded by* 1, *i.e.,* $\max_{x\in[a,b]} |u_0(x)| \leq 1$. *Assume that all the conditions in Theorem 1 are valid. Moreover, assume that* $\tau \leq \tau_0$ ($\tau_0 = \frac{\sqrt{25+2\beta}-5}{2\beta}$ *for* $\beta > 0$ *and* $\tau_0 = \frac{1}{10}$ *for* $\beta = 0$). *Then for all* $1 \leq n \leq M$, *we have*

$$\|\boldsymbol{u}^n - U^n\|_\infty \leq C(\epsilon,\beta,T)(\tau^2 + h^2), \tag{16}$$

where $\boldsymbol{u}^n := (u_1^n, u_2^n, \dots, u_N^n)^T$ *represents the vector of exact solution at the nth level.*

Proof First, we discretize (1) in space and time, respectively, to get

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\tau} + \frac{(\boldsymbol{u}^{n+1})^{.3} - \boldsymbol{u}^{n+1}}{2} + \frac{(\boldsymbol{u}^n)^{.3} - \boldsymbol{u}^n}{2} = \frac{\epsilon^2 D_h(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)}{2} + \boldsymbol{\rho}^n, \tag{17}$$

where $\boldsymbol{\rho}^n := (\rho_1^n, \rho_2^n, \dots, \rho_N^n)^T$ and $|\rho_i^n| \leq C(\epsilon)(\tau^2 + h^2)$, $1 \leq i \leq N$, $0 \leq n \leq M-1$.

Next, letting $e^n = \boldsymbol{u}^n - U^n$ and subtracting (8) from (17), we obtain

$$\begin{aligned}
&\frac{e^{n+1} - e^n}{\tau} - \frac{\epsilon^2 D_h\left(e^{n+1} + e^n\right)}{2} \\
&= \boldsymbol{\rho}^n + \frac{e^{n+1} + e^n}{2} + \frac{\left(U^{n+1} - U^n\right)^{.3}}{12} + \beta\tau\left(U^{n+1} - U^n\right) \\
&\quad + \frac{\left(U^{n+1}\right)^{.3} + \left(U^{n+1}\right)^{.2} U^n + U^{n+1}\left(U^n\right)^{.2} + \left(U^n\right)^{.3}}{4} - \frac{\left(\boldsymbol{u}^{n+1}\right)^{.3} + (\boldsymbol{u}^n)^{.3}}{2}.
\end{aligned}$$

We rewrite it as

$$\begin{aligned}
&\left(1 - \frac{\tau}{2}\right) e^{n+1} - \frac{\tau\epsilon^2}{2} D_h e^{n+1} \\
&= \left[\left(1 + \frac{\tau}{2}\right) e^n + \frac{\tau\epsilon^2}{2} D_h e^n\right] + \left[\frac{\tau}{2}\left(U^n\right)^{.3} - \frac{\tau}{2}\left(\boldsymbol{u}^n\right)^{.3}\right] \\
&\quad + \left[\frac{\tau}{2}\left(U^{n+1}\right)^{.3} - \frac{\tau}{2}\left(\boldsymbol{u}^{n+1}\right)^{.3}\right] - \left[\frac{\tau}{4}\left(U^{n+1} - U^n\right)^{.2}\left(U^{n+1} + U^n\right)\right] \\
&\quad + \frac{\tau}{12}\left(U^{n+1} - U^n\right)^{.3} + \beta\tau\left(U^{n+1} - U^n\right) + \tau\boldsymbol{\rho}^n =: \sum_{i=1}^{7} Q_i.
\end{aligned} \tag{18}$$

For $\tau \leq \frac{1}{2}$, we get

$$\left\| \left(1 - \frac{\tau}{2}\right) e^{n+1} - \frac{\tau \epsilon^2}{2} D_h e^{n+1} \right\|_\infty \geq \left(1 - \frac{\tau}{2}\right) \left\| e^{n+1} \right\|_\infty . \tag{19}$$

Next, we estimate Q_1–Q_7 respectively. For $\tau \leq \frac{h^\alpha \Gamma(\frac{\alpha}{2}+1)^2}{\epsilon^2 \Gamma(\alpha+1)}$, we have

$$\|Q_1\|_\infty \leq \left\| e^n + \frac{\tau}{2} \epsilon^2 D_h e^n \right\|_\infty + \frac{\tau}{2} \left\| e^n \right\|_\infty \leq \left(1 + \frac{\tau}{2}\right) \left\| e^n \right\|_\infty . \tag{20}$$

Since equation (1) preserves the maximum bound principle, we know that $\|\boldsymbol{u}^n\|_\infty \leq 1$ for any $n \geq 0$. So, since $\|\boldsymbol{u}^n\|_\infty \leq 1$, $\|U^n\|_\infty \leq 1$, $\left\|\boldsymbol{u}^{n+1}\right\|_\infty \leq 1$, and $\left\|U^{n+1}\right\|_\infty \leq 1$, we can estimate Q_2 and Q_3 as

$$\begin{aligned} \|Q_2\|_\infty &= \left\| -\frac{\tau}{2} \left(\boldsymbol{u}^n\right)^{.3} + \frac{\tau}{2} \left(U^n\right)^{.3} \right\|_\infty \\ &= \left\| -\frac{\tau}{2} \left[\left(\boldsymbol{u}^n\right)^{.2} + \left(U^n\right)^{.2} + \boldsymbol{u}^n U^n \right] e^n \right\|_\infty \\ &\leq \frac{3}{2} \tau \left\| e^n \right\|_\infty \end{aligned} \tag{21}$$

and

$$\begin{aligned} \|Q_3\|_\infty &= \left\| -\frac{\tau}{2} \left(\boldsymbol{u}^{n+1}\right)^{.3} + \frac{\tau}{2} \left(U^{n+1}\right)^{.3} \right\|_\infty \\ &= \left\| -\frac{\tau}{2} \left[\left(\boldsymbol{u}^{n+1}\right)^{.2} + \left(U^{n+1}\right)^{.2} + \boldsymbol{u}^{n+1} U^{n+1} \right] e^{n+1} \right\|_\infty \\ &\leq \frac{3}{2} \tau \left\| e^{n+1} \right\|_\infty . \end{aligned} \tag{22}$$

Using $\|\boldsymbol{u}^n\|_\infty \leq 1$, $\|U^n\|_\infty \leq 1$, $\left\|\boldsymbol{u}^{n+1}\right\|_\infty \leq 1$, $\left\|U^{n+1}\right\|_\infty \leq 1$, and Cauchy mean value theorem, we conclude that

$$\begin{aligned} \|Q_4\|_\infty &= \left\| \frac{\tau}{4} \left(U^{n+1} - U^n\right)^{.2} \left(U^{n+1} + U^n\right) \right\|_\infty \\ &\leq \left\| \frac{\tau}{4} \left(U^{n+1} + U^n\right) \left[\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.2} - \left(U^{n+1} - U^n\right)^{.2} \right] \right\|_\infty \\ &\quad + \left\| \frac{\tau}{4} \left(U^{n+1} + U^n\right) \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.2} \right\|_\infty \\ &\leq \left\| \frac{\tau}{4} \left(U^{n+1} + U^n\right) \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n + U^{n+1} - U^n\right) \left(e^{n+1} - e^n\right) \right\|_\infty + C\tau^3 \\ &\leq 2\tau \left\| e^{n+1} \right\|_\infty + 2\tau \left\| e^n \right\|_\infty + C\tau^3, \end{aligned} \tag{23}$$

$$\begin{aligned} \|Q_5\|_\infty &\leq \left\| \frac{\tau}{12} \left(U^{n+1} - U^n\right)^{.3} - \frac{\tau}{12} \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.3} \right\|_\infty + \left\| \frac{\tau}{12} \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.3} \right\|_\infty \\ &\leq \left\| \frac{\tau}{12} \left[\left(U^{n+1} - U^n\right)^{.2} + \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.2} + \left(U^{n+1} - U^n\right) \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) \right] \right. \\ &\quad \left. \times \left(e^{n+1} - e^n\right) \right\|_\infty + \left\| \frac{\tau}{12} \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right)^{.3} \right\|_\infty \\ &\leq \tau \left\| e^{n+1} - e^n \right\|_\infty + C\tau^4 \\ &\leq \tau \left(\left\| e^{n+1} \right\|_\infty + \left\| e^n \right\|_\infty \right) + C\tau^3 \end{aligned} \tag{24}$$

and

$$\begin{aligned}\|Q_6\|_\infty &= \beta\tau^2 \left\|\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \left(e^{n+1} - e^n\right)\right\|_\infty \\ &\le \beta\tau^2 \left(\left\|\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right\|_\infty + \left\|e^{n+1}\right\|_\infty + \left\|e^n\right\|_\infty\right) \\ &\le C(\beta)\,\tau^3 + \beta\tau^2 \left\|e^{n+1}\right\|_\infty + \beta\tau^2 \left\|e^n\right\|_\infty .\end{aligned} \tag{25}$$

From (18)–(25), we find that

$$\left(1 - 5\tau - \beta\tau^2\right)\left\|e^{n+1}\right\|_\infty \le \left(1 + 5\tau + \beta\tau^2\right)\left\|e^n\right\|_\infty + C(\epsilon,\beta)\,\tau\left(\tau^2 + h^2\right),$$

namely

$$\left(1 - 5\tau - \beta\tau^2\right)\left(\left\|e^{n+1}\right\|_\infty - \left\|e^n\right\|_\infty\right) \le \left(10\tau + 2\beta\tau^2\right)\left\|e^n\right\|_\infty + C(\epsilon,\beta)\,\tau\left(\tau^2 + h^2\right).$$

Summing over n from 0 to $l-1$, we derive

$$\begin{aligned}&\left(1 - 5\tau - \beta\tau^2\right)\left(\left\|e^l\right\|_\infty - \left\|e^0\right\|_\infty\right) \\ &\le \sum_{n=0}^{l-1}\left(10\tau + 2\beta\tau^2\right)\left\|e^n\right\|_\infty + C(\epsilon,\beta,T)\left(\tau^2 + h^2\right),\end{aligned} \tag{26}$$

where $e^0 = \mathbf{0}$. If $1 - 5\tau - \beta\tau^2 \ge \frac{1}{2}$, namely $\tau \le \frac{\sqrt{25+2\beta}-5}{2\beta}$ for $\beta > 0$ and $\tau \le \frac{1}{10}$ for $\beta = 0$, then (26) becomes

$$\begin{aligned}\left\|e^l\right\|_\infty &\le \sum_{n=0}^{l-1}\left(20\tau + 4\beta\tau^2\right)\left\|e^n\right\|_\infty + C(\epsilon,\beta,T)\left(\tau^2 + h^2\right) \\ &\le \sum_{n=0}^{l-1}\left(10 + 2\sqrt{25 + 2\beta}\right)\tau\left\|e^n\right\|_\infty + C(\epsilon,\beta,T)\left(\tau^2 + h^2\right).\end{aligned} \tag{27}$$

Applying the discrete Gronwall' inequality to (27), we get

$$\left\|e^l\right\|_\infty \le C(\epsilon,\beta,T)\left(\tau^2 + h^2\right),$$

completing the proof of the theorem. □

5 The discrete energy stability

In this section, we consider the discrete energy stability for the scheme (8). Define the following discrete energy:

$$E_h(U) = \frac{h}{4}\sum_{i=1}^{N}(U_i^2 - 1)^2 - \frac{h\epsilon^2}{2}U^T D_h U.$$

Theorem 3 *The scheme* (8) *is unconditionally energy-stable, namely*

$$E_h(U^{n+1}) \le E_h(U^n), \quad n = 0, 1, \ldots, M-1. \tag{28}$$

Proof Taking the difference of the discrete energy between two successive time levels, we get

$$
\begin{aligned}
&E_h(U^{n+1}) - E_h(U^n) \\
&= \frac{h}{4}\sum_{i=1}^{N}\left[\left((U_i^{n+1})^2 - 1\right)^2 - \left((U_i^n)^2 - 1\right)^2\right] \\
&\quad - \frac{h\epsilon^2}{2}\left((U^{n+1})^T D_h U^{n+1} - (U^n)^T D_h U^n\right) \\
&= \frac{h}{4}\sum_{i=1}^{N}\left[(U_i^{n+1})^3 + (U_i^n)^3 + U_i^{n+1}(U_i^n)^2 + U_i^n(U_i^{n+1})^2 - 2(U_i^{n+1} + U_i^n)\right](U_i^{n+1} - U_i^n) \\
&\quad - \frac{h\epsilon^2}{2}(U^{n+1} - U^n)^T D_h (U^{n+1} + U^n),
\end{aligned}
\tag{29}
$$

where we used the symmetry of the matrix D_h.

Taking the L^2 inner product of (8) with $h(U^{n+1} - U^n)^T$, one obtains

$$
\begin{aligned}
&\frac{h}{\tau}\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^2 + \frac{h}{4}\sum_{i=1}^{N}\left[\left((U_i^{n+1})^2 - 1\right)^2 - \left((U_i^n)^2 - 1\right)^2\right] \\
&\quad + \frac{h}{12}\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^4 + h\beta\tau\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^2 \\
&= \frac{h\epsilon^2}{2}\left(U^{n+1} - U^n\right)^T D_h\left(U^{n+1} + U^n\right).
\end{aligned}
\tag{30}
$$

Thus, from (29)–(30), we conclude that

$$
\begin{aligned}
&E_h\left(U^{n+1}\right) - E_h\left(U^n\right) \\
&= \frac{h}{4}\sum_{i=1}^{N}\left[\left(1 - (U_i^{n+1})^2\right)^2 - \left(1 - (U_i^n)^2\right)^2\right] \\
&\quad - \frac{h\epsilon^2}{2}\left[\left(U^{n+1}\right)^T D_h U^{n+1} - \left(U^n\right)^T D_h U^n\right] \\
&= -\frac{h}{\tau}\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^2 - \frac{h}{12}\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^4 - h\beta\tau^2\sum_{i=1}^{N}(U_i^{n+1} - U_i^n)^2 \\
&\leq 0.
\end{aligned}
$$

The proof is ended. □

6 Numerical experiment

In this paper, we provide two numerical examples to validate the theoretical results. The standard Newton method is used to solve the scheme (8).

Table 1 The errors of $\|U^M - U^{2M}\|_\infty$ with $h = 1/500$, $\alpha = 1.5$, and $\beta = 1$.

τ	$\|U^M - U^{2M}\|_\infty$ ($T = 1$)	Order	$\|U^M - U^{2M}\|_\infty$ ($T = 2$)	Order
1/10	8.938867e – 04	–	4.066691e – 03	–
1/20	2.267380e – 04	1.9791	1.032046e – 03	1.9783
1/40	5.717947e – 05	1.9875	2.602393e – 04	1.9876
1/80	1.447069e – 05	1.9824	6.583920e – 05	1.9828

Table 2 The errors of $\|U^M - U^{2M}\|_\infty$ with $h = 1/500$, $\alpha = 1.4$, and $\beta = 2$.

τ	$\|U^M - U^{2M}\|_\infty$ ($T = 1$)	Order	$\|U^M - U^{2M}\|_\infty$ ($T = 2$)	Order
1/10	1.836433e – 03	–	8.388713e – 03	–
1/20	4.713861e – 04	1.9619	2.158389e – 03	1.9585
1/40	1.188754e – 04	1.9875	5.445636e – 04	1.9868
1/80	2.990516e – 05	1.9910	1.369927e – 04	1.9910

Table 3 The errors of $\|U^M - U^{2M}\|_\infty$ with $h = 1/500$, $\alpha = 1.6$, and $\beta = 3$.

τ	$\|U^M - U^{2M}\|_\infty$ ($T = 1$)	Order	$\|U^M - U^{2M}\|_\infty$ ($T = 2$)	Order
1/10	4.467633e – 03	–	2.033645e – 02	–
1/20	1.192390e – 03	1.9057	5.470029e – 03	1.8944
1/40	3.034203e – 04	1.9745	1.394373e – 03	1.9719
1/80	7.633834e – 05	1.9908	3.509387e – 04	1.9903

Example 1 We consider the 1D Allen–Cahn equation with the initial value

$$u_0(x) = 0.05 \sin 2\pi x, \quad x \in (0, 1).$$

For other corresponding data, we set $\epsilon = 0.01$.

We mainly test the convergence rate for temporal discretization. As no analytical solution is available for this numerical experiment, we define the numerical solution errors in the discrete L^∞ norm as $\|U^M - U^{2M}\|_\infty$. First, fix $h = 1/500$ and choose $(\alpha, \beta) = (1.5, 1)$, $(1.4, 2)$, $(1.6, 3)$. We display the errors of $\|U^M - U^{2M}\|_\infty$ for different τ in Tables 1–3, respectively. We find that the convergence orders of the errors are very close to 2. These are consistent with the convergence result obtained in Theorem 2.

Second, letting $(\alpha, \beta) = (1.5, 1)$ and $h = 1/100$, the supremum norm of the numerical solutions and the discrete energy with different τ are checked in Figs. 1–2, respectively. We can clearly see that the numerical solutions preserve the discrete maximum bound principle, and the discrete energy is decreasing for $\tau = 0.02$, 0.05, and 0.1. For $(\alpha, \beta) = (1.8, 4)$ and fixed $h = 1/100$, the same phenomenon can be observed in Figs. 3–4. These numerical results are also consistent with the theoretical results obtained in Theorems 1 and 3.

Example 2 We consider the 2D Allen–Cahn equation with the parameter $\epsilon = 0.01$ and the initial value

$$u_0(x, y) = 0.2(\sin 2x \sin 3y + \sin 4x \sin 5y), \quad x, y \in (0, \pi).$$

For $(\alpha, \beta) = (1.5, 1), (1.8, 0)$ and $h = \pi/50$, the supremum norm of the numerical solutions and the discrete energy with different τ are plotted in Figs. 5–10, respectively. We find that

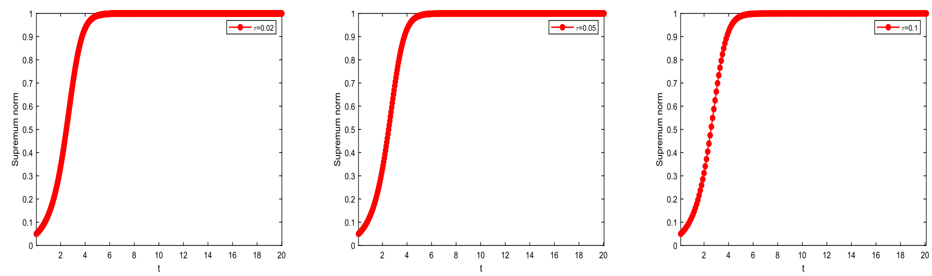

Figure 1 The supremum norm of numerical solutions for different τ

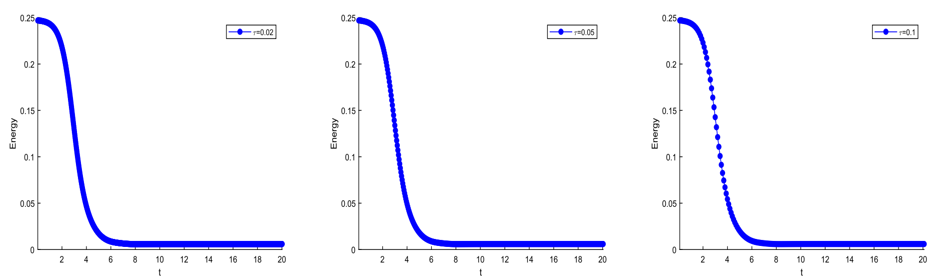

Figure 2 The discrete energy for different τ

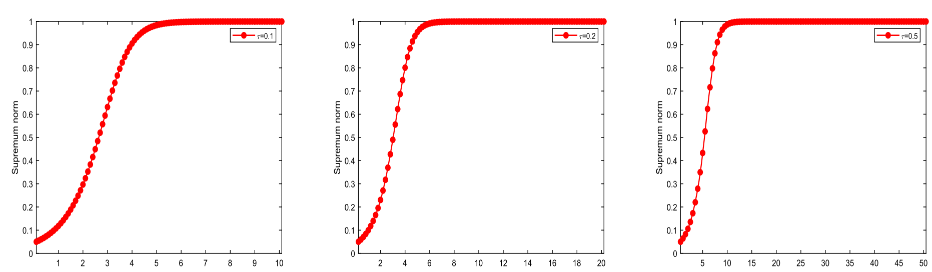

Figure 3 The supremum norm of numerical solutions for different τ

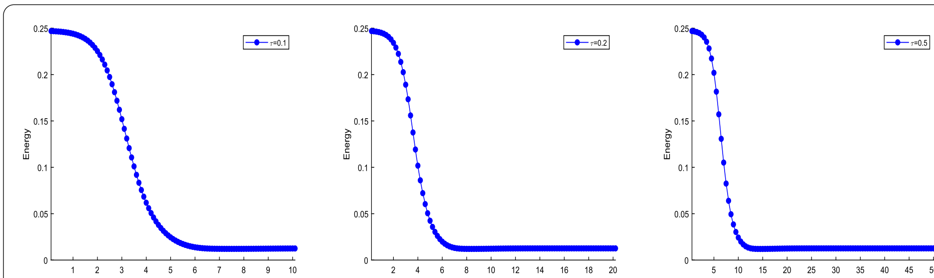

Figure 4 The discrete energy for different τ

the numerical solutions preserve the discrete maximum bound principle in Figs. 5 and 8. Moreover, we see from Figs. 6, 7, 9, and 10 that the discrete energy is decreasing even if $\tau = 1$.

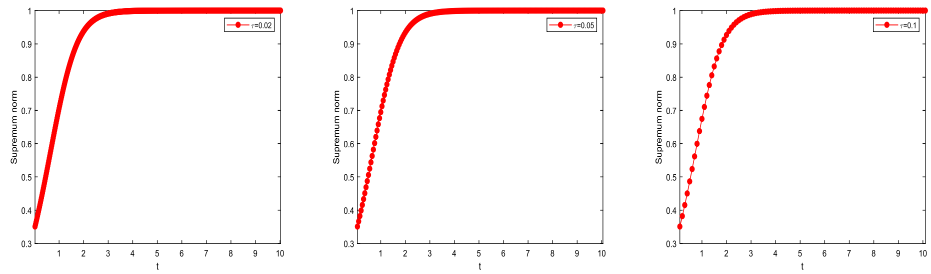

Figure 5 The supremum norm of numerical solutions for different τ

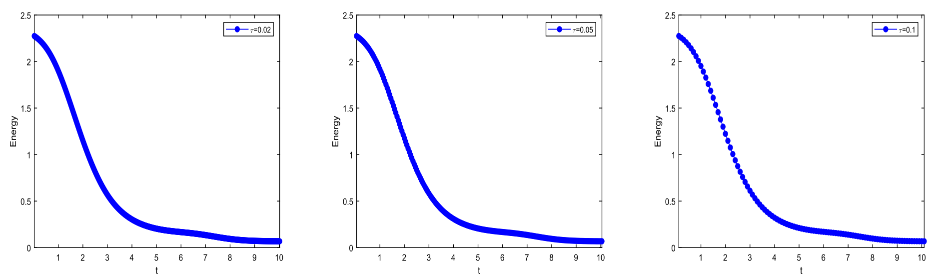

Figure 6 The discrete energy for different τ

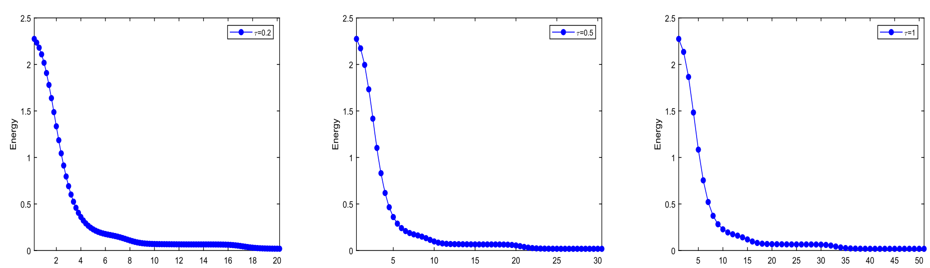

Figure 7 The discrete energy for different τ

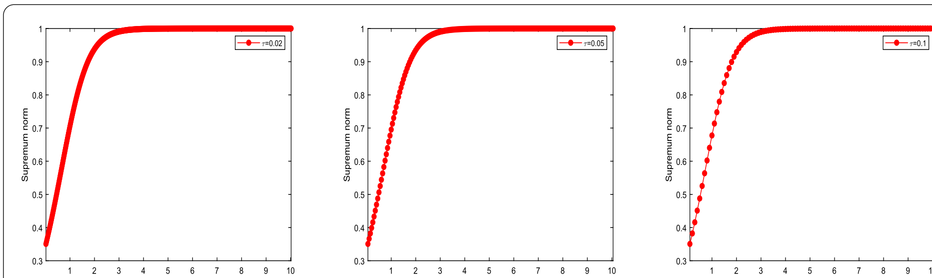

Figure 8 The supremum norm of numerical solutions for different τ

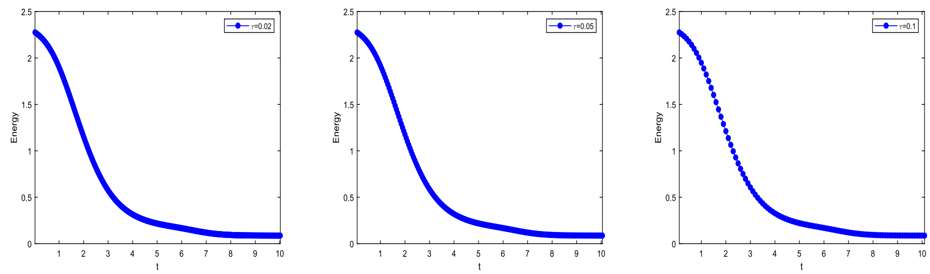

Figure 9 The discrete energy for different τ

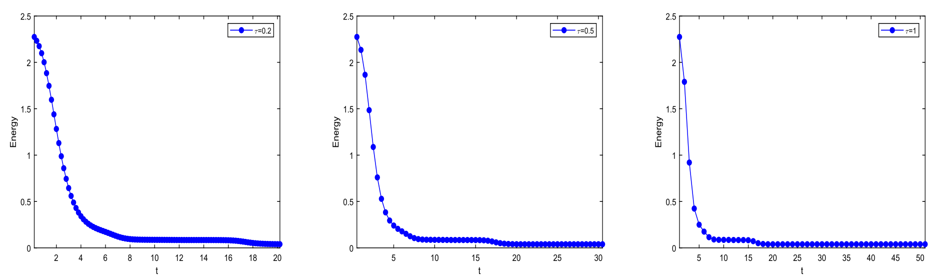

Figure 10 The discrete energy for different τ

Author contributions
Conceptualization, TH; methodology, TH; software, CX and YC; validation, CX and YC; formal analysis, YC and TH; writing-original draft preparation, CX and YC; writing-review and editing, TH; funding acquisition, CX and TH. All authors have read and agreed to the published version of the manuscript.

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Data availability
The data used to support the findings of this study are included within the article.

Declarations

Competing interests
The authors declare no conflict of interest.

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