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On the existence-uniqueness and exponential estimate for solutions to stochastic functional differential equations driven by G-Lévy process

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Abstract

The existence-uniqueness theory for solutions to stochastic dynamic systems is always a significant theme and has received tremendous attention. This article aims to study the theory for stochastic functional differential equations (SFDEs) driven by the G-Lévy process. It derives the existence-uniqueness theorem for solutions to SFDEs driven by the G-Lévy process. Moreover, it shows the error estimation between the exact solution $x(t)$ and Picard approximate solutions $x^n(t), n \geq 1$. Ultimately, the exponential estimate has been derived.

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1 Introduction

Several authors have studied stochastic dynamic equations based on G-Brownian motion [1, 8, 10, 13, 26]. Among them, the existence-uniqueness, stability, moment estimates, continuity and differentiability properties of the solution with respect to the initial data have been explored in detail [4, 7, 14, 17, 19, 22, 27]. Stochastic differential equations (SDEs) based on the Lévy process perform a leading role in a broad range of applications, containing financial mathematics for describing the observed reality of financial markets [2], physics for various phenomena [25], genetics for the movement designs of many animals [9] and biology for modeling the spread of diseases [12]. In [11] Hu and Peng initiated the G-Lévy process. In [18] Ren represented a sublinear expectation related to the framework of G-Lévy process as an upper expectation. Paczka then inaugurated the integrals and Itô formula based on the G-Lévy process [15]. The existence and exponential estimates for solutions to SDEs driven by G-Lévy process were established by Wang and Gao [23]. They also constructed the BDG-type inequality in the stated framework [23]. The existence theory for solutions to SDEs based on G-Lévy process having discontinuous coefficients was given by Wang and Yuan [24]. The quasi-sure exponential stability of

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SDEs in the framework of G-Lévy process was initiated by Shen et. al. [21]. To the best of our knowledge, no text is available on the study of existence-uniqueness and exponential estimate for solutions to stochastic functional differential equations (SFDEs) driven by G-Lévy process. Consequently, the current research aims at working on the stated theme. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$. Consider $BC((-\infty,0];\mathbb{R}^d)$, the family of bounded continuous $\mathbb{R}^d$-valued mappings ψ defined on $(-\infty,0]$ with norm $\|\psi\| = \sup_{-\infty<\theta\leq 0}|\psi(\theta)|$ [20]. Let $\mathcal{F}_t = \sigma\{\mathcal{B}(v) : 0 \leq v \leq t\}$ be the natural filtration defined on a complete probability space $(\mathcal{S},\mathcal{F},P)$. Assume that $\{\mathcal{F}_t : t \geq 0\}$ assures the usual characteristics. Let $f : [0,T] \times BC((-\infty,0];\mathbb{R}^d) \to \mathbb{R}^d$, $g : [0,T] \times BC((-\infty,0];\mathbb{R}^d) \to R^{d\times m}$, $h : [0,T] \times BC((-\infty,0];\mathbb{R}^d) \to R^{d\times m}$ and $K : [0,T] \times BC((-\infty,0];\mathbb{R}^d) \to \mathbb{R}^{d\times m}$ be Borel measurable. We consider the following SFDE driven by G-Lévy process:

$$dx(t) = f(t,x_t)dt + g(t,x_t)d < B,B > (t) + h(t,x_t)dB(t) + \int_{R^d_0} K(t,x_{t-},z)L(dt,dz), \quad (1.1)$$

on $t \in [0,T]$ with initial condition $\varsigma(0) \in \mathbb{R}^d$, $x_t = \{x(t+\theta), -\infty < \theta \leq 0\}$ and x_{t-} indicates the left limits of x_t. $B(t)$ is a d-dimensional G-Brownian motion. The coefficients $f(.,x), g(.,x), h(.,x) \in \mathbb{M}^2_G((-\infty,T];\mathbb{R}^d)$ and $K(.,x,.) \in \mathcal{H}^2_G((-\infty,T] \times \mathbb{R}^d_0;\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. Please see [23]. Equation (1.1) has the following initial condition.

$$x_0 = \zeta = \{\zeta(\theta) : -\infty < \theta \leq 0\}, \quad (1.2)$$

is $\mathcal{F}_0$-measurable, $BC((-\infty,0];\mathbb{R}^d)$-value random variable such that $\zeta \in \mathbb{M}^2_G((-\infty,T];\mathbb{R}^d)$.

The rest of the article is arranged as follows. Basic results and definitions of the G-framework are given in Sect. 2. The existence and uniqueness of solutions to SFDEs driven by G-Lévy process is studied in Sect. 3. Here, the boundedness of solutions is determined. The error estimation between the exact and approximate solutions is shown. The exponential estimate for solutions to SFDEs driven by G-Lévy process is constructed in Sect. 4.

2 Fundamental settings

In this section, we include preliminary results and notions of the G-framework required for the subsequent sections of this article [3, 5, 6, 17]. Consider $\mathcal{S}_T = C_0([0,T],\mathbb{R}^d)$, the space of real-valued continuous mappings on $[0,T]$ such that $w(0) = 0$ endowed with the distance

$$\rho(w^1,w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i}\Big(\max_{t\in[0,i]} |w^1(t) - w^2(t)| \wedge 1\Big).$$

Let for any $w \in \mathcal{S}_T$ and $t \geq 0$, $B(t,w) = w(t)$ be the canonical process. Let $\mathcal{F}_t = \sigma\{B(v), 0 \leq v \leq t\}$ be the filtration generated by canonical process $\{B(t), t \geq 0\}$ and $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. For any $T > 0$, define $\mathcal{L}_{ip}(\mathcal{S}_T) = \{\phi(B(t_1),B(t_2),\dots,B(t_d)) : d \geq 1, t_1,t_2,\dots,t_d \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{d\times m}))\}$, where $C_{b.Lip}(\mathbb{R}^{d\times m})$ is a space of bounded Lipschitz functions. A functional $\mathbb{E}$ defined on $\mathcal{L}_{ip}(\mathcal{S}_T)$ is known as a sublinear expectation if it ensures the characteristics given as follows. For every $x, y \in \mathcal{L}_{ip}(\mathcal{S}_T)$

(**1**) Monotonicity: $\mathbb{E}[x] \geq \mathbb{E}[y]$ if $x \geq y$.

(**2**) Constant Preserving: For all $c \in \mathbb{R}$, $\mathbb{E}[c] = c$.

(**3**) Sub-additivity: $\mathbb{E}[x] + \mathbb{E}[y] \geq \mathbb{E}[x + y]$.

(**4**) Positive homogeneity: For all $\kappa > 0$, $\mathbb{E}[\kappa x] = \kappa\mathbb{E}[x]$.

For $t \leq T$, $\mathcal{L}_{ip}(\mathcal{S}_t) \subseteq \mathcal{L}_{ip}(\mathcal{S}_T)$ and $\mathcal{L}_{ip}(\mathcal{S}) = \cup_{n=1}^{\infty}\mathcal{L}_{ip}(\mathcal{S}_n)$. For $p \geq 1$, $\mathcal{L}_G^p(\mathcal{S})$ indicates the completion of $\mathcal{L}_{ip}(\mathcal{S})$ endowed with the Banach norm $\hat{\mathbb{E}}[|.|^p]^{\frac{1}{p}}$ and $\mathcal{L}_G^p(\mathcal{S}_t) \subseteq \mathcal{L}_G^p(\mathcal{S}_T) \subseteq \mathcal{L}_G^p(\mathcal{S})$ for $0 \leq t \leq T < \infty$. The triple $(\mathcal{S}, \mathcal{L}_{ip}(\mathcal{S}_T), \mathbb{E})$ is recognized as a sublinear expectation space. For $p \geq 1$, a partition of $[0, T]$ is a finite order subset $\{\mathcal{A}_T^{\mathbb{N}} : \mathbb{N} \geq 1\}$ so that $\mathcal{A}_T^{\mathbb{N}} : 0 = t_0 < t_1 < \cdots < t_{\mathbb{N}} = T\}$. The space $\mathbb{M}_G^{p,0}([0, T])$, $p \geq 1$ of simple processes is defined by

$$\mathbb{M}_G^{p,0}([0,T]) = \left\{\eta_t(z) = \sum_{i=0}^{\mathbb{N}-1}\xi_{t_i}(z)I_{[t_i,t_{i+1}]}(t);\ \xi_{t_i}(z) \in \mathcal{L}_G^p(\Omega_{t_i})\right\}. \tag{2.1}$$

The completion of space (2.1) equipped with the norm $\|\eta\| = \left\{\int_0^T \mathbb{E}[|\eta(s)|^p]ds\right\}^{1/p}$ is indicated by $\mathbb{M}_G^p(0, T)$, $p \geq 1$.

Definition 2.1 Let $\eta_t \in \mathbb{M}_G^p(0, T)$, $p \geq 1$. Then the G-Itô's integral is defined by

$$\int_0^T \eta(s)dB(s) = \sum_{i=0}^{\mathbb{N}-1}\xi_i\Big(B(t_{i+1}) - B(t_i)\Big).$$

Definition 2.2 [16] For a partition $0 = t_0 < t_1 < \cdots < t_{\mathbb{N}-1} = t$, the quadratic variation process $\{\langle B\rangle(t)\}_{t\geq 0}$ is defined by

$$\langle B\rangle(t) = \lim_{\mathbb{N}\to\infty}\sum_{i=0}^{\mathbb{N}-1}\Big(B(t_{i+1}^{\mathbb{N}}) - B(t_i^{\mathbb{N}})\Big)^2 = B(t)^2 - 2\int_0^t B(s)dB(s).$$

A mapping $\Pi_{0,T} : \mathbb{M}_G^{0,1}(0, T) \mapsto \mathcal{L}_G^2(\mathcal{F}_T)$ is given by

$$\Pi_{0,T}(\eta) = \int_0^T \eta(s)d\langle B\rangle(s) = \sum_{i=0}^{\mathbb{N}-1}\xi_i\Big(\langle B\rangle_{(t_{i+1})} - \langle B\rangle(t_i)\Big).$$

It can be extended to $\mathbb{M}_G^1(0, T)$ and for $\eta \in \mathbb{M}_G^1(0, T)$ this is still given by

$$\int_0^T \eta(s)d\langle B\rangle(s) = \Pi_{0,T}(\eta).$$

Let $\mathcal{Q}$ be a weakly compact set representing $\mathbb{E}$. The capacity $\hat{\nu}$ is given as the following

$$\hat{\nu}(A) = \sup_{\mathbb{P}\in\mathcal{Q}}\mathbb{P}(A),\quad A \in \mathcal{F}_T.$$

The set A is polar if $\hat{\nu}(A) = 0$. A characteristic holds quasi-surely (q.s) if it sustains outside a polar set.

Lemma 2.3 [3] *Let $x \in \mathcal{L}_G^p$ and $\hat{\mathbb{E}}|x|^p < \infty$. Then*

$$\hat{\nu}(|x| > c) \leq \frac{\mathbb{E}[|x|^p]}{c},$$

for any $c > 0$.

The proof of the Lemmas 2.4 and 2.5 can be seen in [10].

Lemma 2.4 *Let* $\lambda \in \mathbb{M}_G^p(0,T)$, $p \geq 2$. *Then*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \lambda(s)dB(s)\Big|^p\Big] \leq \alpha\mathbb{E}\Big[\int_0^t |\lambda(s)|^2 ds\Big]^{\frac{p}{2}},$$

where $0 < \alpha = k_2 T^{\frac{p}{2}-1} < \infty$, k_2 *is a positive constant depending on* p.

Lemma 2.5 *Let* $\lambda \in \mathbb{M}_G^p(0,T)$, $p \geq 1$. *Then*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \lambda(s)d\langle B,B\rangle(s)\Big|^p\Big] \leq \beta\mathbb{E}\Big[\int_0^t |\lambda(s)|^2 ds\Big]^{\frac{p}{2}},$$

where $0 < \beta = k_1 T^{p-1} < \infty$ *and* k_1 *is a positive constant depending on* p.

Definition 2.6 [15, 21] A stochastic process $\{x(t), t \geq 0\}$ defined on a sublinear expectation space $(\mathcal{S}, \mathcal{L}_{ip}(\mathcal{S}_T), \mathbb{E})$ is known as a a G-Lévy process if it ensures the upcoming five characteristics:

(**1**) $x(t) = 0$.
(**2**) For any $t, s \geq 0$, the increment $x(t+s) - x(s)$ is independent of $x(t_1), x(t_2), \dots, x(t_n)$, $\forall$ $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2, \dots, \leq t_n \leq t$.
(**3**) For every $s, t \geq 0$, the distribution $x(t+s) - x(s)$ does not depend on t.
(**4**) For each $t \geq 0$ there exists a decomposition $x(t) = x^c(t) + x^d(t)$.
(**5**) $(x^c(t), x^d(t))_{t\geq 0}$ is a $2d$-dimensional Lévey process satisfying

$$\lim_{t\downarrow 0}\frac{\mathbb{E}[|x^c(t)|^3]}{t} = 0, \quad \mathbb{E}[|x^d(t)|] \leq \alpha t, \quad t \geq 0,$$

where α is a constant depends on x.

If $\{x(t), t \geq 0\}$ satisfies only the first three properties, i.e., (1)–(3), then it is the classical Lévy process. It is known that $x^c(t)$ is generalized G-Brownian motion and $x^d(t)$ is of finite variation, where $x^c(t)$ and $x^d(t)$ are continuous part and jump part, respectively. Let $\mathcal{H}_G^\delta([0,T]\times\mathbb{R}_0^d)$ be a collection of all basic fields defined on $[0,T]\times\mathbb{R}_0^d\times\mathcal{S}$ of the form

$$K(u,z)(w) = \sum_{i=1}^{n-1}\sum_{j=1}^{m}\Lambda_{i,j}1_{(t_i,t_{i+1}]}(u)\psi_j(z),$$

where $n, m \in \mathbb{N}$ and $0 \leq t_1 < t_2 \dots . < t_n \leq T$, $\{\psi_j\}_{j=1}^m \subset C_{b.lip}(\mathbb{R}^d)$ are mappings with disjoint supports such that $\psi_j(0) = 0$ and $\Lambda_{i,j} = \phi_{i,j}(x_{t_1}, \dots, x_{t_i} - x_{t_{i-1}})$, $\phi_{i,j} \in C_{b.lip}(\mathbb{R}^{d\times i})$. The norm on this space is given by

$$\|K\|_{\mathcal{H}_G^p([0,T]\times\mathbb{R}_0^d)} = \mathbb{E}\Big[\int_0^T \sup_{v\in\nu}\int_{\mathbb{R}_0^d}|K(s,z)|^p v(dz)ds\Big]^{\frac{1}{p}}, \quad p = 1, 2.$$

Definition 2.7 The Itô integral of $K \in \mathcal{H}_G^{\delta}([0,T] \times \mathbb{R}_0^d)$ w. r. t. jump measure L is given as follows

$$\int_0^t \int_{\mathbb{R}_0^d} K(s,z)L(ds,dz) = \sum_{v<s\leq t} K\Big(s, \triangle x(s)\Big), \quad q.s.,$$

where $0 \leq v < t \leq T$.

Let $\mathcal{H}_G^p([0,T] \times \mathbb{R}_0^d)$ be the topological completion of $\mathcal{H}_G^{\delta}([0,T] \times \mathbb{R}_0^d)$ under the norm $\|K\|_{H_G^p([0,T]\times\mathbb{R}_0^d)}$, $p = 1,2$. We can sill extend the Itô integral to the space $\|K\|_{H_G^p([0,T]\times\mathbb{R}_0^d)}$, $p = 1,2$, where the extended integral has valves in $\mathcal{L}_G^p(\mathcal{S}_T)$, $p = 1,2$. For the above integrals, we have the following BDG-type inequality. For the proof, see [23].

Lemma 2.8 *Let* $K(s,z) \in \mathcal{H}_G^2([0,T] \times \mathbb{R}_0^d)$. *Then a càdlàg modification* $\hat{x}(t)$ *of* $x(t) = \int_0^t \int_{\mathbb{R}_0^d} K(s,z)L(ds,dz)$ *exists such that for all* $t \in [0,T]$ *and* $p \geq 2$

$$\mathbb{E}\Big[\sup_{0\leq s\leq t} |\hat{x}(t)|^2\Big] \leq k_3 \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}_0^d} K^2(s,z)\nu(dz)ds\Big],$$

where k_3 *is a positive constant depending on* T.

3 Bounded-ness and existence-uniqueness results for SFDEs driven by G-Lévy process

In this section, we shall determine the boundedness and existence-uniqueness results for solutions to problem (1.1). Let us first see the definition of solutions to equation (1.1).

Definition 3.1 An $\mathcal{F}_t$-adopted càdlàg process $x(t) \in \mathbb{M}_G^2((-\infty,T];\mathbb{R}^d)$ is called a solution to (1.1) with the initial data (1.2) if it satisfies

$$x(t) = \zeta(0) + \int_0^t f(s,x_s)ds + \int_0^t g(s,x_s)d\langle B,B\rangle(s) + \int_0^t h(s,x_s)dB(s) + \int_0^t \int_{\mathbb{R}_0^d} K(s,x_s-,z)L(ds,dz).$$

A solution $x(t)$ of (1.1) is said to be unique if it is identical to any other solution $y(t)$ of the stated equation, i.e.,

$$\mathbb{E}[|x(t) - y(t)|^2] = 0,$$

holds q.s.

This article proposes the upcoming linear growth and Lipschitz conditions, respectively.
($\mathbf{A_1}$) For every $x \in BC((-\infty,0];\mathbb{R}^d)$, a positive number c_1 exists so that

$$|f(t,x)|^2 \vee |g(t,x)|^2 \vee |h(t,x)|^2 \vee \int_{\mathbb{R}_0^d} |K(t,x,z)|^2 \upsilon(dz) \leq c_1(1+|x|^2)$$

($\mathbf{A_2}$) For all $x, y \in BC((-\infty, 0]; \mathbb{R}^d)$, a positive number c_2 exists so that

$$| f(t,y) - f(t,x) |^2 \vee | g(t,y) - g(t,x) |^2 \vee | h(t,y) - h(t,x) |^2$$
$$\vee \int_{\mathbb{R}_0^d} | K(t,y,z) - K(t,x,z |^2 \upsilon(dz) \leq c_2 | y - x |^2 .$$

In the forthcoming lemma, we prove that any solution $x(t)$ of equation (1.1) is bounded, in particular $x(t) \in \mathbb{M}_G^2\Big((-\infty, T]; \mathbb{R}^d\Big)$.

Lemma 3.2 *Let $x(t)$ be a solution of equation (1.1) with initial data (1.2) such that $\mathbb{E}\|x\|^2 \leq \infty$. Assume that the growth condition A_1 holds. Then*

$$\mathbb{E}\Big[\sup_{-\infty \leq s \leq t} | x(s) |^2 \Big] \leq \mathbb{E}\| \zeta \|^2 + 5(1 + c_1 kT)e^{5c_1 kT}, \tag{3.1}$$

where $k = (1 + k_1)T + k_2 + k_3$ and k_1, k_2, k_3 are positive constants.

Proof Consider equation (1.1) and use the basic inequality $| \sum_{i=1}^5 a_i|^2 \leq 5 \sum_{i=1}^5 | a_i|^2$ to derive

$$| x(t) |^2 \leq 5 | \zeta(0) |^2 + 5\Big| \int_0^t f(s,x_s)ds\Big|^2 + 5\Big| \int_0^t g(s,x_s)d\langle B,B\rangle(s)\Big|^2 +$$
$$5\Big| \int_0^t h(s,x_s)dB(s)\Big|^2 + 5\Big| \int_0^t \int_{\mathbb{R}_0^d} K(s,x_s-,z)L(ds,dz)\Big|^2.$$

From the G- expectation, Lemmas 2.4, 2.5, 2.8 and the Cauchy inequality, we get

$$\mathbb{E}\Big[\sup_{0 \leq s \leq t} | x(s) |^2 \Big] \leq 5\mathbb{E} | \zeta(0) |^2 + 5t\mathbb{E} \int_0^t |f(s,x_s)ds|^2 + 5k_1 t\mathbb{E} \int_0^t |g(s,x_s)|^2 ds$$
$$+ 5k_2 \mathbb{E} \int_0^t |h(s,x_s)|^2 ds + 5k_3 \mathbb{E} \int_0^t \int_{\mathbb{R}_0^d} |K(s,x_s-,z)|^2 \upsilon(dz)ds.$$

In view of assumption A_1, we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0 \leq s \leq t} | x(s) |^2 \Big] \\
&\leq 5\mathbb{E}\| \zeta \|^2 + 5c_1(T + k_1 T + k_2 + k_3)T + 5c_1(T + k_1 T + k_2 + k_3)T \int_0^t \mathbb{E}|x_s|^2 ds \\
&\leq 5\mathbb{E}\| \zeta \|^2 + 5c_1(T + k_1 T + k_2 + k_3)T \\
&\quad + 5c_1(T + k_1 T + k_2 + k_3) \int_0^t \Big[\mathbb{E}\| \zeta \|^2 + \mathbb{E}\Big(\sup_{0 \leq u \leq s} |x(u)|^2\Big)\Big]ds \\
&\leq 5\mathbb{E}\| \zeta \|^2 + 5c_1 kT + 5c_1 kT\mathbb{E}\| \zeta \|^2 + 5c_1 k \int_0^t \mathbb{E}\Big[\sup_{0 \leq u \leq s} |x(u)|^2\Big]ds
\end{aligned}$$

where $\hat{k} = T + k_1 T + k_2 + k_3$. From the Grownwall inequality, it follows

$$\mathbb{E}\Big[\sup_{0 \leq s \leq t} |x(s)|^2\Big] \leq 5[(1 + c_1 \hat{k} T)\mathbb{E}\| \zeta \|^2 + c_1 \hat{k} T]e^{5c_1 \hat{k} T} \tag{3.2}$$

Noticing that

$$\mathbb{E}\Big[\sup_{-\infty<s\leq t}|x(s)|^2\Big]\leq \mathbb{E}\|\zeta\|^2+\mathbb{E}\Big[\sup_{0\leq s\leq t}|x(s)|^2\Big],$$

it yields

$$\mathbb{E}\Big[\sup_{-\infty\leq s\leq t}|x(s)|^2\Big]\leq \mathbb{E}\|\zeta\|^2+5[(1+c_1\hat{k}T)\mathbb{E}\|\zeta\|^2+c_1\hat{k}T]e^{5c_1\hat{k}T}.$$

Letting $t=T$, we deduce the desired expression. □

For $t\in[0,T]$, define $x^0(t)=\zeta(0)$ and $x_0^0=\zeta$. For each $n=1,2,\dots$, we set $x_0^n=\zeta$ and define the Picard iteration,

$$\begin{aligned} x^n(t)&=\zeta(0)+\int_0^t f(s,x_s^{n-1})ds+\int_0^t g(s,x_s^{n-1})d\langle B,B\rangle(s)+\int_0^t h(s,x_s^{n-1})dB(s)\\ &\quad+\int_0^t\int_{\mathbb{R}_0^d}K(s,x_{s-}^{n-1},z)L(ds,dz)\quad t\in[0,T]. \end{aligned}\tag{3.3}$$

Next, we prove the existence-uniqueness result and error estimation between the exact solution $x(t)$ and Picard approximate solutions $x^n(t), n\geq 1$.

Theorem 3.3 *Let assumptions A_1 and A_2 hold and $\mathbb{E}\|\zeta\|^2<\infty$. Then equation (1.1) admits a unique càdlàg solution $x(t)\in\mathbb{M}_G^2((-\infty,T];\mathbb{R}^d)$. Moreover, for all $n\geq 1$, the Picard approximate solutions $x^n(t)$ and unique exact solution $x(t)$ of (1.1) satisfy that*

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)-x(s)|^2\Big]\leq \frac{C[Mt]^n}{n!}e^{Mt},$$

where $C=4c_2[(T+Tk_1+k_2+k_3)(1+\mathbb{E}\|\zeta\|^2)T, M=4c_2[(T+Tk_1+k_2+k_3)$ and k_1,k_2,k_3 are positive constants.

Proof Consider the Picard iteration sequence $\{x^n, n\geq 1\}$ given by (3.3). Obviously, $x^0(t)\in\mathbb{M}_G^2\Big((-\infty,T];\mathbb{R}^d\Big)$. From the fundamental inequality $|\sum_{i=1}^5 a_i|^2\leq 5\sum_{i=1}^5|a_i|^2$, Lemmas 2.4, 2.5, 2.8, the Cauchy inequality and assumption A_1, we deduce

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)|^2\Big]&\leq 5\mathbb{E}\|\zeta\|^2+5c_1\hat{k}T+5c_1\hat{k}T\mathbb{E}\|\zeta\|^2\\ &\quad+5c_1\hat{k}\int_0^t\mathbb{E}\Big[\sup_{0\leq u\leq s}|x^{n-1}(u)|^2\Big]ds, \end{aligned}$$

where $\hat{k}=(1+k_1)T+k_2+k_3$. Noticing that

$$\begin{aligned} \max_{1\leq n\leq j}\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^{n-1}(s)|^2\Big]&\leq \max\Big\{\mathbb{E}\|\zeta\|^2,\max_{1\leq n\leq j}\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)|^2\Big]\Big\}\\ &\leq \mathbb{E}\|\zeta\|^2+\max_{1\leq n\leq j}\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)|^2\Big], \end{aligned}$$

it follows

$$\max_{1\le n\le j}\mathbb{E}\Big[\sup_{0\le s\le t}|x^n(s)|^2\Big]$$
$$\le 5\mathbb{E}\|\zeta\|^2+5c_1\hat{k}T+10c_1\hat{k}T\mathbb{E}\|\zeta\|^2+5c_1\hat{k}\int_0^t\max_{1\le n\le j}\mathbb{E}\Big[\sup_{0\le u\le s}|x^n(u)|^2\Big]ds.$$

The Grownwall inequality yields

$$\max_{1\le n\le j}\mathbb{E}\Big[\sup_{0\le s\le t}|x^n(s)|^2\Big]\le 5[(1+2c_1\hat{k}T)\mathbb{E}\|\zeta\|^2+c_1\hat{k}T]e^{c_1\hat{k}t},$$

but j is arbitrary and letting $t=T$ it follows

$$\mathbb{E}\Big[\sup_{0\le s\le T}|x^n(s)|^2\Big]\le 5[(1+2c_1\hat{k}T)\mathbb{E}\|\zeta\|^2+c_1\hat{k}T]e^{c_1\hat{k}T}. \tag{3.4}$$

From the sequence $\{x^n(t); t\ge 0\}$ defined by (3.3), we have

$$x^1(t)-x^0(t)=\int_0^t f(s,x_s^0)ds+\int_0^t g(s,x_s^0)d\langle B,B\rangle(s)+\int_0^t h(s,x_s^0)dB(s)$$
$$+\int_0^t\int_{\mathbb{R}_0^d}K(s,x_{s-}^0,z)L(ds,dz).$$

In view of G-expectation, Lemmas 2.4, 2.5, 2.8, the Cauchy inequality and assumption A_1, we derive

$$\mathbb{E}\Big[\sup_{0\le s\le t}|x^1(s)-x^0(s)|^2\Big]\le 4c_1[(T+Tk_1+k_2+k_3)\int_0^t(1+\mathbb{E}\|\zeta\|^2]ds\le C,$$

where $C=4c_1[(T+Tk_1+k_2+k_3)(1+E\|\zeta\|^2)T$. Next, by similar arguments and assumption A_2, it follows

$$\mathbb{E}\Big[\sup_{0\le s\le t}|x^2(s)-x^1(s)|^2\Big]\le 4c_2(T+Tk_1+k_2+k_3)\int_0^t\mathbb{E}|x_s^1-x_s^0|^2ds$$
$$\le 4c_2(T+Tk_1+k_2+k_3)\int_0^t\mathbb{E}\Big[\sup_{0\le v\le s}|x^1(v)-x^0(v)|^2\Big]ds$$
$$\le 4c_2(T+Tk_1+k_2+k_3)Ct.$$

Similarly, we derive

$$\mathbb{E}\Big[\sup_{0\le s\le t}|x^3(s)-x^2(s)|^2\Big]\le C[4c_2[(T+Tk_1+k_2+k_3)]^2\frac{t^2}{2!}.$$

Thus for all $n\ge 0$, we claim that

$$\mathbb{E}\Big[\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2\Big]ds\le C\frac{[Mt]^n}{n!}, \tag{3.5}$$

where $C = 4c_2[(T + Tk_1 + k_2 + k_3)(1 + E\|\zeta\|^2)T$ and $M = 4c_2(T + Tk_1 + k_2 + k_3)$. With the mathematical induction, we verify that for all $n \geq 0$, (3.5) holds. For $n = 0$, it has been proved. Let (3.5) holds for some $n \geq 0$. By using similar arguments as above, we derive

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2\Big]ds &\leq 4c_2(T+Tk_1+k_2+k_3)\int_0^t \mathbb{E}|x_s^{n+1}-x_s^n|^2 ds\\ &\leq M\int_0^t \mathbb{E}\Big[\sup_{0\leq v\leq s}|x^{n+1}(v)-x^n(v)|^2\Big]ds\\ &\leq M\int_0^t \frac{C[Mt]^n}{n!}ds \leq C[M]^{n+1}\frac{t^{n+1}}{(n+1)!} = \frac{C[Mt]^{n+1}}{(n+1)!}.\end{aligned}$$

This shows that (3.5) holds for $n + 1$. Thus by, induction (3.5) holds for all $n \geq 0$. By virtue of Lemma 2.3, we acquire

$$\hat{\nu}\Big\{\sup_{0\leq s\leq T}|x^{n+1}(s)-x^n(s)|^2 > \frac{1}{2^n}\Big\} \leq 2^n\mathbb{E}\Big[\sup_{0\leq s\leq T}|x^{n+1}(s)-x^n(s)|^2\Big] \leq \frac{K[2Mt]^n}{n!}.$$

Since $\sum_{n=0}^{\infty}\frac{K[2Mt]^n}{n!} < \infty$, from the Borel–Cantelli lemma, we get that for almost all w a positive integer $n_0 = n_0(w)$ exists so that

$$\sup_{0\leq t\leq T}|x^{n+1}(t)-x^n(t)|^2 \leq \frac{1}{2^n}, \text{ as } n \geq n_0. \tag{3.6}$$

It implies that q.s., the partial sums

$$x^0(t) + \sum_{i=0}^{n-1}[x^{i+1}(t)-x^i(t)] = x^n(t),$$

are uniformly convergent on $t \in (-\infty, T]$. Denote the limit by $x(t)$. Then the sequence $x^n(t)$ converges uniformly to $x(t)$ on $t \in (-\infty, T]$. It follows that $x(t)$ is $\mathcal{F}_t$-adapted and càdlàg. Also, from (3.5), we can see that $\{x^n(t) : n \geq 1\}$ is a cauchy sequence in $\mathcal{L}_G^2$. Hence $x^n(t)$ converges to $x(t)$ in $\mathcal{L}_G^2$, that is,

$$\mathbb{E}|x^n(t)-x(t)|^2 \to 0, \text{ as } n \to \infty.$$

Taking limits $n \to \infty$ from (3.4), we deduce

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}|x(s)|^2\Big] \leq 5[(1+2c_1\hat{k}T)\mathbb{E}\|\zeta\|^2 + c_1\hat{k}T]e^{c_1\hat{k}T}. \tag{3.7}$$

Next, we need to verify that $x(t)$ satisfies equation (1.1). In view of assumption A_2 and using similar arguments as above, we derive

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq s\leq T}\int_0^t\Big|[f(s,x_s^n)-f(s,x_s)]ds\Big|^2\Big] + \mathbb{E}\Big[\sup_{0\leq s\leq T}\int_0^t\Big|[g(s,x_s^n)-g(s,x_s)]d\langle B,B\rangle(s)\Big|^2\Big]\\ &\quad + \mathbb{E}\Big[\sup_{0\leq s\leq T}\int_0^t\Big|[h(s,x_s^n)-h(s,x_s)]dB(s)\Big|^2\Big]\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\sup_{0\le s\le T}\int_0^t\int_{\mathbb{R}_0^d}\Big|[K(s,x^n_{s-},z)-K(s,x_{s-},z)]L(ds,dz)\Big|^2\Big]\\
&\le T\int_0^t\mathbb{E}|f(s,x^n_s)-f(s,x_s)|^2ds+Tk_1\int_0^t\mathbb{E}|g(s,x^n_s)-g(s,x_s)|^2ds\\
&\quad+k_2\int_0^t\mathbb{E}|h(s,x^n_s)-f(s,x_s)|^2ds+k_3\int_0^t\int_{\mathbb{R}_0^d}\mathbb{E}|K(s,x^n_{s-},z)-K(s,x_{s-},z)|^2\nu(dz)ds\\
&\le c_2(T+Tk_1+k_2+k_3)\int_0^t\mathbb{E}[\sup_{0\le v\le s}|x^n(v)-x(v)|^2]ds\to 0\text{ as, } n\to\infty,
\end{aligned}
$$

in other words

$$
\begin{aligned}
&\int_0^t f(s,x^n_s)\to\int_0^t f(s,x_s),\ \text{in } \mathcal{L}^2_G,\\
&\int_0^t h(s,x^n_s)d\langle B,B\rangle(s)\longrightarrow\int_0^t h(s,x_s)d\langle B,B\rangle(s),\ \text{in } \mathcal{L}^2_G,\\
&\int_0^t g(s,x^n_s)\longrightarrow\int_0^t g(s,x_s),\ \text{in } \mathcal{L}^2_G,\\
&\int_0^t h(s,x^n_s)dB(s)\longrightarrow\int_0^t h(s,x_s)dB(s),\ \text{in } \mathcal{L}^2_G,\\
&\int_0^t\int_{\mathbb{R}_0^d}K(s,x^{n-1}_{s-},z)L(ds,dz)ds\longrightarrow\int_0^t\int_{\mathbb{R}_0^d}K(s,x_{s-},z)L(ds,dz),\ \text{in } \mathcal{L}^2_G.
\end{aligned}
$$

For $t\in[0,T]$ taking limits $n\to\infty$ in (3.3), we derive

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}x^n(t)&=\zeta(0)+\int_0^t\lim_{n\longrightarrow\infty}f(s,x^{n-1}_s)ds+\int_0^t\lim_{n\longrightarrow\infty}g(s,x^{n-1}_s)d\langle B,B\rangle(s)\\
&\quad+\int_0^t\lim_{n\longrightarrow\infty}h(s,x^{n-1}_s)dB(s)+\int_0^t\lim_{n\longrightarrow\infty}K(s,x^{n-1}_{s-},z)L(ds,dz),
\end{aligned}
$$

which yields

$$
\begin{aligned}
x(t)&=\zeta(0)+\int_0^t f(s,x_s)ds+\int_0^t g(s,x_s)d\langle B,B\rangle(s)+\int_0^t h(s,x_s)dB(s)\\
&\quad+\int_0^t\int_{\mathbb{R}_0^d}K(s,x_s-,z)L(ds,dz),
\end{aligned}
$$

$t\in[0,T]$. This shows that $x(t)$ is the solution of (1.1). To prove the uniqueness, let us assume that equation (1.1) admits two solutions $x(t)$ and $y(t)$. Following similar arguments, we derive

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t}|y(s)-x(s)|^2\Big]\le 4t\int_0^t\mathbb{E}|f(s,y_s)-f(s,x_s)|^2ds+4k_1t\int_0^t\mathbb{E}|g(s,y_s)-g(s,x_s)|^2ds\\
&+4k_2\int_0^t\mathbb{E}|f(s,y_s)-f(s,x_s)|^2ds+4k_3\int_0^t\int_{\mathbb{R}_0^d}\mathbb{E}|K(s,y_{s-},z)-K(s,x_{s-},z)|^2\upsilon(dz)ds
\end{aligned}
$$

By virtue of assumption A_2, we deduce

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|y(s)-x(s)|^2\Big] \leq 4c_2(T+k_1T+k_2+k_3)\int_0^t \mathbb{E}\Big[\sup_{0\leq u\leq s}|y(u)-x(u)|^2\Big]ds.$$

We notice the fact $\sup_{-\infty<u\leq T}|y(u)|^2 \leq |\zeta|^2 + \sup_{0<u\leq T}|y(u)|^2$ [10]. From the Grownwall inequality and the same initial data, one can derive

$$\mathbb{E}\Big[\sup_{-\infty<s\leq t}|y(s)-x(s)|^2\Big] = 0 \tag{3.8}$$

which means $x(t) = y(t)$ quasi-surely, for all $t \in (-\infty, T]$. Finally, we have to prove the error estimation. From equations (1.1) and (3.3), using similar arguments as earlier it follows

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)-x(s)|^2\Big] \\
&\leq 4T\int_0^t \mathbb{E}\Big|f(s,x_s^n)-f(s,x_s)\Big|^2 ds + 4k_1T\int_0^t \mathbb{E}\Big|g(s,x_s^n)-g(s,x_s)\Big|^2 ds \\
&\quad + 4k_2\int_0^t \mathbb{E}\Big|h(s,x_s^n)-h(s,x_s)\Big|^2 ds + 4k_3\int_0^t\int_{\mathbb{R}_0^d} \mathbb{E}\Big|K(s,x_{s-}^n,z)-K(s,x_{s-},z)\Big|^2 \nu(dz)ds \\
&\leq 4c_2[(T+Tk_1+k_2+k_3)\int_0^t \mathbb{E}\Big[\sup_{0\leq v\leq s}\Big|x^n(v)-x(v)\Big|^2\Big]ds \\
&\leq M\int_0^t \mathbb{E}\Big[\sup_{0\leq v\leq s}\Big|x^n(v)-x^{n-1}(v)\Big|^2\Big]ds + M\int_0^t \mathbb{E}\Big[\sup_{0\leq v\leq s}\Big|x^{n-1}(v)-x(v)\Big|^2\Big]ds.
\end{aligned}$$

In view of (3.5), we obtain

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)-x(s)|^2\Big] \leq \frac{C[Mt]^n}{n!} + M\int_0^t \mathbb{E}\Big[\sup_{0\leq v\leq s}|x^n(v)-x(v)|^2\Big]ds$$

Consequently,

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|x^n(s)-x(s)|^2\Big] \leq \frac{C[Mt]^n}{n!}e^{Mt},$$

which yields the error estimation between the Picard approximate solutions $x^n(t)$, $n \geq 0$ and exact solution $x(t)$ of problem (1.1). □

4 Exponential estimates for SFDEs driven by G-Levy process

To show the exponential estimates, let us assume that problem (1.1) has a unique solution $x(t)$ on $t \in [0,\infty)$. Now we derive the exponential estimate for (1.1) as follows.

Theorem 4.1 *Let assumptions A_1 and A_2 hold. Then,*

$$\lim_{n\longrightarrow\infty} sup\frac{1}{t}\log|y(t)| \leq \frac{5}{2}c_1k,$$

where $k = (1+k_1)T + k_2 + k_3$ and k_1, k_2, k_3 are positive constants.

Proof From assertion (3.7), we know that

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}|x(s)|^2\Big] \leq 5[(1+2c_1\hat{k}T)\mathbb{E}\|\zeta\|^2 + c_1\hat{k}T]e^{c_1\hat{k}T}. \tag{4.1}$$

In view of (4.1), for each $m = 1, 2, 3, \ldots$, we have

$$\mathbb{E}\Big[\sup_{m-1\leq t\leq m}|x(t)|^2\Big] \leq 5[(1+2c_1\hat{k}T)\mathbb{E}\|\zeta\|^2 + c_1\hat{k}T]e^{c_1\hat{k}m}.$$

By using Lemma 2.3 for any $\epsilon > 0$, we derive

$$\begin{aligned}\hat{\nu}\left\{w : \sup_{m-1\leq t\leq m}|x(t)|^2 > e^{(5c_1k+\epsilon)m}\right\} &\leq \frac{\mathbb{E}\Big[\sup_{m-1\leq t\leq m}|x(t)|^2\Big]}{e^{(5c_1k+\epsilon)m}} \\ &\leq 5[(1+c_1\hat{k}T)\mathbb{E}\|\zeta\|^2 + c_1\hat{k}T]e^{-\epsilon m}.\end{aligned}$$

Since the series $\sum_{m=1}^{\infty} 5[(1+c_1\hat{k}T)\mathbb{E}\|\zeta\|^2 + c_1\hat{k}T]e^{-\epsilon m}$ is convergent; from the Borel–Cantelli lemma, we derive that for almost all $w \in \Omega$, a random integer $m_0 = m_0(w)$ exists so that

$$\sup_{m-1\leq t\leq m}|x(t)|^2 \leq e^{(5c_1k+\epsilon)m}, \text{ as } m \geq m_0.$$

This implies that for $m - 1 \leq t \leq m$ and $m \geq m_0$ we have

$$|x(t)| \leq e^{\frac{1}{2}}(5c_1k+\epsilon)m.$$

Hence,

$$\lim_{t\to\infty} sup\frac{1}{t}\log|x(t)| \leq \frac{1}{2}(5c_1\hat{k}+\epsilon),$$

the desired expression follows because ϵ is arbitrary. □

Example 4.2 Consider the following scalor FSDE driven by G-Lévy process

$$dx_t = x_t dt + sin(x_t)dB_t + x_t d\langle B\rangle_t + \int_{z\geq 1} x_t L(dt, dz), \tag{4.2}$$

with $x(0) = \zeta(0)$ and the Lêvy measure satisfies $\nu(dz) = \frac{dz}{1+|z|^2}$. Observe that SDE (4.2) possesses a unique solution if it is shown to meet conditions A_1 and A_2. Here $f = x_t$, $g = sin(x_t)$, $h = x_t$ and $K = x_t$. First, let's derive the growth condition.

$$\begin{aligned}&|f(t,x(t))|^2 + |g(t,x(t))|^2 + |h(t,x(t))|^2 + \int_{u\geq 1}|K(t,x,u)|^2\nu(du) \\ &= |x_t|^2 + |sin(x_t)|^2 + |x_t|^2 + \int_{u\geq 1}|x_t|^2\nu(dz) \\ &\leq 3|x|^2 + \int_{u\geq 1}|x|^2\frac{dz}{1+|z|^2} \\ &\leq (3+\frac{\pi}{2}) + (3+\frac{\pi}{2})|x|^2 = (3+\frac{\pi}{2})[1+|x|^2].\end{aligned}$$

The Lipschitz condition can be demonstrated as follows

$$
\begin{aligned}
&| f(t,y) - f(t,x) |^2 + | g(t,y) - g(t,x) |^2 + | h(t,y) - h(t,x) |^2 \\
&\quad + \int_{z\geq 1} | K(t,y,z) - K(t,x,z) |^2 \, \nu(dz) \\
&\leq 3 \, | x_t - y_t |^2 + \int_{z\geq 1} | x_t - y_t |^2 \, \nu(dz) \\
&= 3 \, | x_t - y_t |^2 + \int_{z\geq 1} | x_t - y_t |^2 \, \frac{dz}{1+|z|^2} \\
&\leq (3 + \frac{\pi}{2}) \, | x_t - y_t |^2
\end{aligned}
$$

Hence, the scalar SDE (4.2) admits a unique solution.

5 Conclusion

The G-Brownian motion theory extends classical Brownian motion in a non-trivial manner. G-expectation, characterized by its monotonicity and constant-preserving properties, introduces a non-linear expectation. In their work [1], the authors delve into the existence and stability of G-stochastic differential equations (G-SDEs). Expanding upon this, [19] extends the existence-uniqueness theory to G-stochastic functional differential equations (G-SFDEs). The previous approach [19] relied on Lyapunov type functions and the G-Itô formula to establish exponential estimates. But what if we introduce a jump process, particularly the Lévy jump? In this paper, the existence-uniqueness result has been established for stochastic functional differential equations incorporating G-Lévy jumps. The existing article demonstrates that exponential estimates can be explored without the need for Lyapunov-type functions and the G-Itô formula. Additionally, we provide error estimates between exact and approximate solutions for SFDEs with G-Lévy jumps. We anticipate that the ideas presented in this article will significantly contribute to future research, particularly in exploring the existence, uniqueness, and stability of solutions to backward stochastic differential equations influenced by G-Lévy processes.

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