Chen and Li *Advances in Continuous and Discrete Models* (2024) 2024:58
https://doi.org/10.1186/s13662-024-03855-y

Advances in Continuous and Discrete Models

RESEARCH Open Access

# Complex bifurcation phenomena seized in a discrete ratio-dependent Holling-Tanner predator-prey system

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**Abstract**

Using the semi-discretization method, one revisits a predator-prey model with a ratio-dependent Holling-Tanner functional response, which was previously explored using the forward Euler method. Some complex bifurcation phenomena are found in a new discrete system. In particular, one observes some dynamical differences between the two discrete methods not only in the type of fixed point but also in the existence of bifurcation. Numerical simulations are presented to illustrate the derived results.

**Mathematics Subject Classification:** 39A28; 39A30

**Keywords:** Discrete predator-prey system; Semi-discretization method; Flip bifurcation; Neimark-Sacker bifurcation

## 1 Introduction and preliminaries

In the past few decades, the Leslie predator-prey model has been widely studied in [1–5]. Generally, the Leslie predator-prey model takes the following form:

$$\begin{cases} \frac{dx}{dt} = xg(x) - p(x)y, \\ \frac{dy}{dt} = sy(1 - \frac{y}{K(x)}), \end{cases} \tag{1.1}$$

where $x$ and $y$ represent the population sizes (or densities) of prey and predator, respectively, as functions of time; the predator growth equation belongs to the logistic type; the carrying capacity $K(x)$ of environment to predator is a function on the population size of prey; $s$ is the intrinsic growth rate of predator.

It is assumed that the carrying capacity of the environment for the predator is proportional to the prey abundance, i.e., $K(x) = \frac{x}{h}$, where $h$ is the conversion factor of prey into predator. Thus, the following model, first introduced by Leslie [6], is obtained:

$$\begin{cases} \frac{dx}{dt} = xg(x) - p(x)y, \\ \frac{dy}{dt} = sy(1 - h\frac{y}{x}), \end{cases} \tag{1.2}$$

where $h\frac{y}{x}$ is called the Leslie-Gower term. Leslie and Gower [7] and Pielou [8] explored model (1.2).

In system (1.2), $g(x)$ describes the specific growth rate of the prey in the absence of predator. It is assumed that the prey grows logistically with growth rate $r$ and carrying capacity $k$ in the absence of predator, i.e., $g(x) = r(1 - \frac{x}{k})$. From this, one derives the following model from system (1.2):

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - p(x)y, \\ \frac{dy}{dt} = sy(1 - h\frac{y}{x}). \end{cases} \tag{1.3}$$

Here, the function $p(x)$ is the predator's functional response to the prey. If one takes $p(x) = \frac{mx}{Ay+x}$, it is referred to as a ratio-dependent functional response, where the parameter $m$ is the maximal predator per capita consumption rate, and the parameter $A$ is the number of prey necessary to achieve one-half of the maximum rate $m$. Accordingly, the Leslie predator-prey model with a ratio-dependent functional response is expressed as follows:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - \frac{mx}{Ay+x}y, \\ \frac{dy}{dt} = sy(1 - h\frac{y}{x}), \end{cases} \tag{1.4}$$

where $r, k, A, m, s, h$ are positive constants. In [9], this model is referred to as a ratio-dependent Holling-Tanner model.

For a complex model, one seeks to study its equivalent and simple form mathematically. To achieve equivalence and simplicity, the system (1.4) is nondimensionalized using the following scaling:

$$rt \to t, \frac{x}{k} \to x, \frac{m}{rk}y \to y,$$

which results in the following system

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \frac{xy}{\alpha y+x}, \\ \frac{dy}{dt} = \delta y(\beta - \frac{y}{x}), \end{cases} \tag{1.5}$$

where $\alpha = \frac{rA}{m}, \delta = \frac{sh}{m}, \beta = \frac{m}{hr}$.

Generally, solving a complicated system of ordinary differential equations without computational assistance is challenging. As a result, one often relies on computers to analyze such systems. This naturally leads to the consideration of discrete versions of continuous systems. How can one discretize a continuous system? Various discrete methods and theories, such as the forward Euler method, backward Euler method, and semi-discretization, can be employed. For a detailed discussion, interested readers are referred to [10–19] and the references therein.

S. Md and S. Rana in [20] employed the forward Euler method to the system (1.5) to get and study the following discrete system

$$\begin{cases} x_{n+1} = x_n + \delta x_n[(1 - x_n) - \frac{y_n}{x_n+ay_n}], \\ y_{n+1} = y_n + \delta y_n[d(b - \frac{y_n}{x_n})]. \end{cases} \tag{1.6}$$

How about applying other discrete methods to the system (1.5)? The goal is to identify dynamical differences arising from the application of different discrete methods to the same continuous system. Such an approach highlights the significance of various discrete methods. This serves as a motivation for this paper. To do this, one uses the semi-discretization method that does not require consideration of the step size and reduces the number of parameters in the discrete model to derive the discrete model of the system (1.5). For this, see also [15, 18, 19, 21, 22]. To do this, suppose that $[t]$ denotes the greatest integer not exceeding $t$. Consider the average change rate of the system (1.5) at integer number points

$$\begin{cases} \frac{1}{x(t)}\frac{dx(t)}{dt} = (1 - x([t])) - \frac{y([t])}{\alpha y([t]) + x([t])}, \\ \frac{1}{y(t)}\frac{dy(t)}{dt} = \delta(\beta - \frac{y([t])}{x([t])}). \end{cases} \tag{1.7}$$

It is easy to see that the system (1.7) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of the system (1.7) for $t \in [0, +\infty)$ has the following characteristics:

1. On the interval $[0, +\infty)$, $x(t)$ and $y(t)$ are continuous;
2. $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist for $t \in [0, +\infty)$ except the points $\{0, 1, 2, 3, \dots\}$.

The following system can be obtained by integrating the system (1.7) over the interval $[n, t]$ for any $t \in [n, n + 1)$ and $n = 0, 1, 2, \dots$

$$\begin{cases} x(t) = x_n e^{1 - x_n - \frac{y_n}{\alpha y_n + x_n}(t - n)}, \\ y(t) = y_n e^{\delta(\beta - \frac{y_n}{x_n})(t - n)}, \end{cases} \tag{1.8}$$

where $x_n = x(n)$ and $y_n = y(n)$. Letting $t \to (n + 1)^-$ in the system (1.8) produces

$$\begin{cases} x_{n+1} = x_n e^{1 - x_n - \frac{y_n}{\alpha y_n + x_n}}, \\ y_{n+1} = y_n e^{\delta(\beta - \frac{y_n}{x_n})}, \end{cases} \tag{1.9}$$

where $\alpha, \beta, \delta > 0$.

In the following, we mainly examine the dynamical properties of the system (1.9), formulating the differences compared to known results.

The rest of the paper is organized as follows: In Sect. 2, the existence and stability of the fixed points of the system (1.9) are studied. In Sect. 3, we derive the sufficient conditions for flip bifurcation and Neimark-Sacker bifurcation in the system (1.9) to occur. In Sect. 4, numerical simulations are presented to display the above theoretical results obtained. In Sect. 5, we provide concluding remarks.

## 2 Existence and stability of fixed point

In this section, one considers the existence and stability of the fixed points of the system (1.9), whose fixed points meet

$$x = x e^{1 - x - \frac{y}{\alpha y + x}}, \quad y = y e^{\delta(\beta - \frac{y}{x})}.$$

Due to the biological meanings of the system (1.9), one only considers its nonnegative fixed points. It is easy to find that the system (1.9) has exactly two nonnegative fixed points $E_1 = (1, 0)$ and $E_2 = (x_0, y_0)$ for $\alpha > \frac{\beta - 1}{\beta}$, where

$$x_0 = \frac{1 + \alpha\beta - \beta}{1 + \alpha\beta}, \quad y_0 = \beta x_0 = \frac{\beta(1 + \alpha\beta - \beta)}{1 + \alpha\beta}.$$

The Jacobian matrix of the system (1.9) at a fixed point $E(x,y)$ is

$$J(E)=\begin{pmatrix} \frac{(1-x)(x+\alpha y)^2+xy}{(x+\alpha y)^2}e^{1-x-\frac{y}{x+\alpha y}} & -\frac{x^2}{(x+\alpha y)^2}e^{1-x-\frac{y}{x+\alpha y}} \\ \frac{\delta y^2}{x^2}\mathrm{e}^{\delta(\beta-\frac{y}{x})} & (1-\frac{\delta y}{x})\mathrm{e}^{\delta(\beta-\frac{y}{x})} \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E)$ reads as

$$F(\lambda)=\lambda^2-p\lambda+q,$$

where

$$p=Tr(J(E))=\frac{(1-x)(x+\alpha y)^2+xy}{(x+\alpha y)^2}e^{1-x-\frac{y}{x+\alpha y}}+(1-\frac{\delta y}{x})\mathrm{e}^{\delta(\beta-\frac{y}{x})},$$
$$q=Det(J(E))=\frac{(1-x)(x+\alpha y)^2(x-\delta y)+x^2y}{x(x+\alpha y)^2}e^{1-x-\frac{y}{x+\alpha y}+\delta(\beta-\frac{y}{x})}.$$

Before analyzing the properties of the fixed points of the system (1.9), one recalls the following definition and lemma (see [21, pp1682], [22, pp422]).

**Definition 2.1** Let $E(x,y)$ be a fixed piont of the system (1.9) with multipliers $\lambda_1$ and $\lambda_2$.
($i$) If $|\lambda_1|<1$ and $|\lambda_2|<1$, $E(x,y)$ is called a sink, so a sink is locally asymptotically stable.
($ii$) If $|\lambda_1|>1$ and $|\lambda_2|>1$, $E(x,y)$ is called a source, so a source is locally asymptotically unstable.
($iii$) If $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$), $E(x,y)$ is called a saddle.
($iv$) If either $|\lambda_1|=1$ or $|\lambda_2|=1$, $E(x,y)$ is called to be non-hyperbolic.

**Lemma 2.2** *Let $F(\lambda)=\lambda^2+B\lambda+C$, where $B$ and $C$ are two real constants. Suppose $\lambda_1$ and $\lambda_2$ are two roots of $F(\lambda)=0$. Then, the following statements hold.*
($i$) *If $F(1)>0$, then*
($i.1$) $|\lambda_1|<1$ *and* $|\lambda_2|<1$ *if and only if* $F(-1)>0$ *and* $C<1$;
($i.2$) $\lambda_1=-1$ *and* $\lambda_2\neq-1$ *if and only if* $F(-1)=0$ *and* $B\neq 2$;
($i.3$) $|\lambda_1|<1$ *and* $|\lambda_2|>1$ *if and only if* $F(-1)<0$;
($i.4$) $|\lambda_1|>1$ *and* $|\lambda_2|>1$ *if and only if* $F(-1)>0$ *and* $C>1$;
($i.5$) $\lambda_1$ *and* $\lambda_2$ *are a pair of conjugate complex roots and,* $|\lambda_1|=|\lambda_2|=1$ *if and only if* $-2<B<2$ *and* $C=1$;
($i.6$) $\lambda_1=\lambda_2=-1$ *if and only if* $F(-1)=0$ *and* $B=2$.
($ii$) *If* $F(1)=0$, *namely,* $1$ *is one root of* $F(\lambda)=0$, *then another root* $\lambda$ *satisfies* $|\lambda|=(<,>)1$ *if and only if* $|C|=(<,>)1$.
($iii$) *If* $F(1)<0$, *then* $F(\lambda)=0$ *has one root lying in* $(1,\infty)$. *Moreover,*
($iii.1$) *the other root* $\lambda$ *satisfies* $\lambda<(=)-1$ *if and only if* $F(-1)<(=)0$;
($iii.2$) *the other root* $-1<\lambda<1$ *if and only if* $F(-1)>0$.

Now one formulates some results about the stability of the fixed points $E_1$ and $E_2$ in the following theorems. First, consider the fixed point $E_1$.

**Theorem 2.3** *The fixed point $E_1 = (1,0)$ of the system (1.9) is a saddle.*

*Proof* The Jacobian matrix $J(E_1)$ of the system (1.9) at the fixed point $E_1$ is given by

$$J(E_1) = \begin{pmatrix} 0 & -1 \\ 0 & e^{\delta\beta} \end{pmatrix}.$$

Obviously, $|\lambda_1| = 0 < 1$ and $|\lambda_2| = e^{\delta\beta} > 1$, so $E_1$ is a saddle. □

*Remark* 2.4 In [20], the authors used the forward Euler method to the system (1.5) to produce the system (1.6). They obtained that the fixed point $E_1$ of the system (1.6) has the following properties: if $\delta < 2$, then $E_1$ is a saddle; if $\delta = 2$, then $E_1$ is non-hyperbolic; if $\delta > 2$, then $E_1$ is a source. However, our results display that the fixed point $E_1$ is always a saddle. This is the first difference between these two different discrete methods.

Now consider the stability of fixed point $E_2$. Let $\delta_0 = \frac{2\beta+2(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)(1+\beta+\alpha\beta)\beta}$ and $\delta_1 = \frac{\beta(2+\alpha\beta)-(1+\alpha\beta)^2}{\beta^2(1+\alpha\beta)}$. Obviously, $\delta_1 < \delta_0$.

**Theorem 2.5** *For $\alpha > \frac{\beta-1}{\beta}$, $E_2$ is a positive fixed point of the system (1.9). Moreover, the following statements about the fixed point $E_2$ hold.*

1. When $\delta < \delta_0$,
   a) if $\delta > \delta_1$, then $E_2$ is a sink;
   b) if $\delta = \delta_1$, then $E_2$ is non-hyperbolic;
   c) if $\delta < \delta_1$, then $E_2$ is a source.
2. When $\delta = \delta_0$, $E_2$ is non-hyperbolic.
3. When $\delta > \delta_0$, $E_2$ is a saddle.

*Proof* The Jacobian matrix $J(E_2)$ of the system (1.9) at the fixed point $E_2$ is given by

$$J(E_2) = \begin{pmatrix} \frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2} & -\frac{1}{(1+\alpha\beta)^2} \\ \delta\beta^2 & 1-\delta\beta \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E_2)$ can be written as

$$F(\lambda) = \lambda^2 - p_2\lambda + q_2,$$

where $p_2 = \frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2} + 1 - \beta\delta$ and

$$q_2 = \frac{\beta(1+\alpha\beta)(1-\delta\beta)+\beta}{(1+\alpha\beta)^2} = \frac{\beta(2+\alpha\beta)-\delta\beta^2(1+\alpha\beta)}{(1+\alpha\beta)^2} = \frac{\beta^2(\delta_1-\delta)}{1+\alpha\beta} + 1.$$

It is easy to see that

$$F(1) = \beta\delta(1 - \frac{\beta}{1+\alpha\beta}) = \frac{\beta\delta(1+\alpha\beta-\beta)}{1+\alpha\beta} > 0,$$

$$F(-1) = \frac{2\beta}{(1+\alpha\beta)^2} + (2-\beta\delta)(1+\frac{\beta}{1+\alpha\beta})$$

**Table 1** Properties of the fixed point $E_2$

| Conditions | | Eigenvalues | Properties | Reference |
|---|---|---|---|---|
| $\delta < \delta_0$ | $\delta > \delta_1$ | $\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$ | sink | $(i.1)$ |
| | $\delta = \delta_1$ | $\lvert\lambda_1\rvert = \lvert\lambda_2\rvert = 1$ | non-hyperbolic | $(i.5)$ |
| | $\delta < \delta_1$ | $\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1$ | source | $(i.4)$ |
| $\delta = \delta_0$ | | $\lambda_1 = -1, \lambda_2 \neq -1$ | non-hyperbolic | $(i.2)$ |
| $\delta > \delta_0$ | | $\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$ | saddle | $(i.3)$ |

$$= \frac{2\beta + 2(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)^2} - \frac{(1+\beta+\alpha\beta)\beta\delta}{1+\alpha\beta}$$
$$= \frac{\beta(1+\beta+\alpha\beta)}{1+\alpha\beta}(\delta_0 - \delta).$$

If $\delta < \delta_0$, then $F(-1) > 0$. For $\delta > \delta_1$, $q_2 < 1$. It follows from Lemma 2.2($i.1$) that $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so $E_2$ is a sink. For $\delta = \delta_1$, $q_2 = 1$. It follows from Lemma 2.2($i.5$) that $|\lambda_1| = |\lambda_2| = 1$; therefore, $E_2$ is non-hyperbolic. For $\delta < \delta_1$, $q_2 > 1$, which reads $|\lambda_1| > 1$ and $|\lambda_2| > 1$ by Lemma 2.2($i.4$), so $E_2$ is a source.

If $\delta = \delta_0$, then $F(-1) = 0$. Namely, $-1$ is a root of the characteristic polynomial; therefore, $E_2$ is non-hyperbolic.

If $\delta > \delta_0$, then $F(-1) < 0$. Lemma 2.2($i.3$) shows that $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so $E_2$ is a saddle. □

The results summarized above are presented in Table 1 for illustration.

## 3 Bifurcation analysis

In this section, one uses the center manifold theorem and bifurcation theory to analyze the local bifurcation problems of the system (1.9) in the fixed points $E_1$ and $E_2$. For related work, see [14–19, 23–28].

### 3.1 Bifurcation of the system (1.9) in the fixed point $E_1$

Theorem 2.3 shows that $E_1$ is always a saddle, so the system (1.9) has no bifurcation in the fixed point $E_1$.

*Remark* 3.1 In [20], the authors produced the system (1.6) using the forward Euler method. They found that a flip bifurcation of the system (1.6) exists in the fixed point $E_1$. However, our results show that the system (1.9) has no bifurcation in the fixed point $E_1$. So, this is the second dynamical difference between these two different discrete methods.

### 3.2 Bifurcation of the system (1.9) in the fixed point $E_2$

One can see from Theorem 2.5 that the fixed point $E_2$ is non-hyperbolic when $\delta = \delta_0$ or $\delta = \delta_1$. When the parament $\delta$ goes through these critical values $\delta_0$ or $\delta_1$, the dimensional numbers for the stable manifold and the unstable manifold of the fixed point $E_2$ vary. Thus, a bifurcation may occur in each case. One is concerned with the paraments

$$(\alpha,\beta,\delta) \in S_{E_+} = \{(\alpha,\beta,\delta) \in R_+^3 \,\Big|\, \alpha > 0, \beta > 0, \delta > 0, \alpha > \frac{\beta-1}{\beta}\}.$$

*3.2.1 Flip bifurcation*

Notice that $F(-1) = 0$ is a necessary condition for a flip bifurcation to occur. When $\delta = \delta_0$, $F(-1) = 0$. So, a flip bifurcation may occur in the fixed point $E_2$. Actually, one has the following result.

**Theorem 3.2** *Suppose the parameters $(\alpha, \beta, \delta) \in S_{E_+}$. Then, the system (1.9) undergoes a flip bifurcation in the fixed point $E_2$ when the parameter $\delta$ varies in a small neighborhood of $\delta_0$.*

*Proof* First, let $u_n = x_n - x_0, v_n = y_n - y_0$, which transforms the fixed point $E_2 = (x_0, y_0)$ to the origin $O(0,0)$ and the system (1.9) to

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{1-(u_n+x_0)-\frac{v_n+y_0}{\alpha(v_n+y_0)+u_n+x_0}} - x_0, \\ v_{n+1} = (v_n + y_0)e^{\delta(\beta-\frac{v_n+y_0}{u_n+x_0})} - y_0. \end{cases} \tag{3.1}$$

Second, giving a small perturbation $\delta^*$ of the parameter $\delta$ around $\delta_0$, i.e., $\delta^* = \delta - \delta_0$ with $0 < |\delta^*| \ll 1$, the system (3.1) is perturbed into

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{1-(u_n+x_0)-\frac{v_n+y_0}{\alpha(v_n+y_0)+u_n+x_0}} - x_0, \\ v_{n+1} = (v_n + y_0)e^{(\delta^*+\delta_0)(\beta-\frac{v_n+y_0}{u_n+x_0})} - y_0. \end{cases} \tag{3.2}$$

Letting $\delta^*_{n+1} = \delta^*_n = \delta^*$, we can write the system (3.2) as

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{1-(u_n+x_0)-\frac{v_n+y_0}{\alpha(v_n+y_0)+u_n+x_0}} - x_0, \\ v_{n+1} = (v_n + y_0)e^{(\delta^*_n+\delta_0)(\beta-\frac{v_n+y_0}{u_n+x_0})} - y_0, \\ \delta^*_{n+1} = \delta^*_n. \end{cases} \tag{3.3}$$

Third, performing a Taylor expansion of the system (3.3) at $(\alpha_n, \beta_n, \delta^*_n) = (0,0,0)$ gives

$$\begin{cases} u_{n+1} = & a_{100}u_n + a_{010}v_n + a_{200}u_n^2 + a_{020}v_n^2 + a_{110}u_nv_n \\ & + a_{300}u_n^3 + a_{030}v_n^3 + a_{210}u_n^2v_n + a_{120}u_nv_n^2 + o(\rho_1^3), \\ v_{n+1} = & b_{100}u_n + b_{010}v_n + b_{001}\delta^*_n + b_{200}u_n^2 + b_{020}v_n^2 \\ & + b_{002}\delta^{*2}_n + b_{110}u_nv_n + b_{101}u_n\delta^*_n + b_{011}v_n\delta^*_n \\ & + b_{300}u_n^3 + b_{030}v_n^3 + b_{003}\delta^{*3}_n + b_{210}u_n^2v_n \\ & + b_{120}u_nv_n^2 + b_{021}v_n^2\delta^*_n + b_{201}u_n^2\delta^*_n + b_{102}u_n\delta^{*2}_n \\ & + b_{012}v_n\delta^{*2}_n + b_{111}u_nv_n\delta^*_n + o(\rho_1^3), \\ \delta^*_{n+1} = & \delta^*_n, \end{cases} \tag{3.4}$$

where $\rho_1 = \sqrt{u_n^2 + v_n^2 + \delta^{*2}_n}$,

$$a_{100} = \frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2}, a_{010} = -\frac{1}{(1+\alpha\beta)^2}, a_{110} = \frac{1-2\beta-\alpha\beta^2-\alpha^2\beta^2}{(1+\alpha\beta)^4-\beta(1+\alpha\beta)^3},$$

$$a_{200} = \frac{-(1+\alpha\beta)^4+\beta^2(1+\alpha\beta)^2-2\beta(1+\alpha\beta)+2\beta^2(1+\alpha\beta)+\beta^2}{2[(1+\alpha\beta)^4-\beta(1+\alpha\beta)^3]},$$

$$a_{020} = \frac{2\alpha(1+\alpha\beta)+1}{2[(1+\alpha\beta)^4-\beta(1+\alpha\beta)^3]},$$

$$a_{030}=\frac{6\alpha^2(1+\alpha\beta)^2+6\alpha(1+\alpha\beta)+1}{6[(1+\alpha\beta)^6-2\beta(1+\alpha\beta)^5+\beta^2(1+\alpha\beta)^4]},$$

$$a_{210}=\frac{9\alpha^3\beta^3+21\alpha^2\beta^2+12\alpha\beta-\alpha^2\beta^4-2\alpha\beta^3-3\alpha^2\beta^2-6\alpha\beta^2+\beta+\beta^3}{6[(1+\alpha\beta)^6-2\beta(1+\alpha\beta)^5+\beta^2(1+\alpha\beta)^4]},$$

$$a_{120}=\frac{2\alpha^4\beta^3+6\alpha^3\beta^2+4\alpha^2\beta^2-4\alpha^2\beta+2\alpha^3\beta^3+\alpha\beta^2-8\alpha+4\beta-2}{6[(1+\alpha\beta)^6-2\beta(1+\alpha\beta)^5+\beta^2(1+\alpha\beta)^4]},$$

$$a_{300}=\frac{2(1+\alpha\beta)^6-3\beta(1+\alpha\beta)^5-3\beta(1+\alpha\beta)^4+\beta^3(1+\alpha\beta)^3-3\beta(1+\alpha\beta)^2}{6[(1+\alpha\beta)^6-2\beta(1+\alpha\beta)^5+\beta^2(1+\alpha\beta)^4]}$$
$$+\frac{-6\beta^2(1+\alpha\beta)^2+3\beta^3(1+\alpha\beta)^2-4\beta^2(1+\alpha\beta)+\beta^3(1+\alpha\beta)+\beta^3}{6[(1+\alpha\beta)^6-2\beta(1+\alpha\beta)^5+\beta^2(1+\alpha\beta)^4]},$$

$$b_{100}=\frac{2\beta^2+2\beta(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)(1+\beta+\alpha\beta)},b_{010}=\frac{-2\beta-(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)(1+\beta+\alpha\beta)},$$

$$b_{200}=\frac{2\beta^3+2(1+\beta+\alpha\beta)(1+\alpha\beta)}{(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2-\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)},$$

$$b_{020}=\frac{2\beta+2(1+\beta+\alpha\beta)(1+\alpha\beta)}{(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2-\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)},$$

$$b_{110}=-\frac{4\beta^2+4\beta(1+\beta+\alpha\beta)(1+\alpha\beta)}{(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2-\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)},$$

$$b_{101}=\beta^2,b_{011}=-\beta,b_{102}=b_{012}=b_{002}=b_{003}=0,$$

$$b_{030}=\frac{2(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3-6\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta)-4\beta^3}{3\beta^2(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2},$$

$$b_{210}=\frac{2(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3-2\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta)}{3(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2}$$
$$+\frac{4\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2-4\beta^3}{3(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2},$$

$$b_{201}=\frac{2\beta^3+2\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta)-\beta^2}{3[(1+\beta+\alpha\beta)(1+\alpha\beta)-\beta(1+\beta+\alpha\beta)]},$$

$$b_{021}=\frac{2\beta+2(1+\beta+\alpha\beta)(1+\alpha\beta)-1}{3[(1+\beta+\alpha\beta)(1+\alpha\beta)-\beta(1+\beta+\alpha\beta)]},$$

$$b_{111}=-\frac{2\beta^2+2\beta(1+\beta+\alpha\beta)(1+\alpha\beta)-\beta}{3[(1+\beta+\alpha\beta)(1+\alpha\beta)-\beta(1+\beta+\alpha\beta)]},$$

$$b_{300}=\frac{10\beta^2(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2+6(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3}{3(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2}$$
$$+\frac{-4\beta^3(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2+4\beta^2+4\beta^3(1+\beta+\alpha\beta)(1+\alpha\beta)}{3(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2},$$

$$b_{120}=\frac{(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3-8\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta)}{3\beta(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2}$$
$$+\frac{-3\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2-\beta^2(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2}{3\beta(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2}$$

$$+\frac{-\beta(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3-4\beta^3}{3\beta(1+\beta+\alpha\beta)^3(1+\alpha\beta)^3[1-\frac{\beta}{(1+\alpha\beta)}]^2}.$$

Let

$$J(E_2)=\begin{pmatrix} a_{100} & a_{010} & 0\\ b_{100} & b_{010} & 0\\ 0 & 0 & 1\end{pmatrix},$$

i.e.,

$$J(E_2)=\begin{pmatrix} \frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2} & -\frac{1}{(1+\alpha\beta)^2} & 0\\ \frac{2\beta^2+2\beta(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)(1+\beta+\alpha\beta)} & \frac{-2\beta-(1+\alpha\beta)(1+\beta+\alpha\beta)}{(1+\alpha\beta)(1+\beta+\alpha\beta)} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Compute three eigenvalues of $J(E_2)$ to obtain

$$\lambda_{1,2}=\frac{\beta(2+\alpha\beta)(1+\beta+\alpha\beta)-2\beta(1+\alpha\beta)-(1+\alpha\beta)^2(1+\beta+\alpha\beta)\mp\mu}{2(1+\alpha\beta)^2(1+\beta+\alpha\beta)},\qquad \lambda_3=1,$$

and their corresponding eigenvectors are

$$\begin{aligned}
(\xi_1,\eta_1,\theta_1)^T&=(2(1+\beta+\alpha\beta),K+\mu,0)^T,\\
(\xi_2,\eta_2,\theta_2)^T&=(2(1+\beta+\alpha\beta),K-\mu,0)^T,\\
(\xi_3,\eta_3,\theta_3)^T&=(0,0,1)^T,
\end{aligned}$$

where

$$\begin{aligned}
K&=(1+\beta+\alpha\beta)[(1+\alpha\beta)^2+\alpha\beta^2+2\beta]+2\beta(1+\alpha\beta),\\
\mu&=(1+\beta+\alpha\beta)[(1+\alpha\beta)^2+\alpha\beta^2]+2\beta^2.
\end{aligned}$$

One can see that $\lambda_1=\frac{\beta(2+\alpha\beta)(1+\beta+\alpha\beta)-2\beta(1+\alpha\beta)-(1+\alpha\beta)^2(1+\beta+\alpha\beta)-\mu}{2(1+\alpha\beta)^2(1+\beta+\alpha\beta)}=-1$.

Take $T_1=\begin{pmatrix}\xi_1 & \xi_2 & \xi_3\\ \eta_1 & \eta_2 & \eta_3\\ \theta_1 & \theta_2 & \theta_3\end{pmatrix}$, namely,

$$T_1=\begin{pmatrix} 2(1+\beta+\alpha\beta) & 2(1+\beta+\alpha\beta) & 0\\ K+\mu & K-\mu & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then, its inverse matrix is

$$T_1^{-1}=\begin{pmatrix} \frac{\mu-K}{4\mu(1+\beta+\alpha\beta)} & \frac{1}{2\mu} & 0\\ \frac{\mu+K}{4\mu(1+\beta+\alpha\beta)} & -\frac{1}{2\mu} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Taking the following transformation

$$(u_n, v_n, \delta_n^*)^T = T_1(X_n, Y_n, \omega_n)^T,$$

the system (3.4) is changed into the following form

$$\begin{cases} X_{n+1} = -X_n + F(X_n, Y_n, \omega_n) + o(\rho_2^3), \\ Y_{n+1} = \lambda_2 Y_n + G(X_n, Y_n, \omega_n) + o(\rho_2^3), \\ \omega_{n+1} = \omega_n, \end{cases} \tag{3.5}$$

where $\rho_2 = \sqrt{X_n^2 + Y_n^2 + \omega_n^2}$,

$$\begin{aligned} F(X_n, Y_n, \omega_n) = {} & m_{200}X_n^2 + m_{020}Y_n^2 + m_{002}{\omega_n}^2 + m_{110}X_nY_n + m_{101}X_n\omega_n \\ & + m_{011}Y_n\omega_n + m_{300}X_n^3 + m_{030}Y_n^3 + m_{003}{\omega_n}^3 + m_{210}X_n^2Y_n \\ & + m_{120}X_nY_n^2 + m_{201}X_n^2\omega_n + m_{102}X_n{\omega_n}^2 + m_{021}Y_n^2\omega_n \\ & + m_{012}Y_n{\omega_n}^2 + m_{111}X_nY_n\omega_n, \\ G(X_n, Y_n, \omega_n) = {} & l_{200}X_n^2 + l_{020}Y_n^2 + l_{002}{\omega_n}^2 + l_{110}X_nY_n + l_{101}X_n\omega_n \\ & + l_{011}Y_n\omega_n + l_{300}X_n^3 + l_{030}Y_n^3 + l_{003}{\omega_n}^3 + l_{210}X_n^2Y_n \\ & + l_{120}X_nY_n^2 + l_{201}X_n^2\omega_n + l_{102}X_n{\omega_n}^2 + l_{021}Y_n^2\omega_n \\ & + l_{012}Y_n{\omega_n}^2 + l_{111}X_nY_n\omega_n, \end{aligned}$$

$$\begin{aligned} m_{102} &= m_{012} = m_{002} = m_{003} = 0, \\ m_{200} &= 4(Aa_{200} + \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 \\ &\quad + 2(Aa_{110} + \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)(K+\mu) + (Aa_{020} + \frac{b_{020}}{2\mu})(K+\mu)^2, \\ m_{110} &= 8(Aa_{200} + \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 \\ &\quad + 4(Aa_{110} + \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)K + 2(Aa_{020} + \frac{b_{020}}{2\mu})(K^2-\mu^2), \\ m_{020} &= 4(Aa_{200} + \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 \\ &\quad + 2(Aa_{110} + \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)(K-\mu) + (Aa_{020} + \frac{b_{020}}{2\mu})(K-\mu)^2, \\ m_{101} &= \frac{b_{011}}{2\mu}(K+\mu) + \frac{b_{101}}{\mu}(1+\beta+\alpha\beta), \end{aligned}$$

$$\begin{aligned} m_{011} &= \frac{b_{011}}{2\mu}(K-\mu) + \frac{b_{101}}{\mu}(1+\beta+\alpha\beta), \\ m_{300} &= 8(Aa_{300} + \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + (Aa_{030} + \frac{b_{030}}{2\mu})(K+\mu)^3 \end{aligned}$$

$$
\begin{aligned}
&+ 4(Aa_{210} + \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(K+\mu) \\
&+ 2(Aa_{120} + \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)(K+\mu)^2, \\
m_{030} &= 8(Aa_{300} + \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + (Aa_{030} + \frac{b_{030}}{2\mu})(K-\mu)^3 \\
&+ 4(Aa_{210} + \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(K-\mu) \\
&+ 2(Aa_{120} + \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)(K-\mu)^2, \\
m_{210} &= 24(Aa_{300} + \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + 3(Aa_{030} + \frac{b_{030}}{2\mu})(K-\mu)(K+\mu)^2 \\
&+ 4(Aa_{210} + \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(3K+\mu) \\
&+ 2(Aa_{120} + \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)[(K+\mu)^2 + 2(K^2-\mu^2)], \\
m_{120} &= 24(Aa_{300} + \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + 3(Aa_{030} + \frac{b_{030}}{2\mu})(K-\mu)^2(K+\mu) \\
&+ 4(Aa_{210} + \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(3K-\mu) \\
&+ 2(Aa_{120} + \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)[(K-\mu)^2 + 2(K^2-\mu^2)], \\
m_{201} &= \frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 + \frac{b_{021}}{2\mu}(K+\mu)^2 + \frac{b_{111}}{\mu}(1+\beta+\alpha\beta)(K+\mu), \\
m_{021} &= \frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 + \frac{b_{021}}{2\mu}(K-\mu)^2 + \frac{b_{111}}{\mu}(1+\beta+\alpha\beta)(K-\mu), \\
m_{111} &= \frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 + \frac{b_{021}}{2\mu}(K^2-\mu^2) + \frac{b_{111}}{\mu}2(1+\beta+\alpha\beta)K, \\
l_{102} &= l_{012} = l_{002} = l_{003} = 0, \\
l_{200} &= 4(Ba_{200} - \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 \\
&+ 2(Ba_{110} - \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)(K+\mu) + (Ba_{020} - \frac{b_{020}}{2\mu})(K+\mu)^2, \\
l_{110} &= 8(Ba_{200} - \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 + 4(Ba_{110} - \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)K \\
&+ 2(Ba_{020} - \frac{b_{020}}{2\mu})(K^2-\mu^2), \\
l_{020} &= 4(Ba_{200} - \frac{b_{200}}{2\mu})(1+\beta+\alpha\beta)^2 \\
&+ 2(Ba_{110} - \frac{b_{110}}{2\mu})(1+\beta+\alpha\beta)(K-\mu) + (Ba_{020} - \frac{b_{020}}{2\mu})(K-\mu)^2, \\
l_{101} &= -\frac{b_{011}}{2\mu}(K+\mu) - \frac{b_{101}}{\mu}(1+\beta+\alpha\beta),
\end{aligned}
$$

$$\begin{aligned}
l_{011} &= -\frac{b_{011}}{2\mu}(K-\mu) - \frac{b_{101}}{\mu}(1+\beta+\alpha\beta),\\
l_{300} &= 8(Ba_{300} - \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + (Ba_{030} - \frac{b_{030}}{2\mu})(K+\mu)^3\\
&\quad + 4(Ba_{210} - \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(K+\mu)\\
&\quad + 2(Ba_{120} - \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)(K+\mu)^2,\\
l_{030} &= 8(Ba_{300} - \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + (Ba_{030} - \frac{b_{030}}{2\mu})(K-\mu)^3\\
&\quad + 4(Ba_{210} - \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(K-\mu)\\
&\quad + 2(Ba_{120} - \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)(K-\mu)^2,\\
l_{210} &= 24(Ba_{300} - \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + 3(Ba_{030} - \frac{b_{030}}{2\mu})(K-\mu)(K+\mu)^2\\
&\quad + 4(Ba_{210} - \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(3K+\mu)\\
&\quad + 2(Ba_{120} - \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)[(K-\mu)^2 + 2(K^2-\mu^2)],\\
l_{120} &= 24(Ba_{300} - \frac{b_{300}}{2\mu})(1+\beta+\alpha\beta)^3 + 3(Ba_{030} - \frac{b_{030}}{2\mu})(K-\mu)^2(K+\mu)\\
&\quad + 4(Ba_{210} - \frac{b_{210}}{2\mu})(1+\beta+\alpha\beta)^2(3K-\mu)\\
&\quad + 2(Ba_{120} - \frac{b_{120}}{2\mu})(1+\beta+\alpha\beta)[(K+\mu)^2 + 2(K^2-\mu^2)],\\
l_{201} &= -\frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 - \frac{b_{021}}{2\mu}(K+\mu)^2 - \frac{b_{111}}{\mu}(1+\beta+\alpha\beta)(K+\mu),\\
l_{021} &= -\frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 - \frac{b_{021}}{2\mu}(K-\mu)^2 - \frac{b_{111}}{\mu}(1+\beta+\alpha\beta)(K-\mu),\\
l_{111} &= -\frac{b_{201}}{\mu}2(1+\beta+\alpha\beta)^2 - \frac{b_{021}}{2\mu}(K^2-\mu^2) - \frac{b_{111}}{\mu}2(1+\beta+\alpha\beta)K,
\end{aligned}$$

where $A = -\frac{\beta(1+\alpha\beta)}{\mu(1+\beta+\alpha\beta)}, B = \frac{(1+\alpha\beta)^2+\beta(2+\alpha\beta)}{2\mu}$.

Next, suppose on the center manifold

$$Y_n = h(X_n,\omega_n) = h_{20}X_n^2 + h_{11}X_n\omega_n + h_{02}\omega_n^2 + o(\rho_3^2),$$

where $\rho_3 = \sqrt{X_n^2+\omega_n^2}$. According to

$$\begin{aligned}
Y_{n+1} = h(X_{n+1},\omega_{n+1}) &= \lambda_2 h(X_n,\omega_n) + G(X_n,h(X_n,\omega_n),\omega_n) + o(\rho_3^3),\\
h(X_{n+1},\omega_{n+1}) &= h_{20}(-X_n + F(X_n,h(X_n,\omega_n),\omega_n))^2\\
&\quad + h_{11}(-X_n + F(X_n,h(X_n,\omega_n),\omega_n))\omega_n + h_{02}\omega_n^2 + o(\rho_3^3),
\end{aligned}$$

comparing the corresponding coefficients of terms with the same orders in the above center manifold equation, one gets

$$h_{20} = 0, h_{11} = 0, h_{02} = 0.$$

So the system (3.5) restricted to the center manifold takes as

$$\begin{aligned} X_{n+1} &= f(X_n, \omega_n) = -X_n + F(X_n, h(X_n, \omega_n), \omega_n) + o(\rho_3^3) \\ &= -X_n + l_{200}X_n^2 + l_{101}X_n\omega_n + l_{300}X_n^3 + l_{201}X_n^2\omega_n + o(\rho_3^3), \end{aligned}$$

and

$$\begin{aligned} f^2(X_n, \omega_n) &= f(f(X_n, \omega_n), \omega_n) \\ &= X_n - 2l_{101}X_n\omega_n - (2l_{300} + 2l_{200}^2)X_n^3 \\ &+ (l_{101} - 2l_{200}l_{101})X_n^2\omega_n + l_{101}^2X_n\omega_n^2 + o(\rho_3^3). \end{aligned}$$

Thereout, one has

$$\begin{aligned} &f(X_n, \omega_n)|_{(0,0)} = 0,\ \frac{\partial f}{\partial X_n}|_{(0,0)} = -1,\ \frac{\partial f^2}{\partial \omega_n}|_{(0,0)} = 0,\ \frac{\partial^2 f^2}{\partial X_n^2}|_{(0,0)} = 0, \\ &\frac{\partial^2 f^2}{\partial X_n \partial \omega_n}|_{(0,0)} = \frac{b_{011}(K+\mu) + 2b_{101}(1+\beta+\alpha\beta)}{\mu} \\ &\qquad = -\frac{2\beta(1+\alpha\beta)(1+\beta+\alpha\beta)(1+\beta^2+\alpha\beta)}{\mu} < 0(\neq 0), \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 f^2}{12\partial X_n^3}|_{(0,0)} &= -l_{300} - l_{200}^2 \\ &< -\frac{(1+\beta+\alpha\beta)^3}{6\mu^2[(1+\alpha\beta)^8 - 2\beta(1+\alpha\beta)^7 + \beta^2(1+\alpha\beta)^6]} \\ &\times \{[\beta(2+\alpha\beta) + (1+\alpha\beta)^2]^4[6\alpha^2(1+\alpha\beta)^2 + 6\alpha(1+\alpha\beta) + 1] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2]^3 \\ &* [2\alpha^4\beta^3 + 6\alpha^3\beta^2 + 4\alpha^2\beta^2 + 2\alpha^3\beta^3 + \alpha\beta^3 + 4\beta] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2]^3[12\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta) + 8\beta^3] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2]^2[9\alpha^3\beta^3 + 18\alpha^2\beta^2 + 9\beta^3 + 12\alpha\beta + 5\beta] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2]^2[6\beta(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2] \\ &* [2(1+\alpha\beta)^6 + \beta^3(1+\alpha\beta)^3 + 3\beta^3(1+\alpha\beta)^2] \\ &+ [\beta(2+\alpha\beta) + (1+\alpha\beta)^2][4\beta^2(1+\beta+\alpha\beta)(1+\alpha\beta) + 8\beta^3] \\ &+ [8\beta^3(1+\beta+\alpha\beta)^2(1+\alpha\beta)^2\} < 0(\neq 0). \end{aligned}$$

According to (21.1.42)–(21.1.46) in [26, pp507], all the conditions for the occurrence of a flip bifurcation are satisfied; hence, it is valid for the occurrence of a flip bifurcation in the fixed point $E_2$. The proof is over. □

*3.2.2 Neimark-Sacker bifurcation*

When $\delta = \delta_1$, a pair of imaginary roots with $|\lambda_1| = |\lambda_2| = 1$ occur, implying a necessary condition holds for a Neimark-Sacker bifurcation to occur, hence, there may be a Neimark-Sacker bifurcation in the fixed point $E_2$. In fact, one has the following result.

**Theorem 3.3** *Suppose the parameters* $(\alpha,\beta,\delta) \in S_{E_+}$ *and* $\alpha < \frac{\beta-2+\sqrt{\beta^2+4\beta}}{2\beta}$. *Then, the system* (1.9) *undergoes a Neimark-Sacker bifurcation in the fixed point* $E_2$ *when the parament* $\delta$ *varies in a small neighborhood of* $\delta_1$. *Furthermore, if in* (3.9) $L < (>)0$, *then an attracting* (*repelling*) *invariant closed curve bifurcates from the fixed point* $E_2$ *for* $\delta > (<)\delta_1$.

*Proof* Giving a small perturbation $\delta^{**}$ of the parameter $\delta$ around $\delta_1$ in the system (3.1), i.e., $\delta^{**} = \delta - \delta_1$ with $0 < |\delta^{**}| \ll 1$, the perturbation of the system (3.1) reads

$$\begin{cases} u_{n+1} = (u_n + x_0)e^{1-(u_n+x_0)-\frac{v_n+y_0}{\alpha(v_n+y_0)+u_n+x_0}} - x_0, \\ v_{n+1} = (v_n + y_0)e^{(\delta^{**}+\delta_1)(\beta-\frac{v_n+y_0}{u_n+x_0})} - y_0. \end{cases} \tag{3.6}$$

The characteristic equation of the linearized equation of the system (3.6) at the equilibrium point $(0,0)$ is

$$F(\lambda) = \lambda^2 - p(\delta^{**})\lambda + q(\delta^{**}) = 0,$$

where

$$p(\delta^{**}) = \frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2} + 1 - \beta(\delta^{**}+\delta_1) \quad \text{and} \quad q(\delta^{**}) = -\frac{\beta^2\delta^{**}}{1+\alpha\beta} + 1.$$

It is easy to derive $p^2(0) - 4q(0) < 0$ when $\alpha < \frac{\beta-2+\sqrt{\beta^2+4\beta}}{2\beta}$, so, when $0 < |\delta^*| \ll 1$, the two roots of $F(\lambda) = 0$ are as follows:

$$\lambda_{1,2}(\delta^{**}) = \frac{p(\delta^{**}) \pm \sqrt{p^2(\delta^{**}) - 4q(\delta^{**})}}{2} = \frac{p(\delta^{**}) \pm i\sqrt{4q(\delta^{**}) - p^2(\delta^{**})}}{2}.$$

Moreover,

$$(|\lambda_{1,2}(\delta^{**})|)\big|_{\delta^{**}=0} = \sqrt{q(\delta^{**})}\big|_{\delta^{**}=0} = 1.$$

The occurrence of the Neimark-Sacker bifurcation requires the following two conditions to be satisfied:

1. $\left(\frac{d|\lambda_{1,2}(\delta^{**})|}{d\delta^{**}}\right)\Big|_{\delta^{**}=0} \neq 0$;
2. $\lambda_{1,2}^i(0) \neq 1, i = 1,2,3,4$.

By calculation, one finds

$$\left(\frac{d|\lambda_{1,2}(\delta^{**})|}{d\delta^{**}}\right)\Big|_{\delta^{**}=0} = -\frac{\beta^2}{\alpha\beta+1} < 0(\neq 0).$$

Obviously $\lambda_{1,2}^i(0) \neq 1, i = 1,2,3,4$. So, the two conditions are satisfied.

In order to derive the normal form of the system (3.6), one expands (3.6) in power series up to the third-order term around the origin to get

$$\begin{cases} u_{n+1} = & c_{10}u_n + c_{01}v_n + c_{20}u_n^2 + c_{11}u_nv_n + c_{02}v_n^2 \\ & +c_{30}u_n^3 + c_{21}u_n^2v_n + c_{12}u_nv_n^2 + c_{03}v_n^3 + o(\rho_4^3), \\ v_{n+1} = & d_{10}u_n + d_{01}v_n + d_{20}u_n^2 + d_{11}u_nv_n + d_{02}v_n^2 \\ & +d_{30}u_n^3 + d_{21}u_n^2v_n + d_{12}u_nv_n^2 + d_{03}v_n^3 + o(\rho_4^3), \end{cases} \tag{3.7}$$

where $\rho_4 = \sqrt{u_n^2 + v_n^2}$,

$$\begin{aligned}
&c_{10} = a_{100}, c_{01} = a_{010}, c_{20} = a_{200}, c_{11} = a_{110}, \\
&c_{02} = a_{020}, c_{30} = a_{300}, c_{21} = a_{210}, c_{12} = a_{120}, c_{03} = a_{030}, \\
&d_{10} = \frac{\beta(2+\alpha\beta) - (1+\alpha\beta)^2}{1+\alpha\beta}, d_{01} = \frac{-\beta + (1+\alpha\beta)^2}{\beta(1+\alpha\beta)}, \\
&d_{20} = \frac{(1+\alpha\beta)^4 + \beta(1+\alpha\beta)^3 - \alpha\beta^3(2+\alpha\beta) - \beta(1+\alpha\beta)^2(3+\alpha\beta)}{2[\beta(1+\alpha\beta)^2 - \beta^2(1+\alpha\beta)]}, \\
&d_{02} = \frac{\alpha\beta^2(1+\alpha\beta)^2 + (1+\alpha\beta)^4 - \alpha\beta^3(2+\alpha\beta) - \beta(2+\alpha\beta)(1+\alpha\beta)^2}{2[\beta^3(1+\alpha\beta)^2 - \beta^4(1+\alpha\beta)]}, \\
&d_{11} = \frac{\alpha\beta^3(2+\alpha\beta) + \beta(2+\alpha\beta)(1+\alpha\beta)^2 - \alpha\beta^2(1+\alpha\beta)^2 - (1+\alpha\beta)^4}{\beta^2(1+\alpha\beta)^2 - \beta^3(1+\alpha\beta)}, \\
&d_{30} = \frac{(6\beta^2 - 4\beta^3 + 2\alpha\beta^3)(1+\alpha\beta)(2+\alpha\beta) + 4\beta^2(1+\alpha\beta)^3}{6[\beta(1+\alpha\beta)^2 - 2\beta^2(1+\alpha\beta) + \beta^3]} \\
&\quad + \frac{-\alpha\beta^4(2+\alpha\beta)^2 + 2\beta(1+\alpha\beta)^3(\alpha\beta - 1) - \beta^2(2+\alpha\beta)^2(1+\alpha\beta)^2}{6[\beta(1+\alpha\beta)^2 - 2\beta^2(1+\alpha\beta) + \beta^3]} \\
&\quad + \frac{-\alpha\beta^2(1+\alpha\beta)^2 - (1+\alpha\beta)^4}{6[\beta(1+\alpha\beta)^2 - 2\beta^2(1+\alpha\beta) + \beta^3]}, \\
&d_{03} = \frac{[\beta(2+\alpha\beta) - (1+\alpha\beta)^2]^2[\beta(1+2\alpha\beta) + (1+\alpha\beta)^2]}{6[\beta^5(1+\alpha\beta)^3 - 2\beta^6(1+\alpha\beta)^2 + \beta^7(1+\alpha\beta)]}, \\
&d_{21} = \frac{[3\beta^3(2-\alpha+\alpha\beta)(1+\alpha\beta) + 3\alpha\beta^3(2+\alpha\beta)][\beta(2+\alpha\beta) - (1+\alpha\beta)^2]}{2[\beta^3(1+\alpha\beta)^3 - 2\beta^4(1+\alpha\beta)^2 + \beta^5(1+\alpha\beta)]} \\
&\quad + \frac{[(3\beta - 4\beta^2)(1+\alpha\beta)^2 - 3(1+\alpha\beta)^4][\beta(2+\alpha\beta) - (1+\alpha\beta)^2]}{2[\beta^3(1+\alpha\beta)^3 - 2\beta^4(1+\alpha\beta)^2 + \beta^5(1+\alpha\beta)]}, \\
&d_{12} = \frac{[(1+\alpha\beta)^4 + 2\beta(1+\alpha\beta)^3][\beta(2+\alpha\beta) - (1+\alpha\beta)^2]}{2[\beta^4(1+\alpha\beta)^3 - 2\beta^5(1+\alpha\beta)^2 + \beta^6(1+\alpha\beta)]} \\
&\quad + \frac{[\beta^2(1+\alpha\beta)^2 - 4\beta^2(2+\alpha\beta)(1+\alpha\beta)][\beta(2+\alpha\beta) - (1+\alpha\beta)^2]}{2[\beta^4(1+\alpha\beta)^3 - 2\beta^5(1+\alpha\beta)^2 + \beta^6(1+\alpha\beta)]}.
\end{aligned}$$

Let

$$J(E_2) = \begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix},$$

namely,

$$J(E_2)=\begin{pmatrix}\frac{\beta(2+\alpha\beta)}{(1+\alpha\beta)^2} & -\frac{1}{(1+\alpha\beta)^2}\\ \frac{\beta(2+\alpha\beta)-(1+\alpha\beta)^2}{1+\alpha\beta} & \frac{-\beta+(1+\alpha\beta)^2}{\beta(1+\alpha\beta)}\end{pmatrix}.$$

It is not difficult to derive that the two eigenvalues of the matrix $J(E_2)$ are

$$\lambda_{1,2}=\frac{\beta^2(2+\alpha\beta)-\beta(1+\alpha\beta)+(1+\alpha\beta)^3\pm i\gamma}{2\beta(1+\alpha\beta)^2},$$

where $\gamma=\sqrt{4\beta^2(1+\alpha\beta)^4-[\beta^2(2+\alpha\beta)-\beta(1+\alpha\beta)+(1+\alpha\beta)^3]^2}$.

Their corresponding eigenvectors are

$$v_{1,2}=\begin{pmatrix}-2\beta\\ M\end{pmatrix}\pm i\begin{pmatrix}0\\ \gamma\end{pmatrix},$$

where $M=\beta^2(2+\alpha\beta)-\beta(1+\alpha\beta)+(1+\alpha\beta)^3$.

Let

$$T_2=\begin{pmatrix}0 & -2\beta\\ \gamma & M\end{pmatrix},\ \text{then}\ T_2{}^{-1}=\begin{pmatrix}\frac{M}{2\beta\gamma} & \frac{1}{\gamma}\\ -\frac{1}{2\beta} & 0\end{pmatrix}.$$

Take the transformation of variables:

$$(u,v)^T=T_2(X,Y)^T,$$

then the system (3.7) is transformed into the following form

$$\begin{cases}X\to & e_{10}X+e_{01}Y+\overline{F}(X,Y)+o(\rho_5^3),\\ Y\to & f_{10}X+f_{01}Y+\overline{G}(X,Y)+o(\rho_5^3),\end{cases}\tag{3.8}$$

where $\rho_5=\sqrt{X^2+Y^2}$,

$$\overline{F}(X,Y)=e_{20}X^2+e_{11}XY+e_{02}Y^2+e_{30}X^3+e_{21}X^2Y+e_{12}XY^2+e_{03}Y^3,$$

$$\overline{G}(X,Y)=f_{20}X^2+f_{11}XY+f_{02}Y^2+f_{30}X^3+f_{21}X^2Y+f_{12}XY^2+f_{03}Y^3,$$

$$e_{10}=(\frac{M}{2\beta\gamma}c_{01}+\frac{1}{\gamma}d_{01})\gamma,e_{01}=(\frac{M}{2\beta\gamma}c_{01}+\frac{1}{\gamma}d_{01})M-(\frac{M}{2\beta\gamma}c_{10}+\frac{1}{\gamma}d_{10})2\beta,$$

$$f_{10}=-\frac{\gamma}{2\beta}c_{01}\gamma,f_{01}=-\frac{M}{2\beta}c_{01}+c_{10},$$

$$e_{20}=(\frac{M}{2\beta\gamma}c_{02}+\frac{1}{\gamma}d_{02})\gamma^2,e_{30}=(\frac{M}{2\beta\gamma}c_{03}+\frac{1}{\gamma}d_{03})\gamma^3,$$

$$e_{02}=(\frac{M}{2\beta\gamma}c_{02}+\frac{1}{\gamma}d_{02})M^2+4(\frac{M}{2\beta\gamma}c_{20}+\frac{1}{\gamma}d_{20})\beta^2$$

$$
\begin{aligned}
&- 2(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta M,\\
e_{11} &= 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma M - 2(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta\gamma,\\
e_{03} &= (\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})M^3 - 8(\frac{M}{2\beta\gamma}c_{30} + \frac{1}{\gamma}d_{30})\beta^3\\
&- 2(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta M^2 + 4(\frac{M}{2\beta\gamma}c_{21} + \frac{1}{\gamma}d_{21})\beta^2 M,\\
e_{21} &= 3(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})\gamma^2 M - 2(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta\gamma^2,\\
e_{12} &= 3(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})\gamma M^2 - 4(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta L\gamma\\
&+ 4(\frac{M}{2\beta\gamma}c_{21} + \frac{1}{\gamma}d_{21})\beta^2\gamma,\\
f_{02} &= -\frac{1}{2\beta}(4\beta^2 c_{20} + M^2 c_{02} - 2\beta M c_{11}),\\
f_{11} &= -\frac{1}{2\beta}(2\gamma M c_{02} - 2\beta\gamma c_{11}), f_{30} = -\frac{\gamma^3}{2\beta}c_{03},\\
f_{03} &= -\frac{1}{2\beta}(-8\beta^3 c_{30} + M^3 c_{03} + 4\beta^2 M c_{21} - 2\beta M^2 c_{12}),\\
f_{21} &= -\frac{1}{2\beta}(3\gamma^2 M c_{03} - 2\beta\gamma^2 c_{12}),\\
f_{12} &= -\frac{1}{2\beta}(3M^2\gamma c_{03} + 4\beta^2\gamma c_{21} - 4\beta M\gamma c_{12}).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\overline{F}_{XX} &= 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma^2, \overline{F}_{XXX} = 6(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})\gamma^3,\\
\overline{F}_{XY} &= 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma M - 2(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta\gamma,\\
\overline{F}_{YY} &= 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})M^2 + 8(\frac{M}{2\beta\gamma}c_{20} + \frac{1}{\gamma}d_{20})\beta^2\\
&- 4(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta M,
\end{aligned}
$$

$$
\begin{aligned}
\overline{F}_{XXY} &= 6(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})\gamma^2 M - 4(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta\gamma^2,\\
\overline{F}_{XYY} &= 6(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})\gamma M^2 - 8(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta M\gamma\\
&+ 8(\frac{M}{2\beta\gamma}c_{21} + \frac{1}{\gamma}d_{21})\beta^2\gamma,\\
\overline{F}_{YYY} &= 3(\frac{M}{2\beta\gamma}c_{03} + \frac{1}{\gamma}d_{03})M^3 - 24(\frac{M}{2\beta\gamma}c_{30} + \frac{1}{\gamma}d_{30})\beta^3\\
&- 6(\frac{M}{2\beta\gamma}c_{12} + \frac{1}{\gamma}d_{12})\beta M^2 + 12(\frac{M}{2\beta\gamma}c_{21} + \frac{1}{\gamma}d_{21})\beta^2 M,
\end{aligned}
$$

$$\overline{G}_{XX} = -\frac{\gamma^2}{\beta}c_{02}, \overline{G}_{XY} = -\frac{1}{2\beta}(2\gamma Mc_{02} - 2\beta\gamma c_{11}),$$
$$\overline{G}_{YY} = -\frac{1}{\beta}(4\beta^2 c_{20} + M^2 c_{02} - 2\beta Mc_{11}),$$
$$\overline{G}_{XXX} = -\frac{3\gamma^3}{\beta}c_{03}, \overline{G}_{XXY} = -\frac{1}{\beta}(3\gamma^2 Mc_{03} - 2\beta\gamma^2 c_{12}),$$
$$\overline{G}_{XYY} = -\frac{1}{\beta}(3M^2\gamma c_{03} + 4\beta^2\gamma c_{21} - 4\beta M\gamma c_{12}),$$
$$\overline{G}_{YYY} = -\frac{3}{\beta}(-8\beta^3 c_{30} + M^3 c_{03} + 4\beta^2 Mc_{21} - 2\beta M^2 c_{12}).$$

In order to determine the direction and the stability of an invariant closed orbit bifurcated from Neimark-Sacker bifurcation of the system (3.8), one needs to calculate the discriminating quantity

$$L = -Re\Big(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\zeta_{20}\zeta_{11}\Big) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\lambda_2\zeta_{21}), \tag{3.9}$$

where

$$\zeta_{20} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} + 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} - 2\overline{F}_{XY})],$$
$$\zeta_{11} = \frac{1}{4}[\overline{F}_{XX} + \overline{F}_{YY} + i(\overline{G}_{XX} + \overline{G}_{YY})],$$
$$\zeta_{02} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} - 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} + 2\overline{F}_{XY})],$$
$$\zeta_{21} = \frac{1}{16}[\overline{F}_{XXX} + \overline{F}_{XYY} + \overline{G}_{XXY} + \overline{G}_{YYY}$$
$$+ i(\overline{G}_{XXX} + \overline{G}_{XYY} - \overline{F}_{XXY} - \overline{F}_{YYY})],$$

and $L$ is required not to be zero [25–28]. Some calculations display

$$\zeta_{20} = \frac{1}{8}[2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma^2 - 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})L^2 - 8(\frac{M}{2\beta\gamma}c_{20} + \frac{1}{\gamma}d_{20})\beta^2$$
$$+ 4(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta M - \frac{1}{\beta}(2\gamma Mc_{02} - 2\beta\gamma c_{11})]$$
$$+ \frac{1}{8}i[-\frac{\gamma^2}{\beta}c_{02} + \frac{1}{\beta}(4\beta^2 c_{20} + M^2 c_{02} - 2\beta Mc_{11})],$$
$$\zeta_{11} = \frac{1}{4}[2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma^2 + 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})M^2$$
$$+ 8(\frac{M}{2\beta\gamma}c_{20} + \frac{1}{\gamma}d_{20})\beta^2 - 4(\frac{M}{2\beta\gamma}c_{11} + \frac{1}{\gamma}d_{11})\beta M]$$
$$+ \frac{1}{4}i[-\frac{\gamma^2}{\beta}c_{02} - \frac{1}{\beta}(4\beta^2 c_{20} + M^2 c_{02} - 2\beta Mc_{11})],$$
$$\zeta_{02} = \frac{1}{8}[2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})\gamma^2 - 2(\frac{M}{2\beta\gamma}c_{02} + \frac{1}{\gamma}d_{02})M^2$$
$$- 8(\frac{M}{2\beta\gamma}c_{20} + \frac{1}{\gamma}d_{20})\beta^2$$

$$+4(\frac{M}{2\beta\gamma}c_{11}+\frac{1}{\gamma}d_{11})\beta M+\frac{1}{\beta}(2\gamma Mc_{02}-2\beta\gamma c_{11})]$$

$$+\frac{1}{8}i[-\frac{\gamma^2}{\beta}c_{02}+\frac{1}{\beta}(4\beta^2c_{20}+M^2c_{02}-2\beta Mc_{11})$$

$$+4(\frac{M}{2\beta\gamma}c_{02}+\frac{1}{\gamma}d_{02})\gamma M-4(\frac{M}{2\beta\gamma}c_{11}+\frac{1}{\gamma}d_{11})\beta\gamma],$$

$$\zeta_{21}=\frac{1}{16}[6(\frac{M}{2\beta\gamma}c_{03}+\frac{1}{\gamma}d_{03})\gamma^3-6(\frac{M}{2\beta\gamma}c_{03}+\frac{1}{\gamma}d_{03})\gamma M^2$$

$$+8(\frac{M}{2\beta\gamma}c_{12}+\frac{1}{\gamma}d_{12})\beta M\gamma$$

$$-8(\frac{M}{2\beta\gamma}c_{21}+\frac{1}{\gamma}d_{21})\beta^2\gamma-\frac{1}{\beta}(3\gamma^2Mc_{03}-2\beta\gamma^2c_{12})$$

$$-\frac{3}{\beta}(-8\beta^3c_{30}+M^3c_{03}+4\beta^2Mc_{21}-2\beta M^2c_{12})]$$

$$+\frac{1}{16}i[-\frac{3\gamma^3}{\beta}c_{03}-\frac{1}{\beta}(3M^2\gamma c_{03}+4\beta^2\gamma c_{21}-4\beta M\gamma c_{12})$$

$$-6(\frac{M}{2\beta\gamma}c_{03}+\frac{1}{\gamma}d_{03})\gamma^2M+4(\frac{M}{2\beta\gamma}c_{12}+\frac{1}{\gamma}d_{12})\beta\gamma^2$$

$$-3(\frac{M}{2\beta\gamma}c_{03}+\frac{1}{\gamma}d_{03})M^3+24(\frac{M}{2\beta\gamma}c_{30}+\frac{1}{\gamma}d_{30})\beta^3$$

$$+6(\frac{M}{2\beta\gamma}c_{12}+\frac{1}{\gamma}d_{12})\beta M^2-12(\frac{M}{2\beta\gamma}c_{21}+\frac{1}{\gamma}d_{21})\beta^2M].$$

Based on the above analysis, one can see that the system (1.9) undergoes a Neimark-Sacker bifurcation in the fixed point $E_2$ when the parament $\delta$ varies in a small neighborhood of $\delta_1$ for $\alpha < \frac{\beta-2+\sqrt{\beta^2+4\beta}}{2\beta}$. In addition, if in (3.9) $L < (>)0$, then an attracting (repelling) invariant closed curve bifurcates from the fixed point $E_2$ for $\delta > (<)\delta_1$. So, the proof of the Theorem 3.3 is completed. □

*Remark* 3.4 The occurrence of a Neimark-Sacker bifurcation causes the system to jump from stable window to chaotic states through periodic and quasi-periodic states, and trigger a route to chaos.

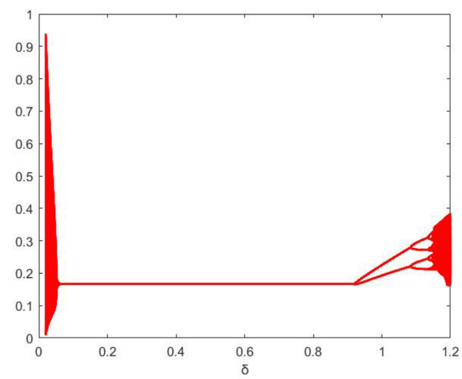
## 4 Numerical simulation

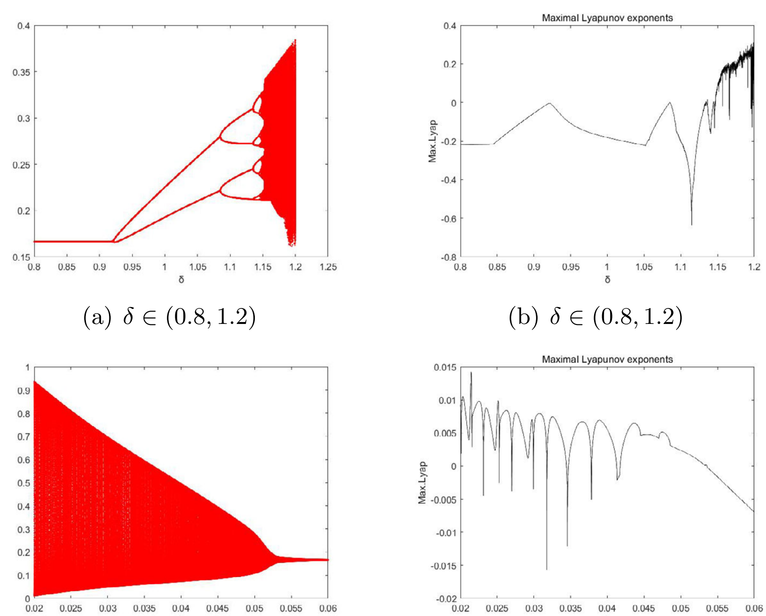
In this section, to illustrate theoretical analysis derived above, we present the bifurcation diagrams, phase portraits, and Lyapunov exponents for specific parameter values using Matlab software with an automatic resolution ratio. The following cases of bifurcation parameters are considered.

Vary $\delta$ in the range $(0.02, 1.2)$ and fix $\alpha = 0.8, \beta = 2.5$ with the initial values $(x_0, y_0) = (0.16, 0.42)$. Figure 1 shows that there are two bifurcations in the system (1.9). Now we explore the details.

First, let $\delta \in (0.8, 1.2)$. One can obtain Fig. 2(a) and observe the existence of a flip bifurcation in the fixed point $E_2 = (0.16667, 0.41667)$ when $\delta = \delta_0 = 0.92$, which is in accordance with the result in Theorem 3.2. Figure 2(b) means the spectrum of maximum Lyapunov exponent.

**Figure 1** Bifurcation of the system (1.9) in ($\delta$, $x$)-plane for $\delta \in (0.02, 1.2)$



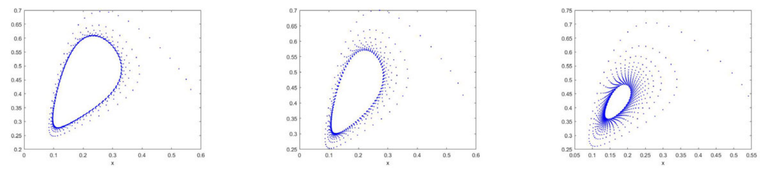

(a) $\delta \in (0.8, 1.2)$ (b) $\delta \in (0.8, 1.2)$

(c) $\delta \in (0.02, 0.06)$ (d) $\delta \in (0.02, 0.06)$

**Figure 2** Bifurcation of the system (1.9) in $(\delta, x)$-plane and maximal Lyapunov exponent

Then, let $\delta \in (0.02, 0.06)$. Figure 2(c) shows the bifurcation diagram in $(\delta, x)$-plane from which the fixed point $E_2$ is stable when $\delta > \delta_1 = 0.053$ and unstable when $\delta < \delta_1$. Hence, a Neimark-Sacker bifurcation occurs in $E_2$ when $\delta = \delta_1$, whose multipliers are $\lambda_{1,2} = \frac{89\pm\sqrt{179}i}{90}$ with $|\lambda_{1,2}| = 1$. The corresponding maximum Lyapunov exponent diagram of the system (1.9) is plotted in Fig. 2(d).

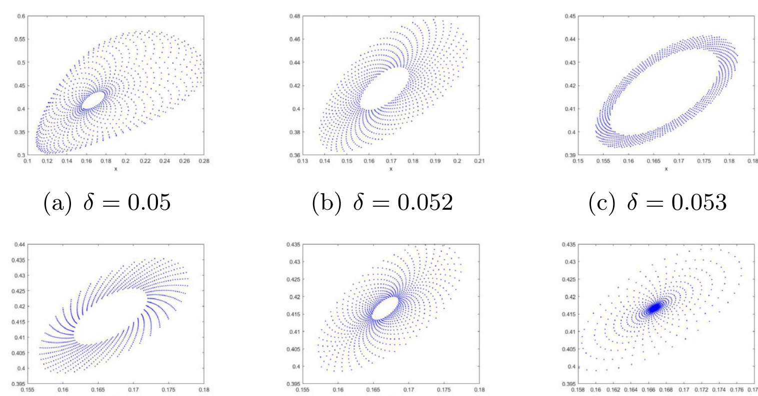
Take the initial values $(x_0, y_0) = (0.16, 0.42), (0.16, 0.2)$ in Fig. 3 and Fig. 4, respectively. These figures show that the dynamical properties of the fixed point $E_2$ change from unstable to stable as the value of the parameter $\delta$ increases, and there is an occurrence of invariant closed curve around $E_2$ when $\delta = \delta_1$. Figure 3 displays that the bifurcated closed orbit is stable outside while Fig. 4 indicates that the bifurcated closed orbit is stable inside, which agrees with the result of Theorem 3.3.



(a) $\delta = 0.048$ (b) $\delta = 0.05$ (c) $\delta = 0.053$

(d) $\delta = 0.055$ (e) $\delta = 0.058$ (f) $\delta = 0.06$

**Figure 3** Phase portraits of the system (1.9) with $\alpha = 0.8$, $\beta = 2.5$ and different $\delta$ and the initial value $(x_0, y_0) = (0.16, 0.2)$ inside the closed orbit



(a) $\delta = 0.05$ (b) $\delta = 0.052$ (c) $\delta = 0.053$

(d) $\delta = 0.054$ (e) $\delta = 0.055$ (f) $\delta = 0.058$

**Figure 4** Phase portraits for the system (1.9) with $\alpha = 0.8$, $\beta = 2.5$ and different $\delta$ and the initial value $(x_0, y_0) = (0.16, 0.42)$ outside the closed orbit

## 5 Conclusion

In this paper, we revisit a predator-prey model with a ratio-dependent Holling-Tanner functional response. By applying the semi-discretization method instead of the forward Euler method, the system (1.5) is transformed into the system (1.9). Under given parametric conditions, we comprehensively demonstrate the existence and stability of two nonnegative fixed points $E_1 = (1, 0)$ and $E_2 = (\frac{1+\alpha\beta-\beta}{\alpha\beta+1}, \frac{\beta(1+\alpha\beta-\beta)}{\alpha\beta+1})$. Moreover, one derives the sufficient conditions for the occurrence of the flip bifurcation and Neimark-Sacker bifurcation. In particular, the positive equilibrium $E_2$ is shown to be asymptotically stable when $\delta > \delta_1 = \frac{\beta(\alpha\beta+2)-(\alpha\beta+1)^2}{\beta^2(\alpha\beta+1)}$ and unstable when $\delta < \delta_1$. Hence, the system (1.9) undergoes a Neimark-Sacker bifurcation when the parameter $\delta$ goes through the critical value $\delta_1$. This displays the coexistence of prey and predator when the parameter $\delta = \delta_1$.

We made a surprising discovery: for the same differential system (1.5), different discrete methods—the forward Euler method used in [20] and the semi-discretization method employed in this paper—can lead to different conclusions. Specifically, the fixed point $E_1$ is a saddle for $\delta < 2$ in [20], whereas our results show that the fixed point $E_1$ is always a saddle. Additionally, a flip bifurcation is reported at $E_1$ in [20], while our findings indicate that no bifurcation occurs at $E_1$.

This finding highlights the importance of approaching the problem from different angles or directions to gain a comprehensive understanding. Considering various perspectives may sometimes lead to differing or entirely new results.

**Acknowledgements**
This work is finished with the discussion of Master Xingming Shao.

**Author contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

**Funding**
This work is partly supported by the National Natural Science Foundation of China (61473340) and the Distinguished Professor Foundation of Qianjiang Scholar in Zhejiang Province (F708108P02).

**Data availability**
There is no applicable data associated with this manuscript.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

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Received: 11 June 2024 Accepted: 29 November 2024 Published online: 19 December 2024

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