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Well-posedness problem of an anisotropic parabolic equation with a nonstandard growth order

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Abstract

Consider the well-posedness problem of an anisotropic parabolic equation with a nonstandard growth order. The weak solution is introduced, an L^∞-estimate is provided, and the existence is established using the parabolically regularized method. However, the weak solution under consideration is not in $L^1(0,T;W_0^{1,\vec{p}(x)}(\Omega))$, and defining the trace becomes a new problem. In this paper, we give a new definition for the trace of $u\in L^\infty(Q_T)$ to solve this problem. Based on the new concept of the generalized trace and the selection of suitable test functions, the stability theorems of the weak solutions are obtained.

Mathematics Subject Classification: 35K65; 35D05; 35B65; 35K99

Keywords: Nonstandard growth order; Trace; L^∞-estimate; Stability

1 Introduction

Consider an anisotropic parabolic equation

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right),\quad (x,t)\in Q_T, \tag{1.1}$$

with the initial value condition

$$u(x,0)=u_0(x),\quad x\in\Omega, \tag{1.2}$$

and the boundary value condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $Q_T=\Omega\times(0,T)$, $d(x)=\mathrm{dist}(x,\partial\Omega)$ is the distance function from the boundary $\partial\Omega$, both $p_i(x)$ and $a_i(x)$ belong to $C^1(\overline{\Omega})$, $1\le i\le N$. Compared with the usual non-Newtonian fluids equation

$$u_t=\mathrm{div}(|\nabla u|^{p-2}\nabla u),\ (x,t)\in Q_T,$$

Equation (1.1) incorporates variable exponents $p_i(x)$ and is referred to as an evolutionary equation with a nonstandard growth order.

Equation (1.1) arises in various physics and biology contexts, such as the dynamics of fluids in porous media [9], anisotropic reaction-diffusion-advection systems [10], and the spread of epidemic diseases in heterogeneous environments [2, 11]. When $p_i(x) = p_i$ is a constant, the earliest study of Equation (1.1) can be traced back to [18] in 1968.

Since the beginning of this century, with the development of the theory of the variable exponent Sobolev space [13, 17], many mathematicians have shifted their focus to the solvability of the parabolic equation

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + f(x,t,u),\ (x,t) \in Q_T, \tag{1.4}$$

or its anisotropic version

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}\right) + f(x,t,u),\ (x,t) \in Q_T.$$

Researchers have investigated the essential changes arising from the nonstandard growth order $p(x)$ or $p_i(x)$ [3, 12, 16]. In recent years, a more general equation than (1.4), namely,

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + f(x,t,u),\ (x,t) \in Q_T, \tag{1.5}$$

has garnered significant attention. This kind of equation arises from the image denoising [15] and the megascopic double porosity model in a periodic fractured medium [1], where $a(x)$ is the diffusion coefficient. If $a(x) > a > 0$, the well-posedness problem and the blow-up phenomena of weak solution to equation (1.5) have been studied in [4, 20, 22]. Moreover, a more complex equation,

$$u_t = \operatorname{div}(a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u) + f(x,t),\ (x,t) \in Q_T, \tag{1.6}$$

has also been analyzed. In particular, if $a(x,t,u) = |u|^{\alpha} + d_0$, $d_0 > 0$, $\alpha \geq 2$, then the existence and the uniqueness of weak solutions to Equation (1.6) were discussed in [14]. When $a(x,t,u) > a > 0$ is not imposed, the existence of weak solution to Equation (1.6) was studied in [6]; however, no uniqueness results were provided in [6]. In summary, when $a(x,t,u) \geq 0$, the uniqueness of weak solution to Equation (1.6) remains an open problem.

As for the anisotropic case, if $a_i(x) > a > 0$,

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right) + f(x,t,u),\quad (x,t) \in Q_T,$$

then the blow-up and extinction in finite time of the weak solutions have been studied in [5, 7]. If

$$a_i(x) > 0,\ x \in \Omega,\quad i = 1,2,\ldots,N,$$

and

$$\int_\Omega a_i(x)^{-\frac{1}{p_i(x)-1}}\,dx < \infty,\ j = 1,2,\ldots,N,$$

the existence of weak solutions to Equation (1.1) with the initial value condition (1.2) was proved in [27]. Moreover, by imposing various restrictions on the diffusion coefficient $a_i(x)$, several stability theorems for weak solutions have been established in [28] using the weak characteristic function method. The key contribution of [28] lies in the authors' attempt to study the stability theorem independently of the boundary value condition (1.3). Naturally, alternative conditions must be imposed on $a_i(x)$ to replace the boundary value condition (1.3) in [28]. However, based on the related references discussed above, we observe an important gap that remains to be addressed. Specifically, if there exists an $i_0 \in \{1, 2, \dots, N\}$ such that

$$\int_\Omega a_{i_0}(x)^{-\frac{1}{p_{i_0}(x)-1}} dx = \infty, \tag{1.7}$$

then the boundary value condition (1.3) cannot be imposed in the sense of trace. How, then, can one prove the stability of weak solutions to equation (1.1)? A typical example is $a_i(x) = d^{p_i(x)}$, which gives Equation (1.1) the important characteristic

$$\int_\Omega d(x)^{-\frac{p_i(x)}{p_i(x)-1}} dx = \infty,\ i = 1, 2, \dots, N.$$

Such a fact invalidates all the stability theorems obtained in [28].

The primary objective of this paper is to provide a method for addressing the well-posedness problem of Equation (1.1) under the condition that (1.7) holds. A secondary contribution of this paper is the relaxation of assumptions regarding the existence result of the weak solution, allowing the diffusion coefficient $a_i(x)$ to vanish within the interior of the domain. In contrast, $a_i(x) > 0$, $x \in \Omega$ was the fundamental assumption in our previous works [27–30]. The novelty of this paper lies in the generalization of the concept of the classical trace for $u \in L^1(0, T; W_0^{1,r}(\Omega))$, $r \geq 1$ to $L^\infty(Q_T)$, when $a_i(x) = d^{p_i(x)}$. This generalization enables the study of the stability of weak solutions.

The paper is arranged as follows. In Sect. 2, we generalize the concept of the classical trace for $u \in L^1(0, T; W_0^{1,r}(\Omega))$ to a weaker functional space, where $r \geq 1$. In Sect. 3, we define the weak solution to equation (1.1) and present the main theorems of the paper. In Sect. 4, we obtain the L^∞-estimate and prove the existence of a weak solution. In Sect. 5, we prove two stability theorems. The first theorem focuses on the case $a_i(x) = d^{\alpha_i}$, with α_i being a constant. The second theorem addresses the question of defining the trace of u when $u \notin W^{1,1}(\Omega)$, in the case where $a_i(x) = d^{p_i(x)}$, and studies the stability under the boundary value condition (1.3).

2 The generalization of the trace

We denote that

$$p_i^+ = \max_{x\in\overline{\Omega}} p_i(x),\ p_i^- = \min_{x\in\overline{\Omega}} p_i(x)$$

$$1 < p_0 = \min_{x\in\overline{\Omega}} \{p_1(x), p_2(x), \dots, p_{N-1}(x), p_N(x)\},$$

$$p^0 = \max_{x\in\overline{\Omega}} \{p_1(x), p_2(x), \dots, p_{N-1}(x), p_N(x)\}.$$

To summarize the introduction, since $a_i(x)$ may be equal to zero both on the boundary $\partial\Omega$ and within the interior of Ω, the weak solution $u(x,t)$ is generally satisfies only the condition

$$\int_\Omega a_i(x)|u_{x_i}|^{p_i(x)}dx < \infty.$$

This inequality is weaker than

$$\int_\Omega |u_{x_i}|^{p_i(x)}dx < \infty,$$

and the boundary value condition (1.3) cannot be imposed in sense of the trace in classical way, i.e., (2.4) below. In some details, for every i, $1 \le i \le N$, we denote that

$$\begin{aligned}
&\Sigma_{1i} = \{x \in \partial\Omega : a_i(x) > 0\},\\
&\Sigma_{2i} = \left\{x \in \partial\Omega :\ a_i(x) = 0,\ \text{there exists } r > 0, \text{such that } \int_{\Omega\bigcap B_r(x)} a_i(y)^{-\frac{1}{p_i(y)-1}}dy < +\infty\right\},\\
&\Sigma_{3i} = \left\{x \in \partial\Omega :\ a_i(x) = 0,\ \text{for any small } r > 0,\ \int_{\Omega\bigcap B_r(x)} a_i(y)^{-\frac{1}{p_i(y)-1}}dy = +\infty\right\}.
\end{aligned}$$

Clearly, for every i, we have

$$\partial\Omega = \Sigma_{1i}\bigcup\Sigma_{2i}\bigcup\Sigma_{3i}.$$

According to the analysis of [26, 29], a part of the boundary value condition of (1.3)

$$u(x,t) = 0,\ (x,t) \in \left\{\left(\bigcap_{i=1}^N \Sigma_{1i}\right)\bigcup\left(\bigcap_{i=1}^N \Sigma_{2i}\right)\right\} \times (0,T),$$

can be imposed in the classical trace. For example, when $a_i(x) = d^{\alpha_i}$, $\alpha_i < p_0 - 1$, then we know $\int_\Omega d(x)^{-\frac{\alpha_i}{p_i(x)-1}}dx < +\infty$, and on Σ_{2i}, the part boundary value condition

$$u(x,t) = 0,\ (x,t) \in \Sigma_{2i} \times (0,T),$$

can be imposed in the classical trace.

The difficulty in determining how to define

$$u(x,t) = 0,\ (x,t) \in \{\partial\Omega \setminus \Sigma_1\} \times (0,T), \tag{2.1}$$

where $\Sigma_1 = \{(\bigcap_{i=1}^N \Sigma_{1i})\bigcup(\bigcap_{i=1}^N \Sigma_{2i})\}$. It appears to be extremely challenging to solve this problem completely and perfectly, and it deserves further discussion. Consider the non-Newtonian fluid equation in the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} + c(x,t)u = f(x,t), \quad (x,t) \in Q_T, \tag{2.2}$$

where $p > 1$, $0 \le a(x) \in C(\overline{\Omega})$, $b_i(x) \in C^1(\overline{\Omega})$, $c(x,t)$ and $f(x,t)$ are continuous functions on $\overline{Q}_T$. Define $\mathbf{B}$ as the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$\|u\|_{\mathbf{B}} = \iint_{Q_T} a(x)\left(|u(x,t)|^p + |\nabla u(x,t)|^p\right) dxdt,\ u \in \mathbf{B},$$

then for $u \in \mathbf{B}$, being a weak solution to Equation (2.2), the authors in [26] claimed that the boundary value condition (1.3) is over determined, and the necessary partial boundary value condition is

$$u(x,t) = 0,\ (x,t) \in \Sigma \times (0,T), \tag{2.3}$$

with $\Sigma = (\Sigma_2 \bigcup \Sigma_3 \bigcup \Sigma_4) \subseteq \partial\Omega$, and

$$\Sigma_3 = \{x \in \partial\Omega : a(x) > 0\},$$

$$\Sigma_4 = \{x \in \partial\Omega : a(x) = 0,\ \text{ and there exists } r > 0 \text{ such that } \int_{\Omega \bigcap B_r(x)} a(x)^{\frac{1}{p-1}} dx < \infty\},$$

$$\Sigma_2 = \{x \in \Sigma^0 : \sum_{i=1}^N b_i(x)n_i(x) < 0\},$$

where $B_r(x)$ is the ball centered at x and with radius r, and $\Sigma^0 = \partial\Omega \setminus (\Sigma_3 \bigcup \Sigma_4)$. Proposition 2.1 in [26] yields that, if $u \in \mathbf{B}$, then

$$u(x,t) = 0,\ (x,t) \in (\Sigma_3 \bigcup \Sigma_4) \times (0,T),$$

can be defined as in the classical way, i.e.,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = 0,\ (x,t) \in (\Sigma_3 \bigcup \Sigma_4) \times (0,T), \tag{2.4}$$

where $u_n \in C_0^\infty(Q_T)$. While

$$u(x,t) = 0,\ (x,t) \in \Sigma_2 \times (0,T),$$

is imposed in the sense of that

$$\limsup_{\lambda\to 0} \int_0^T \int_{\left\{x\in\partial\Omega_\lambda : \sum_{i=1}^N b_i(x)n_i(x)<0\right\}} u^2 \sum_{i=1}^N b_i(x)n_i(x) d\sigma dt = 0, \tag{2.5}$$

where $\lambda > 0$, $\limsup_{\lambda\to 0} f(\lambda) = \inf_{\delta>0}\{\operatorname{ess\,sup}\{f(\lambda) : |\lambda| < \delta\}\}$ is the super limit. One can see that, if u satisfies (2.4), then it also satisfies (2.5). Thus, one can regard that the boundary value condition (2.3), which matches up with the non-Newtonian fluid Equation (2.2), is imposed in a generalized trace defined as (2.5).

For the main Equation (1.1), a new problem arises. Since there is not the convection term $\sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i}$, it is impossible to define the boundary value condition (1.3) in the sense of (2.5). In other words, if one regards that the boundary value condition (1.3) is imposed in a sense of a generalized trace, then such a trace cannot be defined as (2.5).

However, when $u \in L^\infty(Q_T)$, if $p_i(x) = p_i$, in [30], then the author found that the part boundary value condition (2.1) can be imposed in the sense of

$$\limsup_{n\to\infty}\left[n \sup_{x\in D_n\setminus D_{\frac{n}{2}}} a_i(x)^{\frac{1}{p_i}}|\chi_{x_i}||u|\right] = 0,\ i = 1,2,\dots,N, \tag{2.6}$$

where $D_n = \{x \in \Omega : \chi(x) > \frac{1}{n}\}$, and χ is a weak characteristic function of Ω, i.e., it is a continuous function when x is near $\partial\Omega$ and satisfies

$$\chi(x) > 0,\ x \in \Omega \text{ and } \chi(x) = 0,\ x \in \partial\Omega.$$

Now, if $u \in L^\infty(Q_T)$ is a weak solution to Equation (1.1), we obviously can generalize (2.6) as follows:

$$\limsup_{n\to\infty}\left[n \sup_{x\in D_n\setminus D_{\frac{n}{2}}} a_i(x)^{\frac{1}{p_i(x)}}|\chi_{x_i}||u|\right] = 0,\ i = 1,2,\dots,N. \tag{2.7}$$

This allows us to define the boundary value condition (1.3) in the sense of (2.7). However, a significant disadvantage is that both (2.6) and (2.7) depend on the choice of $\chi(x)$. If the choice of $\chi(x)$ is not so good, verifying whether (2.7) holds can become challenging. So, in this paper, we aim to find another definition of the trace of $u \in L^\infty(Q_T)$ to avoid such a trouble. For simplicity, we only consider the typical case where $a_i(x) = d^{p_i(x)}$, and using some idea of [30], we introduce the following new definition about the trace.

Definition 2.1 If $u(x,t) \in L^\infty(Q_T)$, then the generalization of trace of $u = 0$ on $\partial\Omega \setminus \Sigma_1$ can be defined as

$$\limsup_{n\to\infty} \sup_{x\in\Omega_n\setminus\Omega_{\frac{n}{2}}} d^2|u| = 0, \tag{2.8}$$

where $\Omega_n = \left\{x \in \Omega : d^2 > \frac{1}{n}\right\}$.

By such a generalization, when $a_i(x) = d^{p_i(x)}$, if one impose the homogeneous boundary value condition (1.3) in the sense of Definition 2.1, the stability of weak solutions to Equation (1.1) can be proved. This is the main dedication of this paper.

At the end of this section, we give a short explanation of the above generalized trace. It is well-known that, when u is in $W_0^{1,p}(\Omega)$ or $BV(\Omega)$, u is almost everywhere differentiable in Ω, and the classical trace of $u = 0$ on the boundary is defined in the sense of (2.4). Since the weak solution considered in this paper is not in $W_0^{1,p}(\Omega)$ or $BV(\Omega)$, we have to define $u = 0$ on the boundary in a new way as (2.7) or (2.8). Actually, one can see that, when $u(x,t)$ is a continuous function in Q_T, $u(x,t)$ satisfies (2.5), (2.6), (2.7), and (2.8) simultaneously. Certainly, if for every $t \in (0,T)$, $u(x,t)$ in $W_0^{1,p}(\Omega)$ or $BV(\Omega)$, then it satisfies (2.5), (2.6), and (2.8) naturally. Among these generalized traces, (2.5) itself is not comparable with the other definitions of trace listed above. Definitions (2.6) and (2.7) are similar. Definition (2.8) is only can be used in the case when $a_i(x) = d^{p_i(x)}$. If $u(x,t)$ satisfies (2.8), then by a simple calculation, one can see that (2.6) is true only if $\chi(x) = d^2$.

3 The definitions of weak solutions and the main results

We introduce a function space

$$V=\left\{v\in L^{p_0}\left(0,T;W_0^{1,\vec{p}(x)}(\Omega)\right):|v_{x_i}|\in L^{p_i(x)}(Q_T),i=1,2,\dots,N\right\},$$

endowed with the norm $\|u\|_V=|u|_{L^{p_0}\left(0,T;W_0^{1,\vec{p}(x)}(\Omega)\right)}+|\nabla u|_{L^{\vec{p}(x)}(Q_T)}$. Then, V is a separable and reflexive Banach space [5]. We denote V^* by its dual space.

Lemma 3.1 *Assume that $\pi:\mathbb{R}\to\mathbb{R}$ is a piecewise function in C^1, satisfying $\pi(0)=0$, and is out of a compact set $\pi'=0$. Let $\Pi(s)=\int_0^s\pi(\sigma)d\sigma$. If $u\in V$ and $u_t\in V^*$, we have*

$$\begin{aligned}\int_0^T<u_t,\pi(u)>dt:=&<u_t,\pi(u)>_{V^*,V\cap L^\infty(Q_T)}\\=&\int_\Omega\Pi(u(T))dx-\int_\Omega\Pi(u(0))dx.\end{aligned}\tag{3.1}$$

Lemma 3.1 can be found in [19].

Definition 3.2 A function $u(x,t)$ is said to be a weak solution to Equation (1.1) with the initial value condition (1.2) if

$$u\in L^\infty(Q_T),\ \frac{\partial u}{\partial t}\in V^*,\ a_i(x)|u_{x_i}|^{p_i(x)}\in L^1(Q_T),\tag{3.2}$$

and for any function $\varphi\in C_0^1(Q_T)$, such that

$$\int_0^T\left\langle\frac{\partial u}{\partial t},\varphi\right\rangle dt+\sum_{i=1}^N\iint_{Q_T}a_i(x)\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}\varphi_{x_i}dxdt=0.\tag{3.3}$$

The initial value condition (1.2) is satisfied in the sense of

$$\lim_{t\to0}\int_\Omega u(x,t)\psi(x)dx=\int_\Omega u_0(x)\psi(x)dx,\ \forall\psi(x)\in C_0^\infty(\Omega).\tag{3.4}$$

By this definition, we can prove the following existence theorem in next section.

Theorem 3.3 *Suppose that for every i, $1\le i\le N$, the measure of $\Omega_{0i}=\{x\in\Omega:a_i(x)=0\}$ is zero, i.e., $a_i(x)$ is almost everywhere positive on $\overline{\Omega}$, and $a_0(x)\in A_{p_0}$ is a weighted function. If $p_i(x)$ is a log-Hölder continuous function, $p_0>1$, $u_0(x)\in W^{1,p^0}(\Omega)\bigcap L^\infty(Q_T)$, then there exists a weak solution of Equation (1.1) with the initial value condition (1.2).*

Here, we set

$$a_0(x)=\min_{x\in\overline{\Omega}}\{a_1(x),a_2(x),\dots,a_N(x)\},$$

and for any $p>1$, A_p is the weight function space introduced by B. Muckenhoupt, the details are given in the next section.

In fact, $a_0(x) \in A_{p_0}$ is not a necessary condition. In the next section, such a condition is used for L^∞-estimate by the De Giorgi method in Theorem 4.7. We believe that by considering the approximate equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left((a_i(x)+\varepsilon)\left(|u_{x_i}|^2+\varepsilon\right)^{\frac{p_i(x)-2}{2}} u_{x_i}\right),\ (x,t) \in Q_T, \tag{3.5}$$

as in [27], the classical parabolic equation theory [18] can yield a classical solution u_ε of the initial-boundary value problem to Equation (3.5). Using the maximal principle, we have

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c. \tag{3.6}$$

The difference lies in that, in [27], $a_i(x) > 0$, $x \in \Omega$. However, in this paper, $a_i(x)$ may be equal to zero at some points in Ω. Certainly, such a minor difference does not affect the truth of (3.6). The significance of Theorem 4.7 below lies in that, using the De Giorgi method and the embedding theorem in the weighted Sobolev space, one can independently give an L^∞-estimate of an anisotropic parabolic equation with a nonstandard growth order.

In this paper, we are mainly concerned with the stability of the weak solutions. The first stability result is the following.

Theorem 3.4 *Let $a_i(x) = d^{\alpha_i}$, α_i be a constant, $u(x,t)$ and $v(x,t)$ be two solutions to Equation (1.1), $u_0(x)$ and $v_0(x)$ be the corresponding initial values, respectively. If*

$$\alpha_i \geq p_i^+ - 1,\ i = 1,2,\dots,N, \tag{3.7}$$

then there is

$$\int_\Omega |u(x,t)-v(x,t)|dx \leq c\int_\Omega |u_0(x)-v_0(x)|dx. \tag{3.8}$$

We noticed that for the non-Newtonian equation

$$u_t = \operatorname{div}(d^\alpha |\nabla u|^{p-2}\nabla u),\ (x,t) \in Q_T,$$

when $\alpha \geq p-1$, the stability (3.8) was proved by Yin-Wang in 2004 [25]. While, $\alpha < p-1$, the stability (3.8) can be true when the boundary value condition

$$u(x,t) = v(x,t) = 0, (x,t) \in \partial\Omega \times (0,T)$$

is imposed in the sense of the classical trace. Theorem 3.4 has partially improved Yin-Wing's results.

One can see that if condition (3.7) is assumed in Theorem 3.4, then there is nothing to do with the boundary value condition. At the same time, as we have introduced above, if $\alpha_i < p_i^- - 1$, the boundary value condition (1.3) can be imposed, the stability is also true as (3.8). However, since $p_i(x)$ is a function on $\overline{\Omega}$, there is generally a gap between $p_i^- - 1$ and $p_i^+ - 1$. Understanding how to bridge and reconcile this gap is an interesting problem.

In the other words, if $p_i^- - 1 \le \alpha_i \le p_i^+ - 1$, is Theorem 3.4 still valid? In this paper, we directly discuss the stability of weak solutions when $a_i(x) = d^{p_i(x)}$ and obtain the following theorem.

Theorem 3.5 *Let $a_i(x) = d^{p_i(x)}$. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions to Equation (1.1), $u_0(x)$ and $v_0(x)$ are the corresponding initial values, respectively. The same homogeneous boundary value condition*

$$u(x,t) = v(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T), \tag{3.9}$$

is defined as Definition 2.1, then

$$\int_\Omega |\, u(x,s) - v(x,s)\,|^2\, dx \le \int_\Omega |\, u_0(x) - v_0(x)\,|^2\, dx. \tag{3.10}$$

An open problem is whether the stability condition (3.10) holds when $a_i(x) = d^{p_i(x)}$ and the boundary value condition (3.9) is not imposed.

4 The existence

When $p(x)$, $q_1(x)$, $q_2(x)$ are log-Hölder continuous functions, we have the following basic lemma.

Lemma 4.1 (*i*) *The space $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$, $\left(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)}\right)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.*

(*ii*) *Let $q_1(x)$ and $q_2(x)$ be real functions with $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ and $q_1(x) > 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. For any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, there holds*

$$\left|\int_\Omega uv dx\right| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}.$$

(*iii*) *There holds that*

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} = 1,\ \text{then} \int_\Omega |u|^{p(x)} dx = 1.$$

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} > 1,\ \text{then } \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)} dx \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}.$$

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} < 1,\ \text{then } \|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)} dx \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}.$$

(*iv*) *If $q_1(x) \le q_2(x)$, then*

$$L^{q_1(x)}(\Omega) \supset L^{q_2(x)}(\Omega).$$

(*v*) *If $q_1(x) \le q_2(x)$, then*

$$W^{1,q_2(x)}(\Omega) \hookrightarrow W^{1,q_1(x)}(\Omega).$$

(vi) *($p(x)$-Poincarés inequality) If $p(x) \in C(\overline{\Omega})$, then there is a constant $C > 0$, such that*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)}, \ \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $\|\nabla u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent to the norms of $W_0^{1,p(x)}(\Omega)$.

This lemma can be found in [13, 17].

In this section, we will prove Theorem 3.3. We first introduce an embedding theorem related to anisotropic variable exponent space.

Lemma 4.2 *Let $1 \leq m < \frac{N\bar{q}}{N-\bar{q}}$ and $\frac{1}{\bar{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^+}$. Then, $W_0^{1,\vec{p(x)}}(\Omega) \hookrightarrow L^m(\Omega)$ and $\|u\|_m \leq M\left(\prod_{i=1}^{N}\|u_{x_i}\|_{p_i(x)}\right)^{\frac{1}{N}}$, for all $u \in W_0^{1,\vec{p(x)}}(\Omega)$, where M is a constant independent of u.*

This lemma can be generalized from Lemma 1.23 of [8], and when $p_i(x) = p_i$ is constant, it was first proved in [23].

Second, we quote the weighted Sobolev space $W_0^{k,p}(\Omega,\omega)$ from [24, Chap. 17], where ω is the weighted function.

By a weight, we mean a locally integrable function ω on $\mathbb{R}^N$ such that $\omega(x) > 0$ a.e. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will also be denoted by ω. Thus, $\omega(E) = \int_E \omega(x)dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 4.3 Let ω be a weight and Ω be open. For $0 < p < \infty$, we define $L^p(\Omega,\omega)$ as the set of measurable functions u on Ω such that

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\omega(x)dx\right)^{\frac{1}{p}} < \infty.$$

Definition 4.4 Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Let ω be a given family of weight functions ω_α, $|\alpha| \leq k$, $\omega = \{\omega_\alpha = \omega_\alpha(x), x \in \Omega, |\alpha| \leq k\}$. We denote by $W^{k,p}(\Omega,\omega)$ the set of all functions $u \in L^p(\Omega,\omega_0)$ for which the weak derivatives $D^\alpha u$, with $|\alpha| \leq k$, belong to $L^p(\Omega,\omega_\alpha)$. The weighted Sobolev space $W^{k,p}(\Omega,\omega)$ is a normed linear space if equipped with the norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha u|^p\omega_\alpha dx\right)^{\frac{1}{p}}.$$

If $1 < p < \infty$ and $\omega_\alpha^{-\frac{1}{p-1}} \in L^1_{loc}(|\alpha| \leq c)$, then $W^{k,p}(\Omega,\omega)$ is a uniformly convex Banach space. If we additionally suppose that also $\omega_\alpha \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{k,p}(\Omega,\omega)$, and we can introduce the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{k,p}(\Omega,\omega)}$.

The class of A_p weight was introduced by B. Muckenhoupt, where he showed that A_p weights are precisely those weights ω for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N,\omega)$ to $L^p(\mathbb{R}^N,\omega)$ $(1 < p < \infty)$, that is,

$$M : L^p(\mathbb{R}^N,\omega) \to L^p(\mathbb{R}^N,\omega),$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy$$

is bounded if and only if $\omega \in A_p$, i.e., there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B \omega dx\right)\left(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}} dx\right)^{p-1} \le C,$$

for every ball $B \in \mathbb{R}^N$.

The union of all Muckenhoupt classes A_p is denoted by A_∞, $A_\infty = \bigcup_{p>1} A_p$.

If $\omega \in A_p$, then since $\omega^{-\frac{1}{p-1}}$ is a locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega)$ for every open set Ω. It thus makes sense to discuss weak derivatives of functions in $L^p(\Omega,\omega)$. The weighted Sobolev space $W^{k,p}(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $|\alpha| \le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p \omega dx\right)^{\frac{1}{p}}.$$

We have the following:

(i) If $\omega \in A_p$, then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega,\omega)$.

(ii) If $\omega \in A_p$, then we have a weighted Poincare inequality.

Let $1 < p < \infty$ and $\omega \in A_p$. Then, there are positive constants C and δ such that for all Lipchitz conditions function φ defined on $\overline{B}(B = B(x_0,R))$ and for all $1 \le \theta \le \frac{N}{N-1} + \delta$,

$$\left(\frac{1}{\omega(B)}\int_B |\varphi - \varphi_B|^{\theta p} \omega dx\right)^{\frac{1}{\theta p}} \le CR\left(\frac{1}{\omega(B)}\int_B |\nabla\varphi|^p \omega dx\right)^{\frac{1}{p}},$$

where $\varphi_B = \frac{1}{\omega(B)}\int_B \varphi\omega dx$.

Definition 4.5 Let $\Omega \in \mathbb{R}^N$ be a bounded open set, $1 \le p < \infty$, k is a nonnegative integer and $\omega \in A_p$. We denote by $W^{k,p}(\Omega,\omega)$ the set of all functions $u \in L^p(\Omega,\omega)$ for which the weak derivatives $D^\alpha u$, with $|\alpha| \le k$, belong to $L^p(\Omega,\omega)$. The weighted Sobolev space $W^{k,p}(\Omega,\omega)$ is a normed linear space if equipped with the norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)dx + \sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u|^p \omega(x)dx\right)^{\frac{1}{p}}.$$

We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u|^p \omega(x)dx\right)^{\frac{1}{p}}.$$

We need the following basic result.

Theorem 4.6 (*The weighted Sobolev inequality*) [24, *Theorem* 17.3] *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set and* ω *be an* A_p*-weight*, $1 < p < \infty$. *Then, there exist positive constants* C_Ω *and* δ *such that for all* $f \in C_0^\infty(\Omega)$ *and* $1 \le \eta \le \frac{N}{N-1} + \delta$

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \|\nabla f\|_{L^p(\Omega,\omega)}.$$

Now, we consider the regularized parabolic equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left((a_i(x)+\varepsilon)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right),\ (x,t) \in Q_T, \tag{4.1}$$

with the usual initial-boundary value conditions

$$u(x,0) = u_{0\varepsilon}(x),\ x \in \Omega, \tag{4.2}$$

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T), \tag{4.3}$$

where, $u_{0\varepsilon}(x) \in C_0^\infty(\Omega)$ is strongly convergent to $u_0(x)$ in $W_0^{1,p^0}(\Omega)$, and $u_{0\varepsilon}(x) \to u_0(x)$ weakly star in $L^\infty(\Omega)$.

Since $p_i(x)$ is a log-Hölder continuous function for every i, $1 \le i \le N$, using a modified De Giorgi method introduced in Theorem 2.3 of [19], we can prove the following lemma.

Theorem 4.7 *Assume that* $u_\varepsilon \in V \cap L^\infty(Q_T)$ *is a weak solution to the initial-boundary value problem* (4.1)–(4.3). *If* $a_0(x) \in A_{p_0}$ *is a weighted function, then there is a constant* C *that depends only on* p_0, N, T, Ω *such that*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\Omega)} + C.$$

If $p_i(x) = p_i$ is a constant, such a uniform L^∞-estimate also can be found in Theorem 1 in [21].

For simplicity, we only give the proof of Lemma 4.7 when $p_0 \ge 2$. If $1 < p_0 < 2$, one can also prove Lemma 4.7 using a modified De Giorgi method introduced in Theorem 2.3 in [19]; we omit the details here. We need the following lemma (Lemma 2.1 in [19]).

Lemma 4.8 *Let* a, b, λ *be positive constants*, $\lambda > \frac{1}{2} + \frac{b}{a}$. *Define*

$$\varphi(s) = \begin{cases} e^{\lambda s-1}, & s \ge 0, \\ -e^{-\lambda s} + 1, & s \le 0. \end{cases} \tag{4.4}$$

Then we have:

1. *For any* $s \in \mathbb{R}$, *there holds*

$$|\varphi(s)| \ge \lambda|s|, \quad a\varphi'(s) - b\,|\varphi(s)| \ge \frac{a}{2}e^{\lambda|s|}. \tag{4.5}$$

2. *For any* $s \ge d$, *there are constants* $d \ge 0$, $M > 1$ *such that*

$$\varphi'(s) \le \lambda M\left[\varphi\left(\frac{s}{p_0}\right)\right]^{p_0}, \quad \varphi(s) \le M\left[\varphi\left(\frac{s}{p_0}\right)\right]^{p_0}. \tag{4.6}$$

3. *Let* $\Phi(s)=\int_0^s\varphi(\sigma)d\sigma$. *If* $p_0\ge 2$, *then there exists* c^* *such that*

$$\Phi(s)\ge c^*\left[\varphi\left(\frac{s}{p_0}\right)\right]^{p_0},\quad \forall s\ge 0. \tag{4.7}$$

Proof of Theorem 4.7 Let k be a positive constant satisfying $\|u_0\|_{L^\infty(\Omega)}\le k$, and let φ be defined as in (4.1) with $\lambda\ge\frac{1}{2}+2b$.

We define

$$G_k(u)=\begin{cases}u-k, & u>k,\\ u+k, & u<-k,\\ 0, & |u|\le k\end{cases}$$

and denote χ_A as the characteristic function of A. Since $u\in V\cap L^\infty(Q_T)$, we know $\varphi(G_k(u))\in V\cap L^\infty(Q_T)$. For any $\tau\in[0,T]$, we choose $v=\varphi(G_k(u))\chi_{[0,\tau]}$ as the test function in (4.1), by the facts $v_{x_i}=\chi_{[0,\tau]}\chi\{|u|>k\}\varphi'(G_k(u))u_{x_i}$, we have

$$\begin{aligned}&\int_0^\tau <u_t,\varphi(G_k(u))>dt\\&\quad+\sum_{i=1}^N\int_0^\tau\int_\Omega(a_i(x)+\varepsilon)|u_{x_i}|^{p_i(x)}\varphi'(G_k(u))\chi\{|u|>k\}dxdt=0.\end{aligned} \tag{4.8}$$

Denote that $A_k(t)=\{x\in\Omega:|u(x,t)|>k\}$. Since $\|u_0\|_{L^\infty(\Omega)}\le k$, we have

$$\begin{aligned}\int_0^\tau <u_t,\varphi(G_k(u))>dt&=\int_\Omega\Phi(G_k(u))(\tau)dx-\int_\Omega\Phi(G_k(u_0))dx\\&=\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx-\int_{A_k(0)}\Phi(G_k(u_0))dx\\&=\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx.\end{aligned} \tag{4.9}$$

Let $a=1$ in (4.5). We have $\varphi'\ge\varphi'-b|\varphi|\ge\frac{1}{2}e^{\lambda|G_k(u)|}>0$. Since $|u_{x_i}|^{p_i(x)}\ge|u_{x_i}|^{p_0}-1$, $i=1,2,\dots,N$, Equality (4.8) yields

$$\begin{aligned}&\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx+\sum_{i=1}^N\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)|u_{x_i}|^{p_0}\varphi'dxdt\\&\le c\sum_{i=1}^N\int_0^\tau\int_{A_k(t)}\varphi'dxdt.\end{aligned} \tag{4.10}$$

Denoting $\omega_k=\varphi(\frac{|G_k(u)|}{p_0})$, by $p_0\ge 2$, we have

$$\begin{aligned}&\sum_{i=1}^N\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)|u_{x_i}|^{p_0}\varphi'dxdt\\&\quad\ge\frac{1}{2}\sum_{i=1}^N\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)\left|e^{\lambda\frac{|G_k(u)|}{p_0}}u_{x_i}\right|^{p_0}dxdt\end{aligned} \tag{4.11}$$

$$= \frac{1}{2}\sum_{i=1}^{N}\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)\left|\frac{p_0}{\lambda}\right|^{p_0}|\omega_{kx_i}|^{p_0}dxdt$$
$$\geq \frac{1}{2}\sum_{i=1}^{N}\left(\frac{1}{\lambda}\right)^{p_0}\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)|\omega_{kx_i}|^{p_0}dxdt.$$

By the definition of A_k, we have $A_k(t)\backslash A_{k+d}(t) = \{x\in\Omega : k < |u(x,t)| \leq k+d\}$. Thus, on $A_k(t)\backslash A_{k+d}(t)$, we have $0 < |G_k(u)| \leq d$, $\varphi'(G_k(u)) = \lambda e^{\lambda|G_k(u)|} \leq \lambda e^{\lambda d}$. From (4.6), we have

$$\begin{aligned}&\int_0^\tau\int_{A_k(t)}\varphi' dxdt\\ \leq&\lambda M\int_0^\tau\int_{A_{k+d}(t)}|\omega_k|^{p_0}dxdt+\int_0^\tau\int_{A_k(t)\backslash A_{k+d}(t)}\varphi' dxdt\\ \leq&\lambda M\int_0^\tau\int_{A_{k+d}(t)}|\omega_k|^{p_0}dxdt+\lambda e^{\lambda d}\int_0^\tau\int_{A_k(t)\backslash A_{k+d}(t)}dxdt.\end{aligned}\tag{4.12}$$

Combing (4.10), (4.11), and (4.12), we can deduce that

$$\begin{aligned}&\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx+\frac{1}{2}\sum_{i=1}^{N}\left(\frac{1}{\lambda}\right)^{p_0}\int_0^\tau\int_{A_k(t)}(a_i(x)+\varepsilon)|\omega_{kx_i}|^{p_0}dxdt\\ \leq&\lambda M\sum_{i=1}^{N}\int_0^\tau\int_{A_{k+d}(t)}|\omega_k|^{p_0}dxdt+\lambda e^{\lambda d}\sum_{i=1}^{N}\int_0^\tau\int_{A_k(t)\backslash A_{k+d}(t)}dxdt.\end{aligned}\tag{4.13}$$

By (4.7), we have

$$\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx \geq C^*\int_{A_k(\tau)}|\omega_k|^{p_0}dx.\tag{4.14}$$

Substituting (4.14) into (4.13) and taking the supremum for $\tau\in[0,t_1]$ with $t_1\leq T$, we have

$$\begin{aligned}&C^*\int_{A_k(\tau)}|\omega_k|^{p_0}dx+\frac{1}{2}\sum_{i=1}^{N}\left(\frac{1}{\lambda}\right)^{p_0}\int_0^{t_1}\int_{A_k(t)}(a_i(x)+\varepsilon)|\omega_{kx_i}|^{p_0}dxdt\\ \leq&\lambda M\sum_{i=1}^{N}\int_0^{t_1}\int_{A_k(t)}|\omega_k|^{p_0}dxdt+\lambda e^{\lambda d}\sum_{i=1}^{N}\int_0^{t_1}\int_{A_k(t)\backslash A_{k+d}(t)}dxdt.\end{aligned}\tag{4.15}$$

Since $p_0>1$, $1<\frac{N}{N-p_0}\leq\frac{N}{N-1}+\delta$ is obviously true. Moreover, since $p_0\geq 2$, we have

$$a_0(x)|\nabla\omega_k|^{p_0}\leq 2^{(\frac{p_0}{2}-1)(N-1)}\leq\sum_{i=1}^{N}a_i(x)|\omega_{kx_i}|^{p_0}.\tag{4.16}$$

By that $a_0 \in A_{p_0}$ and inequality (4.16), we can use Theorem 4.6 and the Hölder inequality to obtain

$$\begin{aligned}
&\left(\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p_0} \frac{N+p_0}{N} dxdt\right)^{\frac{N}{N+p_0}} \\
&= \left(\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p_0} |\omega_k|^{p_0 \frac{p_0}{N}} dxdt\right)^{\frac{N}{N+p_0}} \\
&\le \left(\int_0^{t_1}\left(\int_{A_k(t)} a_0 |\omega_k|^{p_0} dx\right)^{\frac{p_0}{N}} \left(\int_{A_k(t)} a_0 |\omega_k|^{\frac{Np_0}{N-p_0}} dx\right)^{\frac{N-p_0}{N}} dt\right)^{\frac{N}{N+p_0}} \\
&\le \left(\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} a_0 |\omega_k|^{p_0} dx\right)^{\frac{p_0}{N}} \int_0^{t_1}\left(\int_{A_k(t)} a_0 |\omega_k|^{\frac{Np_0}{N-p_0}} dx\right)^{\frac{N-p_0}{N}} dt\right)^{\frac{N}{N+p_0}} \qquad (4.17)\\
&\le \left(C(p_0,N,|\Omega|)\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} |\omega_k|^{p_0} dx\right)^{\frac{p_0}{N}} \left(\int_0^{t_1}\int_{A_k(t)} a_0 |\nabla\omega_k|^{p_0} dxdt\right)\right)^{\frac{N}{N+p_0}} \\
&\le C(p_0,N,|\Omega|)\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} |\omega_k|^{p_0} dx + \int_0^{t_1}\int_{A_k(t)} a_0 |\nabla\omega_k|^{p_0} dxdt\right) \\
&\le C(p_0,N,|\Omega|) \\
&\quad\times\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} |\omega_k|^{p_0} dx + 2^{(\frac{p_0}{2}-1)(N-1)}\sum_{i=1}^N\int_0^{t_1}\int_{A_k(t)} a_i(x)|\omega_{kx_i}|^{p_0} dxdt\right) \\
&\le C(p_0,N,|\Omega|)\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} |\omega_k|^{p_0} dx + \sum_{i=1}^N\int_0^{t_1}\int_{A_k(t)} (a_i(x)+\varepsilon)|\omega_{kx_i}|^{p_0} dxdt\right),
\end{aligned}$$

where, $C(p_0,N,|\Omega|)$ depends on N, p_0, $|\Omega|$ but is independent of $t_1 \le T$.

Hence, from (4.15), it follows that

$$\begin{aligned}
J_{kt_1} :&= \left(\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p_0 \frac{N+p_0}{N}} dxdt\right)^{\frac{N}{N+p_0}} \\
&\le C\left(\sum_{i=1}^N\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p_0} dxdt + \sum_{i=1}^N\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} dxdt\right),
\end{aligned}$$

where C is independent of t_1.

Consequently, by the Hölder inequality and $r > \frac{N+p_0}{p_0}$, we have

$$\begin{aligned}
J_{kt1} :&\le C\sum_{i=1}^N\left\{\left(\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p_0 \frac{N+p_0}{N}} dxdt\right)^{\frac{N}{N+p_0}} \left(\int_0^{t_1}\int_{A_k(t)} dxdt\right)^{\frac{p_0}{N+p_0}}\right\} \\
&\quad + C\sum_{i=1}^N\left\{\left(\int_0^{t_1}\int_{A_k(t)} dxdt\right)^{\frac{1}{r}} \left(\int_0^{t_1} \mu(A_k(t))dt\right)^{1-\frac{1}{r}}\right\}
\end{aligned}$$

$$\leq C\sum_{i=1}^{N}\left\{\left(\int_0^{t_1}\int_{A_k(t)}|\omega_k|^{p_0\frac{N+p_0}{N}}dxdt\right)^{\frac{N}{N+p_0}}(t_1\mu(\Omega))^{\frac{p_0}{N+p_0}}-\frac{1}{r}\right\}$$
$$+C\sum_{i=1}^{N}\left\{\left(\int_o^{t_1}\mu(A_k(t))dt\right)^{1-\frac{1}{r}}\right\},$$

where $\mu(\Omega)$ is the measure of Ω.

Now, if we choose t_1 small enough such that

$$C\sum_{i=1}^{N}\left\{(t_1\mu(\Omega))^{\frac{p_0}{N+p_0}-\frac{1}{r}}\right\}\leq\frac{1}{2}, \tag{4.18}$$

we have

$$J_{kt_1}\leq c\sum_{i=1}^{N}\left\{\left(\int_0^{t_1}\mu(A_k(t))dt\right)^{1-\frac{1}{r}}\right\}. \tag{4.19}$$

For any $l>k\geq\|u_0\|_{L^\infty(\Omega)}$, using (4.5), we obtain that

$$\begin{aligned}J_{kt_1}&\geq\left(\int_0^{t_1}\int_{A_k(t)}\left|\frac{\lambda G_k(u)}{p_0}\right|^{p_0\frac{N+p_0}{N}}dxdt\right)^{\frac{N}{N+p_0}}\\&\geq\left(\frac{\lambda}{p_0}\right)^{p_0}\left(\int_0^{t_1}\int_{A_k(t)}(|u|-k)^{p_0\frac{N+p_0}{N}}dxdt\right)^{\frac{N}{N+p_0}}\\&\geq\left(\frac{\lambda}{p_0}\right)^{p_0}(l-k)^{p_0}\left(\int_0^{t_1}\mu(A_l(t))dt\right)^{\frac{N}{N+p_0}}.\end{aligned} \tag{4.20}$$

Let $\psi_k=\int_0^{t_1}\mu(A_k(t))dt$. It follows from (4.19) and (4.20) that

$$\psi_l\leq\frac{C}{(l-k)^{\frac{p_0(N+p_0)}{N}}}\psi_k^{(1-\frac{1}{r})\frac{N+p_0}{N}}. \tag{4.21}$$

Since $r>\frac{N+p_0}{N}$ implies $(1-\frac{1}{r})\frac{N+p_0}{N}>1$, according to the iteration lemma [31], we know that $\psi(\|u_0\|_{L^\infty(\Omega)}+C)=0$, where C depends on p_0, N, t_1, r, b, Ω. The above discussion shows that, for any given λ, Lemma 4.8 implies

$$|u(x,t)|_{L^\infty(Q_{t_1})}\leq\|u_0(x)\|_{L^\infty(\Omega)}+C. \tag{4.22}$$

Moreover, we split $[0,T]$ into a series subinterval $[0,t_1]$, $[t_1,t_2],\dots,[t_{n-1},T]$, such that on every $[t_i,t_{i+1}]$, Inequality (4.18) is true. Then by a similar method, we can also obtain Inequality (4.22). By such a consideration, we can deduce that $\|u(x,t)\|_{L^\infty(Q_T)}\leq\|u_0(x)\|_{L^\infty(\Omega)}+C$, where C depends on p_0, N, T, r, b, Ω. □

Proof of Theorem 3.3 By Lemma 4.7, multiplying (4.1) by u_ε yields

$$\sum_{i=1}^{N}\iint_{Q_T}a_i(x)\left|u_{\varepsilon x_i}\right|^{p_i(x)}dxdt\leq c. \tag{4.23}$$

Since for every i, $a_i(x) \in C^1(\overline{\Omega})$ is positive almost everywhere in Ω, if we denote that

$$D_{0i} = \{x \in \Omega : a_i(x) > 0\},\ i = 1, 2, \dots, N,$$

then

$$|D_{0i}| = \text{mes } D_{0i} = \text{mes } \Omega = |\Omega|,\ i = 1, 2, \dots, N,$$

which implies

$$\text{mes} \bigcap_{i=1}^{N} D_{0i} = |\Omega|. \tag{4.24}$$

Thus, for every point $x \in \bigcap_{i=1}^{N} D_{0i}$, there is a neighbourhood $U_x \in \bigcap_{i=1}^{N} D_{0i}$, when $x \in U_x$, $a_i(x) > 0$ for every i. From (4.23), we have

$$\int_0^T \int_{U_x} |u_{\varepsilon x_i}|^{p_i(x)} dxdt \le c,\ i = 1, 2, \dots, N.$$

By Lemma 4.2, we know that there is a function $u \in L^m(U_x \times (0, T))$ and

$$u_\varepsilon \to u,\ \text{in } L^m(U_x \times (0, T))$$

and so

$$u_\varepsilon \to u,\ \text{a.e. } (x, t) \in (U_x \times (0, T)).$$

By (4.24), we know

$$u_\varepsilon \to u,\ \text{a.e. } (x, t) \in Q_T. \tag{4.25}$$

Meanwhile, for each $v \in V$, by the definition of the norm V and $p_i(x)$-Hölder inequality, since $a_i(x) \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}
&\sup_{\|v\| \le 1} \left| < \sum_{i=1}^{N} \left(a_i(x)|u_{x_i}|^{p_i(x)-2} u_{x_i}\right)_{x_i}, v > \right| \\
= &\sup_{\|v\| \le 1} \left| \sum_{i=1}^{N} \iint_{Q_T} a_i(x)|u_{x_i}|^{p_i(x)-2} u_{x_i} v_{x_i} dxdt \right| \\
\le &c \sup_{\|v\| \le 1} \sum_{i=1}^{N} \left(\iint_{Q_T} a_i(x)|u_{x_i}|^{p_i(x)} dxdt \right)^{\frac{1}{p_{1i}}} \|v_{x_i}\|_{L^{p_i(x)}(Q_T)} \\
\le &c,
\end{aligned}$$

where $p_{i1} = p_i^+$ or p_i^- according to $\|a_i(x)u_{x_i}\|_{L^{p_i(x)}(\Omega)} > 1$ or ≤ 1 from (iii) of Lemma 4.1. Thus, we have

$$\left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{V^*} \le c. \tag{4.26}$$

Hence, from (4.23), (4.25), and (4.26), there exist a function u and an N-dimensional vector $\overrightarrow{\zeta}=(\zeta_1,\ldots,\zeta_n)$ such that

$$u\in L^{\infty}(Q_T),\quad \frac{\partial u}{\partial t}\in V^*,\quad \zeta_i\in L^1\left(0,T;L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)\right),$$

and

$$\begin{aligned}
&u_\varepsilon\rightharpoonup u,\ \textit{weakly star in } L^{\infty}(Q_T),\\
&\frac{\partial u_\varepsilon}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}\quad in\quad V^*,\\
&a_i(x)\left|u_{\varepsilon x_i}\right|^{p_i(x)-2}u_{\varepsilon x_i}\rightharpoonup\zeta_i\quad in\quad L^1\left(0,T;L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)\right).
\end{aligned}$$

Similar to the proof of Theorem in [27], we can show that

$$\sum_{i=1}^{N}\iint_{Q_T}a_i(x)\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}\varphi_{x_i}dxdt=\sum_{i=1}^{N}\iint_{Q_T}\zeta_i(x)\varphi_{x_i}dxdt,\tag{4.27}$$

for any function $\varphi\in C_0^1(Q_T)$. Then, we have (3.2) and (3.3).

At last, similar to the general evolutionary $p(x)$-Laplacian equation [8], we are able to show (3.4).

Thus, u satisfies Equation (1.1) with the initial value (1.2) in the sense of Definition 3.2.

□

5 Proofs of Theorem 3.4 and Theorem 3.5

For $n>0$ being a natural number, let

$$h_n(s)=2n(1-n|s|)_+,\ g_n(s)=\int_0^s h_n(\tau)d\tau.$$

Obviously,

$$\lim_{\eta\to 0}g_n(s)=\mathrm{sgn}s,\ \ \lim_{\eta\to 0}sg_n'(s)=0.$$

Proof of Theorem 3.4 Let $u(x,t)$ and $v(x,t)$ be two weak solutions to Equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, but without any boundary value condition. We define

$$\phi_n(x)=\begin{cases}1, & \text{if } x\in\Omega_{\frac{n}{2}},\\ n(d(x)-\frac{1}{n}), & \text{if } x\in\Omega_n\setminus\Omega_{\frac{n}{2}},\\ 0, & \text{if } x\in\Omega\setminus\Omega_n,\end{cases}$$

where $\Omega_n=\left\{x\in\Omega: d(x)>\frac{1}{n}\right\}$.

Then, we can take $\chi_{[\tau,s]}\phi_n g_n(u-v)$ as the test function in which $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s]\subset(0,T)$. From Definition 3.2, we have

$$\begin{aligned}
&\int_\tau^s\int_\Omega \phi_n g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
&\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\\
&\quad\times(u_{x_i}-v_{x_i})g_n'(u-v)\phi_n(x)dxdt\\
&\quad+\sum_{i=1}^N\int_\tau^s\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\\
&\quad\times(u_{x_i}-v_{x_i})g_n(u-v)\phi_{nx_i}dxdt\\
&=0.
\end{aligned}\tag{5.1}$$

First, we notice that

$$\int_\Omega d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)(u_{x_i}-v_{x_i})g_n'(u-v)\phi_n(x)dx\ge 0,\tag{5.2}$$

and then by Lemma 3.1, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\tau^s\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
&=\int_\Omega|u-v|(x,s)dx-\int_\Omega|u-v|(x,\tau)dx.
\end{aligned}\tag{5.3}$$

Second, since $\phi_{nx_i}=nd_{x_i}$ when $x\in\Omega_n\setminus\Omega_{\frac{n}{2}}$, using the fact that $|d_{x_i}|\le|\nabla d|=1$, Lemma 4.1, and (3.7), we deduce that

$$\begin{aligned}
&\left|\int_\Omega d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)dx\right|\\
&=\left|\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)dx\right|\\
&\le n\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-1}+\left|v_{x_i}\right|^{p_i(x)-1}\right)\left|d_{x_i}g_n(u-v)\right|dx\\
&\le cn\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)}+\left|v_{x_i}\right|^{p_i(x)}\right)dx\right)^{\frac{1}{q_i^+}}\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}|d_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{p_i^+}}\\
&\le c\left[\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}|u_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}}+\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}|v_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}}\right]\\
&\quad\cdot n\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}dx\right)^{\frac{1}{p_i^+}}
\end{aligned}\tag{5.4}$$

$$\leq c\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}|u_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}} + c\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} d^{\alpha_i}|v_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}}.$$

Here and the after, $q_i(x) = \frac{p_i(x)}{p_i(x)-1}$ and $q_i^+ = \max_{x\in\overline{\Omega}} q_i(x)$.

Accordingly, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega d^{\alpha_i}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)dxdt\right|\\
&\leq c\lim_{n\to\infty}\left[\left(\int_{\Omega\setminus\Omega_n} d^{\alpha_i}|u_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}}+\left(\int_{\Omega\setminus\Omega_n} d^{\alpha_i}|v_{x_i}|^{p_i(x)}dx\right)^{\frac{1}{q_i^+}}\right]\\
&=0.
\end{aligned} \tag{5.5}$$

Let $\eta\to 0$ in (5.1). From (5.2)–(5.5), we have

$$\int_\Omega |u(x,s)-v(x,s)|\,dx \leq \int_\Omega |u(x,\tau)-v(x,\tau)|\,dx.$$

Due to the arbitrariness of τ, we obtain

$$\int_\Omega |u(x,s)-v(x,s)|\,dx \leq c\int_\Omega |u_0(x)-v_0(x)|\,dx.$$

The proof is complete. □

Proof of Theorem 3.5 Let $u(x,t)$ and $v(x,t)$ be two weak solutions to Equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. Define

$$\varphi_n(x) = \begin{cases} 1, & \text{if } x\in D_{\frac{n}{2}},\\ n(d^2-\frac{1}{n}), & \text{if } x\in D_n\setminus D_{\frac{n}{2}},\\ 0, & \text{if } x\in\Omega\setminus D_n,\end{cases} \tag{5.6}$$

where $D_n = \{x\in\Omega : d^2 > \frac{1}{n}\}$. Take

$$\varphi = \chi_{[\tau,s]}(u-v)\varphi_n(x),$$

where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]$. Then, we have

$$\iint_{Q_T}\left[\left(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t}\right)\varphi+\sum_{i=1}^N d^{p_i(x)}\left(\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x)-2}v_{x_i}\right)\varphi_{x_i}\right]dxdt = 0. \tag{5.7}$$

Let us analyze the next term of the left-hand side of (5.7):

$$\begin{aligned}
&\int_\tau^s\int_\Omega d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)[(u-v)\varphi_n]_{x_i}dxdt\\
&=\int_\tau^s\int_\Omega d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)(u-v)_{x_i}\varphi_n dxdt\\
&+n\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}} d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)(u-v)2dd_{x_i}dxdt.
\end{aligned} \tag{5.8}$$

First, we have

$$\iint_{Q_{\tau s}} d^{p_i(x)}\varphi_n(x)\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)(u-v)_{x_i}dxdt \ge 0. \tag{5.9}$$

Second, by the generalization of the trace defined in Definition 2.1, we know

$$\limsup_{n\to\infty}\left[n\sup_{x\in D_n\setminus D_{\frac{n}{2}}} d^2|u-v|\right]=0,\ i=1,2,\ldots,N,$$

for the second term on the right-hand side of (5.8), using the Young inequality, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\iint_{Q_{\tau s}}(u-v)d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\varphi_{nx_i}dxdt\right|\\
\le&\limsup_{n\to\infty} n\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}}|u-v|d^{p_i(x)-1}\left(|u_{x_i}|^{p_i(x)-1}+|v_{x_i}|^{p_i(x)-1}\right)2d^2|d_{x_i}|dxdt\\
\le& c\limsup_{n\to\infty} n\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}}d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)}\right)|u-v|d^2dxdt\\
&+c\limsup_{n\to\infty} n\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}}d^{p_i(x)}|u-v|d^2dxdt\\
\le& c\limsup_{n\to\infty} n\sup_{x\in D_n\setminus D_{\frac{n}{2}}}\left[d^2|u-v|\right]\\
&\times\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}}d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)}\right)dxdt\\
&+c\limsup_{n\to\infty} n\sup_{x\in D_n\setminus D_{\frac{n}{2}}} d^2|u-v|.
\end{aligned}\tag{5.10}$$

Inequality (5.10) yields

$$\limsup_{n\to\infty}\left|\iint_{Q_{\tau s}}(u-v)d^{p_i(x)}\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\varphi_{nx_i}dxdt\right|=0. \tag{5.11}$$

Third, we have

$$\begin{aligned}
&\lim_{n\to\infty}\iint_{Q_{\tau s}}(u-v)\varphi_n(x)\frac{\partial(u-v)}{\partial t}dxdt\\
=&\int_\Omega[u(x,s)-v(x,s)]^2dx-\int_\Omega[u(x,\tau)-v(x,\tau)]^2dx.
\end{aligned}\tag{5.12}$$

In view of (5.8)–(5.12), letting $n\to\infty$ in (5.7) leads to

$$\int_\Omega |\ u(x,s)-v(x,s)\ |^2\ dx \le \int_\Omega |\ u(x,\tau)-v(x,\tau)\ |^2\ dx.$$

Due to the arbitrariness of τ, we obtain

$$\int_\Omega |\ u(x,s)-v(x,s)\ |^2\ dx \le \int_\Omega |\ u_0(x)-v_0(x)\ |^2\ dx.$$

This means that the stability (3.10) is true. □

6 Conclusion

Equations of the type (1.1) and their elliptic counterparts appear in numerous mathematical models from fluid mechanics, image processing, and virus spread modeling. As noted by the author in the introduction, while the existence of weak solution to Equation (1.1) has been studied extensively in the literature, the uniqueness of weak solution, except some special cases, remains an open problem. The difficulties mainly arise from both the degeneracy of the diffusion coefficient $a_i(x)$ and the integral singularity $\int_\Omega a_i(x)dx = \infty$.

In this paper, using the parabolically regularized method, the existence of such weak solution is proved, even allowing $a_i(x) = 0$ within the interior of the domain Ω. Moreover, a reasonable boundary value condition (1.3) is imposed in the generalized trace defined in Definition 2.1. Two stability theorems for weak solutions are also established, applicable when $a_i(x) = d^{\alpha_i}$, with α_i being a constant, and when $a_i(x) = d^{p_i(x)}$.

Certainly, if $a_i(x)$ is only with $\int_\Omega a_i(x)dx = \infty$, the generalization of the classical trace of $u \in L^1(0, T; W^{1,r}(\Omega))$ to $u \in L^\infty(Q_T)$ remains an unsolved problem. In fact, if $a_i(x) > a_i^- > 0$, then the well-posedness problem of the evolutionary parabolic equation with nonstandard growth order was studied intensively by Antontsev and Shmarev in [2–8]. While $a_i(x)$ exhibits the degeneracy in $\overline{\Omega}$, especially when $\int_\Omega a_i(x)dx = \infty$, the well-posedness problem of the evolutionary parabolic equation with nonstandard growth order is very important and becomes more difficult. A key difficulty lies in formulating a reasonable boundary value condition. This paper makes some essential progress in addressing this issue.

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