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Kolmogorov bounds for maximum likelihood drift estimation for discretely sampled SPDEs

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Abstract

In this paper, we investigate an approximative maximum likelihood estimator (MLE) for the drift coefficient of a stochastic partial differential equation in the case where the corresponding Fourier coefficients $u_k(t)$, $k = 1,\dots,N$ over a finite interval of time $[0,T]$ are observed on a uniform time grid: $0 = t_0 < t_1 < \cdots < t_M = T$, with $\Delta := t_i - t_{i-1} = T/M$, $i = 1,\dots,M$. We provide an explicit Berry–Esseen bound in Kolmogorov distance for this approximative MLE when $N,M,T \to \infty$, assuming that $T^3N^7/M^2 \to 0$ and $N^2/T \to 0$.

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1 Introduction

In recent years, the study of the rate of convergence in the central limit theorem for drift parameter estimation of the solution to certain stochastic partial differential equations (SPDEs) has received growing attention. We refer to the paper [2] and the references therein for an extensive description of the literature on parameter estimation for SPDEs.

In this paper, we consider the stochastic partial differential equation

$$\begin{aligned} du(t,x) &= \theta\Delta u(t,x)dt + dW_Q(t,x), \quad 0<x<1, 0\le t\le T, \\ u(0,x) &= f(x), \quad f\in L^2([0,1]), \\ u(t,0) &= u(t,1) = 0, \quad 0\le t\le T, \end{aligned} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$, and $\theta > 0$ is an unknown parameter, whereas Q is the covariance operator for the Wiener process $W_Q(t,x)$ so that

$$W_Q(t,x) = Q^{1/2}W(t,x),$$

with $W(t,x)$ being a cylindrical Brownian motion in $L^2([0,1])$. It is a standard fact (see, e.g., [12]) that, given Q is nuclear,

$$dW_Q(t,z) = \sum_{i=1}^{\infty} q_k^{1/2} e_k(x)\, dW_k(t),$$

where $W_1, W_2, \dots$ are independent standard Brownian motions, and $\{e_k, k = 1,2,\dots\}$ is a complete orthonormal system in $L^2([0,1])$, which consists of eigenvectors of Q. We denote q_k as the eigenvalue corresponding to e_k. For simplicity, we consider a special covariance operator $Q = (1-\Delta)^{-1}$ and a complete orthonormal system $e_k := \sin k\pi x$, $k = 1,2,\dots$ with $\lambda_k = (\pi k)^2$, $k = 1,2,\dots$. In this case, the corresponding eigenvalues $\{e_k,\ k = 1,2,\dots\}$ are $q_k := (1+\lambda_k)^{-1}$, $i = 1,2\dots$, that is,

$$Qe_k = q_k e_k = (1+\lambda_k)^{-1} e_k,\ k = 1,2\dots.$$

We define a solution $u(t,x)$ to the problem (1.1) as a formal sum (see [12])

$$u(t,x) = \sum_{i=1}^{\infty} u_k(t) e_k(x), \quad k = 1,2\dots,$$

where the Fourier coefficients $u_k(t)$, $k = 1,2\dots$ follow the dynamics of Ornstein–Uhlenbeck processes as follows:

$$du_k(t) = -\lambda_k \theta u_k(t) dt + \frac{1}{\sqrt{\lambda_k + 1}} dW_k(t), \tag{1.2}$$

with initial condition

$$u_k(0) = \alpha_k.$$

Here, α_k, $k = 1,2\dots$ are determined by

$$f(x) = \sum_{k=0}^{\infty} \alpha_k e_k(x), \qquad \alpha_k = \int_0^1 f(x) e_k(x) dx, \quad k = 1,2\dots.$$

It can be shown (see [12]) that $u(t,x)$ belongs to $L^2\left([0,T]\times\Omega; L^2([0,1])\right)$ together with its derivative in x. It vanishes at 0 and 1 in space, and its norm in $L^2([0,1])$ is continuous in t. In addition, $u(t,x)$ is the only solution to (1.1) with the above properties. In what follows, to simplify the notation, we set $u(0,x) = f(x) = 0$ and, consequently, $u_k(0) = 0$ for all $i \geq 1$.

We denote by Π^N the finite dimensional subspace of $L^2(\Omega)$ generated by $\{e_1,\dots,e_N\}$, so the likelihood ratio of the projection of the solution $u(t,x)$ onto the subspace Π^N (see [7, 9])

$$u^N(t,x) = \sum_{i=1}^{N} u_k(t) e_k(x)$$

can be expressed as follows:

$$\frac{dP^N_\theta}{dP^N_{\theta_0}}\left(u^N\right)$$
$$= \exp\left\{-\sum_{k=1}^N \lambda_k\left(\lambda_k+1\right)\left[\left(\theta-\theta_0\right)\int_0^T u_k(t)du_k(t)+\frac{1}{2}\left(\theta^2-\theta_0^2\right)\lambda_k\int_0^T u_k^2(t)dt\right]\right\},$$

where P_θ denotes the probability measure on the space of continuous paths $C([0,T])$ generated by the u^N. Maximizing the log-likelihood ratio with respect to the parameter θ yields the following maximum likelihood estimator (MLE) $\widehat{\theta}_{N,T}$ for θ based on continuous observations of u^N:

$$\widehat{\theta}_{N,T} := -\frac{\sum_{k=1}^N \lambda_k(1+\lambda_k)\int_0^T u_k(s)du_k(s)}{\sum_{i=1}^N \lambda_k^2(1+\lambda_k)\int_0^T u_k^2(s)ds}, \quad N\geq 1, T>0. \tag{1.3}$$

Recall that the estimator $\widehat{\theta}_{N,T}$ is strongly consistent and asymptotically normal in three asymptotic regimes: for the two cases $N\to\infty$ and T fixed, and $T\to\infty$ and N fixed, see, for instance, [2] and the references therein, and for the case when both $N,T\to\infty$, see [3]. On the other hand, in [8], a Berry–Esseen bound in Kolmogorov distance for $\widehat{\theta}_{N,T}$ has been studied in the case where $N\to\infty$ while T is fixed. However, paper [4] provided Berry–Esseen bounds in the Wasserstein distance for $\widehat{\theta}_{N,T}$ when $N\to\infty$ and/or $T\to\infty$.

Here, our aim is to estimate the drift parameter θ based on discrete high-frequency data in time of the Fourier coefficients $u_k(t)$, $k=1,\ldots,N$ of the solution of the SPDE (1.1) by considering the discrete version $\widetilde{\theta}_{N,M,T}$ of the estimator $\widehat{\theta}_{N,T}$:

$$\widetilde{\theta}_{N,M,T} := -\frac{\sum_{k=1}^N \lambda_k(1+\lambda_k)\sum_{i=1}^M u_k\left(t_{i-1}\right)\left[u_k\left(t_i\right)-u_k\left(t_{i-1}\right)\right]}{\Delta\sum_{k=1}^N \lambda_k^2(1+\lambda_k)\sum_{i=1}^M u_k^2\left(t_{i-1}\right)}, \tag{1.4}$$

where the Fourier modes $u_k(t)$, $k\geq 1$ are observed on a uniform time grid:

$$0=t_0<t_1<\cdots<t_M=T, \quad \text{with} \quad \Delta := t_i-t_{i-1}=\frac{T}{M}, i=1,\ldots,M.$$

Recently, the work [3] studied the asymptotic properties of the approximative MLE $\widetilde{\theta}_{N,M,T}$ as follows:

- The estimator $\tilde{\theta}_{N,M,T}$ is weakly consistent, see [3, Theorem 2], namely

  $$\tilde{\theta}_{N,M,T}\to\theta, \text{ in probability,}$$

  as $N,M,T\to\infty$, and assuming that $T^2N^3/M^2\to 0$.
- The estimator $\tilde{\theta}_{N,M,T}$ is asymptotically normal, see [3, Theorem 3]. More precisely,

  $$d_{\mathrm{Kol}}\left(\sqrt{\frac{\pi^2TN^3}{6\theta}}\left(\theta-\widetilde{\theta}_{N,M,T}\right), \mathcal{N}(0,1)\right)\to 0,$$

  as $N,M,T\to\infty$, and such that $T^3N^6/M^2\to 0$.

However, the study of the asymptotic distribution of an estimator is generally not very useful for practical purposes unless the rate of convergence is known. To the best of our knowledge, no Berry–Esseen type result is known for the distribution of the approximative MLE $\widetilde{\theta}_{N,M,T}$. In the present paper, we focus on the framework proposed by [3] and refine their result by deriving estimates for the associated rates of convergence. More precisely, we provide a rate of Kolmogorov distance in the central limit theorem of $\widetilde{\theta}_{N,M,T}$, see Theorem 4.6.

The remainder of the paper is organized as follows. In Sect. 2, we present the basic tools of Malliavin calculus needed throughout the paper. Section 3 provides notation and auxiliary results. Section 4 presents our main result, which gives an explicit upper bound for the Kolmogorov distance in the central limit theorem of the approximative MLE $\widetilde{\theta}_{N,M,T}$.

2 Preliminaries

Here, we recall elements from the analysis on Wiener space and Malliavin calculus for Gaussian processes that will be needed throughout the paper. The interested reader can find more details in [10] and [11]. Let $\mathcal{H} := L^2([0,T])$ and let $\{W(\varphi), \varphi \in \mathcal{H}\}$ be a Wiener process, that is a centered Gaussian family of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. In this case, $W_t = W(1_{[0,t]})$ and $W(\varphi) := \int_0^T \varphi(s)\, dW_s$ for every $\varphi \in \mathcal{H}$.

The Wiener chaos $\mathcal{H}_p$ of order p is defined as the closure in $L^2(\Omega)$ of the linear span of the random variables $H_p(W(\varphi))$, where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_p is the Hermite polynomial of degree p.

• *Multiple Wiener–Itô integral.* The multiple Wiener stochastic integral I_p with respect to W of order p is defined as an isometry between the Hilbert space $\mathcal{H}^{\odot p} = L^2_{sym}([0,T]^p)$ (symmetric tensor product) equipped with the norm $\sqrt{p!}\|\cdot\|_{\mathcal{H}^{\otimes p}}$ and the Wiener chaos of order p, denoted by $\mathcal{H}_p$, under $L^2(\Omega)$'s norm, that is, the multiple Wiener stochastic integral of order p:

$$I_p : \left(\mathcal{H}^{\odot p}, \sqrt{p!}\|\cdot\|_{\mathcal{H}^{\otimes p}}\right) \longrightarrow \left(\mathcal{H}_p, L^2(\Omega)\right)$$

is a linear isometry defined by $I_p(f^{\otimes p}) = H_p(W(f))$.

Fix $T \geq 1$ and an integer $N \geq 1$. Recall that, if $\mathcal{H} = L^2\left([0,T], \mathbb{R}^N\right)$ and $W = (W_1, W_2, \dots, W_N)$ with $W_1, W_2, \dots, W_N$ are independent standard Brownian motions, for every $h = \left(h^1, \dots, h^N\right) \in \mathcal{H}$, the multiple integral $I_1(h)$ is defined by

$$I_1(h) := I_1^W(h) = \sum_{i=1}^N I_1^{W_i}(h^i) = \sum_{i=1}^N \int_0^T h_s^i dW_i(s), \tag{2.1}$$

and

$$\|h\|_{\mathcal{H}}^2 = \sum_{i=1}^N \int_0^T (h_s^i)^2 ds.$$

• *Kolmogorov and Wasserstein distances.* Given two real-valued random variables X, Y, the Kolmogorov distance between the law of X and the law of Y is given by

$$d_{\mathrm{Kol}}(X,Y) := \sup_{z \in \mathbb{R}} \left|\mathbb{P}(X \leq z) - \mathbb{P}(Y \leq z)\right|,$$

and the Wasserstein distance between the law of X and the law of Y is given by

$$d_W(X,Y) := \sup_{f \in Lip(1)} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$

where Lip(1) is the set of all Lipschitz functions with Lipschitz constant ≤ 1.

It is well known that if F is any real-valued random variable and $\mathcal{N}(0,1)$ is standard Gaussian, then

$$d_{\mathrm{Kol}}(F, \mathcal{N}(0,1)) \leq 2\sqrt{d_{\mathrm{W}}(F, \mathcal{N}(0,1))}.$$

(See, for example, [1, Theorem 3.3] or [10, Remark C.2.2]).

• *Third and fourth cumulants.* The third and fourth cumulants are, respectively, defined by

$$\kappa_3(X) = \mathbb{E}\left[X^3\right] - 3\mathbb{E}\left[X^2\right]\mathbb{E}[X] + 2\mathbb{E}[X]^3,$$
$$\kappa_4(X) = \mathbb{E}\left[X^4\right] - 4\mathbb{E}[X]\mathbb{E}\left[X^3\right] - 3\mathbb{E}\left[X^2\right]^2 + 12\mathbb{E}[X]^2\mathbb{E}\left[X^2\right] - 6\mathbb{E}[X]^4.$$

In particular, when $\mathbb{E}[X] = 0$, we have that

$$\kappa_3(X) = \mathbb{E}\left[X^3\right] \text{ and } \kappa_4(X) = \mathbb{E}\left[X^4\right] - 3\mathbb{E}\left[X^2\right]^2.$$

Throughout the paper, $\mathcal{N}(0,1)$ denotes a standard normal random variable, while $\mathcal{N}(\mu, \sigma^2)$ denotes a normal variable with mean μ and variance σ^2. C_θ also denotes a generic positive constant (possibly depending on θ but not on any other parameters), which may change from line to line.

3 Notation and auxiliary results

Here, we introduce the notation and essential facts used throughout the paper.

First, let us recall a quantitative central limit theorem (CLT) for random ratios using the Kolmogorov distance. Recently, using techniques that rely on a combination of the Malliavin calculus and Stein method (see, e.g., [10]), authors of paper [6] provided upper bounds in the Kolmogorov distance for the CLT of a ratio of functionals of Gaussian fields. Here, we state a slight extension of [6, Theorem 3.1], which can be proved using the same arguments as in [6]; thus, its proof is omitted. We now state the required assumptions.

Assumption $(\mathcal{A}_1)$ Suppose that q is a fixed positive integer. Let $\phi_{N,M,T}, N, M \geq 1, T > 0$ be positive constants and $\{G_{N,M,T}, N, M \geq 1, T > 0\}$ be a stochastic process that satisfies, as $N, M, T \to \infty$,

$$\phi_{N,M,T} \to \infty, \quad \left|\frac{1}{\rho\sqrt{\phi_{N,M,T}}}\mathbb{E}G_{N,M,T} - 1\right| \to 0,$$
$$\mathbb{E}\left[(G_{N,M,T} - \mathbb{E}G_{N,M,T})^2\right] \longrightarrow \sigma^2 \tag{3.1}$$

for some positive constants $\rho > 0$, $\sigma > 0$, and

$$G_{N,M,T} - \mathbb{E}G_{N,M,T} = V_{N,M,T} + \frac{1}{\sqrt{\phi_{N,M,T}}}R_{N,M,T}, \quad N, M \geq 1, T > 0, \tag{3.2}$$

where $V_{N,M,T}, R_{N,M,T} \in \mathcal{H}_q$, $N, M \geq 1$, $T > 0$, and as $N, M, T \to \infty$,

$$\frac{\|R_{N,M,T}\|_{L^2(\Omega)}}{\sqrt{\phi_{N,M,T}}} \to 0.$$

Assumption $(\mathcal{A}_2)$ Let $\{(A_{N,M,T}, a_{N,M,T}), N, M \geq 1, T > 0\}$ be a stochastic process that satisfies, for all $N, M \geq 1$, $T > 0$, $A_{N,M,T} \in \mathcal{H}_q$, and $a_{N,M,T}$ is a real constant such that, as $N, M, T \to \infty$,

$$\frac{\|A_{N,M,T}\|_{L^2(\Omega)} + |a_{N,M,T}|}{\sqrt{\phi_{N,M,T}}} \to 0.$$

Theorem 3.1 (A slight extension of [6, Theorem 3.1]) *Let* $\{G_{N,M,T}, N, M \geq 1, T > 0\}$ *and* $\{(A_{N,M,T}, a_{N,M,T}), N, M \geq 1, T > 0\}$ *be stochastic processes satisfying* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *respectively. Then, there exists a constant* $C > 0$ (*independent of* N, M, *and* T) *such that, for all* $N, M \geq 1$, $T > 0$,

$$\begin{aligned}
& d_{\mathrm{Kol}}\left(\frac{\frac{1}{\sigma}(G_{N,M,T} - \mathbb{E}G_{N,M,T}) + \frac{1}{\sqrt{\phi_{N,M,T}}}\left(A_{N,M,T} + a_{N,M,T}\right)}{\frac{1}{\rho\sqrt{\phi_{N,M,T}}}G_{N,M,T}}, \mathcal{N}(0,1)\right) \\
& \leq \max\left(\left|\kappa_3\left(\frac{V_{N,M,T}}{\sigma}\right)\right|, \kappa_4\left(\frac{V_{N,M,T}}{\sigma}\right)\right) + C\phi_{N,M,T}^{\frac{1}{4}}\left|\frac{1}{\rho\sqrt{\phi_{N,M,T}}}\mathbb{E}G_{N,M,T} - 1\right| \\
& \quad + C\left|\mathbb{E}[(G_{N,M,T} - \mathbb{E}G_{N,M,T})^2] - \sigma^2\right| \\
& \quad + \frac{C}{\sqrt{\phi_{N,M,T}}}\left(\|R_{N,M,T}\|_{L^2(\Omega)} + \|A_{N,M,T}\|_{L^2(\Omega)} + |a_{N,M,T}|\right),
\end{aligned}$$

where the constants ρ *and* σ *are defined by* (3.1), *and* $\{V_{N,M,T}, N, M \geq 1, T > 0\}$ *and* $\{R_{N,M,T}, N, M \geq 1, T > 0\}$ *are the processes given by* (3.2).

Note that the approximative MLE of θ defined by (1.4) can be expressed as follows:

$$\begin{aligned}
\widetilde{\theta}_{N,M,T} &= -\frac{\sum_{k=1}^{N}\lambda_k(1+\lambda_k)\sum_{i=1}^{M}u_k(t_{i-1})\left[u_k(t_i) - u_k(t_{i-1})\right]}{\Delta\sum_{k=1}^{N}\lambda_k^2(1+\lambda_k)\sum_{i=1}^{M}u_k^2(t_{i-1})} \\
&= -\frac{\sum_{k=1}^{N}\lambda_k\sum_{i=1}^{M}v_k(t_{i-1})\left[v_k(t_i) - v_k(t_{i-1})\right]}{\Delta\sum_{k=1}^{N}\lambda_k^2\sum_{i=1}^{M}v_k^2(t_{i-1})},
\end{aligned} \tag{3.3}$$

where $u_k(t)$ is the solution of the linear equation (1.2), which can be expressed explicitly as follows:

$$u_k(t) = \frac{1}{\sqrt{1+\lambda_k}}v_k(t), \quad \text{with} \quad v_k(t) := \int_0^t e^{-\theta\lambda_k(t-s)}dW_k(s),\ k = 1, \ldots, N.$$

For each $k = 1, \ldots, N$, if we denote the Gaussian stationary process $e^{-\theta\lambda_k t}\int_{-\infty}^{t}e^{\theta\lambda_k s}dW_k(s)$ by $Z_k(t)$, then $v_k(t)$ can be expressed as

$$v_k(t) = Z_k(t) - e^{-\theta\lambda_k t}Z_k(0), \quad t \geq 0. \tag{3.4}$$

Define

$$S_{N,M,T} := \sum_{k=1}^{N} \lambda_k^2 S_M(v_k), \quad \text{with} \quad S_M(v_k) := \Delta \sum_{i=1}^{M} v_k^2(t_{i-1}),$$

and

$$F_M(v_k) := \frac{\sqrt{T}}{M} \sum_{i=1}^{M} \left(v_k^2(t_{i-1}) - \mathbb{E} v_k^2(t_{i-1}) \right). \tag{3.5}$$

Observe that

$$S_M(v_k) - \mathbb{E} S_M(v_k) = \sqrt{T} F_M(v_k).$$

Let us also introduce

$$\begin{aligned}\Lambda_{N,M,T} &:= \sum_{k=1}^{N} \lambda_k \sum_{i=1}^{M} e^{-\theta\lambda_k t_i} v_k(t_{i-1}) \left[\zeta_k(t_i) - \zeta_k(t_{i-1}) \right] \\ &= \sum_{k=1}^{N} \lambda_k \sum_{i=1}^{M} e^{-\theta\lambda_k (t_i + t_{i-1})} \zeta_k(t_{i-1}) \left[\zeta_k(t_i) - \zeta_k(t_{i-1}) \right] \\ &=: \sum_{k=1}^{N} \lambda_k \Lambda_{\lambda_k, M},\end{aligned} \tag{3.6}$$

where

$$\Lambda_{\lambda_k, M} = \sum_{i=1}^{M} e^{-\theta\lambda_k (t_i + t_{i-1})} \zeta_k(t_{i-1}) \left[\zeta_k(t_i) - \zeta_k(t_{i-1}) \right], \quad \zeta_k(t) = \int_0^t e^{\theta\lambda_k s} dW_k(s). \tag{3.7}$$

Thus, using (3.3) and the fact that

$$\begin{aligned}\sum_{i=1}^{M} v_k(t_{i-1}) \left[v_k(t_i) - v_k(t_{i-1}) \right] &= \sum_{i=1}^{M} v_k(t_{i-1}) v_k(t_i) - \sum_{i=1}^{M} v_k^2(t_{i-1}) \\ &= \sum_{i=1}^{M} e^{-\theta\lambda_k (t_i + t_{i-1})} \zeta_k(t_{i-1}) \left(\zeta_k(t_i) - \zeta_k(t_{i-1}) \right) \\ &\quad + \sum_{i=1}^{M} e^{-\theta\lambda_k (t_i + t_{i-1})} \zeta_k^2(t_{i-1}) - \sum_{i=1}^{M} v_k^2(t_{i-1}) \\ &= \Lambda_{N,M,T} + \left(e^{-\lambda_k \theta \Delta} - 1 \right) \sum_{i=1}^{M} v_k^2(t_{i-1}),\end{aligned}$$

we get

$$-\widetilde{\theta}_{N,M,T} = \frac{\sum_{k=1}^{N} \lambda_k \left(e^{-\lambda_k \theta \Delta} - 1 \right) \sum_{i=1}^{M} v_k^2(t_{i-1}) + \Lambda_{N,M,T}}{\Delta \sum_{k=1}^{N} \lambda_k^2 \sum_{i=1}^{M} v_k^2(t_{i-1})}.$$

This enables us to write

$$\theta - \widetilde{\theta}_{N,M,T} = \frac{\sum_{k=1}^{N} \lambda_k \left(e^{-\lambda_k \theta \Delta} - 1 + \lambda_k \theta \Delta\right) \sum_{i=1}^{M} v_k^2 (t_{i-1})}{S_{N,M,T}} + \frac{\Lambda_{N,M,T}}{S_{N,M,T}}.$$

Moreover, setting

$$f_{N,M,T}(v) := \frac{1}{M \sum\limits_{k=1}^{N} \lambda_k} \sum_{k=1}^{N} \lambda_k^2 \sum_{i=1}^{M} v_k^2 (t_{i-1}), \tag{3.8}$$

$$G_{N,M,T} = \sqrt{T \sum_{k=1}^{N} \lambda_k f_{N,M,T}(v)}, \quad \rho = \frac{1}{2\theta}, \quad \sigma^2 = \frac{1}{2\theta^3}, \tag{3.9}$$

$$\begin{aligned} A_{N,M,T} = \sqrt{2\theta} \Bigg[& \sum_{k=1}^{N} \lambda_k^2 \left(\frac{e^{-\lambda_k \theta \Delta} - 1}{\lambda_k \Delta} + \theta \right) (S_M(v_k) - \mathbb{E} S_M(v_k)) \\ & + \Lambda_{N,M,T} - \theta \sum_{k=1}^{N} \lambda_k^2 (S_M(v_k) - \mathbb{E} S_M(v_k)) \Bigg], \end{aligned} \tag{3.10}$$

and

$$a_{N,M,T} = \sqrt{2\theta} \left[\sum_{k=1}^{N} \lambda_k^2 \left(\frac{e^{-\lambda_k \theta \Delta} - 1}{\lambda_k \Delta} + \theta \right) \mathbb{E} S_M(v_k) \right], \tag{3.11}$$

we can write

$$\begin{aligned} & \sqrt{\frac{T}{2\theta} \sum_{k=1}^{N} \lambda_k} \left(\theta - \widetilde{\theta}_{N,M,T}\right) \\ & = \frac{\sqrt{\frac{2\theta}{T \sum\limits_{k=1}^{N} \lambda_k}} \left[\sum_{k=1}^{N} \lambda_k \left(e^{-\lambda_k \theta \Delta} - 1 + \lambda_k \theta \Delta\right) \sum_{i=1}^{M} v_k^2 (t_{i-1}) + \Lambda_{N,M,T} \right]}{\frac{2\theta}{M \sum\limits_{k=1}^{N} \lambda_k} \sum_{k=1}^{N} \lambda_k^2 \sum_{i=1}^{M} v_k^2 (t_{i-1})} \\ & = \frac{\sqrt{\frac{2\theta}{T \sum\limits_{k=1}^{N} \lambda_k}} \left[\sum_{k=1}^{N} \lambda_k \left(e^{-\lambda_k \theta \Delta} - 1 + \lambda_k \theta \Delta\right) \sum_{i=1}^{M} v_k^2 (t_{i-1}) + \Lambda_{N,M,T} \right]}{\frac{1}{\rho \sqrt{T \sum\limits_{k=1}^{N} \lambda_k}} \left(\sqrt{T \sum\limits_{k=1}^{N} \lambda_k f_{N,M,T}(v)} \right)} \\ & =: \frac{\frac{1}{\sigma} (G_{N,M,T} - \mathbb{E} G_{N,M,T}) + \frac{1}{\sqrt{T \sum\limits_{k=1}^{N} \lambda_k}} \left(A_{N,M,T} + a_{N,M,T}\right)}{\frac{1}{\rho \sqrt{T \sum\limits_{k=1}^{N} \lambda_k}} G_{N,M,T}}, \end{aligned} \tag{3.12}$$

where the last equation follows from the fact that

$$
\begin{aligned}
&\sqrt{2\theta}\left[\sum_{k=1}^{N}\lambda_k\left(e^{-\lambda_k\theta\Delta}-1+\lambda_k\theta\Delta\right)\sum_{i=1}^{M}v_k^2(t_{i-1})+\Lambda_{N,M,T}\right]\\
&-\frac{1}{\sigma}\sqrt{T\sum_{k=1}^{N}\lambda_k}\left(G_{N,M,T}-\mathbb{E}G_{N,M,T}\right)\\
&=\sqrt{2\theta}\left[\sum_{k=1}^{N}\lambda_k\left(e^{-\lambda_k\theta\Delta}-1\right)\sum_{i=1}^{M}v_k^2(t_{i-1})+\Lambda_{N,M,T}+\theta\Delta\sum_{k=1}^{N}\lambda_k^2\sum_{i=1}^{M}\mathbb{E}v_k^2(t_{i-1})\right]\\
&=\sqrt{2\theta}\left[\sum_{k=1}^{N}\lambda_k^2\left(\frac{e^{-\lambda_k\theta\Delta}-1}{\lambda_k\Delta}+\theta\right)\left(S_M(v_k)-\mathbb{E}S_M(v_k)\right)\right.\\
&\quad+\sum_{k=1}^{N}\lambda_k^2\left(\frac{e^{-\lambda_k\theta\Delta}-1}{\lambda_k\Delta}+\theta\right)\mathbb{E}S_M(v_k)\\
&\quad\left.+\Lambda_{N,M,T}-\theta\sum_{k=1}^{N}\lambda_k^2\left(S_M(v_k)-\mathbb{E}S_M(v_k)\right)\right].
\end{aligned}
$$

Moreover,

$$
G_{N,M,T}-\mathbb{E}G_{N,M,T}=V_{N,M,T}+\frac{1}{\sqrt{T\sum_{k=1}^{N}\lambda_k}}R_{N,M,T},\quad N,M\geq 1, T>0,
$$

where

$$
V_{N,M,T}=\frac{1}{\sqrt{\sum_{k=1}^{N}\lambda_k}}\sum_{k=1}^{N}\lambda_k^2F_M(Z_k),\quad\text{with}\quad F_M(Z_k):=\frac{\sqrt{T}}{M}\sum_{i=1}^{M}\left(Z_k^2(t_{i-1})-\mathbb{E}Z_k^2(t_{i-1})\right),\tag{3.13}
$$

and

$$
\begin{aligned}
R_{N,M,T}&=\sqrt{T\sum_{k=1}^{N}\lambda_k}\left(G_{N,M,T}-\mathbb{E}G_{N,M,T}-V_{N,M,T}\right)\\
&=T\sum_{k=1}^{N}\lambda_k\left[\left(f_{N,M,T}(v)-f_{N,M,T}(Z)\right)-\left(\mathbb{E}f_{N,M,T}(v)-\mathbb{E}f_{N,M,T}(Z)\right)\right].
\end{aligned}\tag{3.14}
$$

4 Main results

To derive Berry–Esseen bounds in the Kolmogorov distance for the estimator $\widetilde{\theta}_{N,M,T}$, we will use the following lemmas.

Lemma 4.1 *Let $F_M(v_k)$ be the process defined by (3.5). Then, there exists $C_\theta > 0$ that depends only on θ such that, for every $k, M \geq 1$,*

$$\left|\mathbb{E}\left(F_M^2(v_k)\right) - \frac{1}{2\theta^3\lambda_k^3}\right| \leq C_\theta \left[\frac{\Delta^2}{\lambda_k} + \frac{1}{\lambda_k^2 M\Delta}\right]. \tag{4.1}$$

Proof Since $v_k(t_{i-1})$ is Gaussian, it follows from the Wick formula that

$$\mathbb{E}\left(v_k^2(t_{i-1})v_k^2(t_{j-1})\right) = \mathbb{E}\left(v_k^2(t_{i-1})\right)\mathbb{E}\left(v_k^2(t_{j-1})\right) + 2\left(\mathbb{E}\left(v_k(t_{i-1})v_k(t_{j-1})\right)\right)^2.$$

Hence, we can write

$$\begin{aligned}
\mathbb{E}\left(F_M^2(v_k)\right) &= \frac{2\Delta}{M}\sum_{i,j=1}^{M}\left(\mathbb{E}\left(v_k(t_{i-1})v_k(t_{j-1})\right)\right)^2 \\
&= \frac{2\Delta}{M}\sum_{i,j=1}^{M}\left(e^{-\theta\lambda_k(t_{i-1}+t_{j-1})}\int_0^{t_{i-1}\wedge t_{j-1}} e^{2\theta\lambda_k s}ds\right)^2 \\
&= \frac{2\Delta}{M(2\theta\lambda_k)^2}\sum_{i,j=1}^{M}\left(e^{-\theta\lambda_k|t_{j-1}-t_{i-1}|} - e^{-\theta\lambda_k(t_{i-1}+t_{j-1})}\right)^2 \\
&= \frac{2\Delta}{M(2\theta\lambda_k)^2}\sum_{i,j=1}^{M}\left(e^{-2\theta\lambda_k|t_{j-1}-t_{i-1}|} - 2e^{-2\theta\lambda_k(t_{j-1}\vee t_{i-1})} + e^{-2\theta\lambda_k(t_{i-1}+t_{j-1})}\right) \\
&=: B_{1,k,M} + B_{2,k,M} + B_{3,k,M}.
\end{aligned} \tag{4.2}$$

Furthermore,

$$\begin{aligned}
\left|B_{1,k,M} - \frac{1}{2\theta^3\lambda_k^3}\right| &= \left|\frac{2\Delta}{M(2\theta\lambda_k)^2}\sum_{i,j=1}^{M} e^{-2\theta\lambda_k|t_{j-1}-t_{i-1}|} - \frac{1}{2\theta^3\lambda_k^3}\right| \\
&= \left|\frac{\Delta}{2\theta^2\lambda_k^2} + \frac{\Delta}{M\theta^2\lambda_k^2}\sum_{i<j=1}^{M} e^{-2\theta\lambda_k(j-i)\Delta} - \frac{1}{2\theta^3\lambda_k^3}\right| \\
&= \left|\frac{\Delta}{2\theta^2\lambda_k^2} + \frac{\Delta}{M\theta^2\lambda_k^2}\sum_{l=1}^{M-1}(M-l)e^{-2\theta\lambda_k l\Delta} - \frac{1}{2\theta^3\lambda_k^3}\right| \\
&= \left|\frac{-\Delta}{2\theta^2\lambda_k^2} + \frac{\Delta}{\theta^2\lambda_k^2}\sum_{l=0}^{M-1} e^{-2\theta\lambda_k l\Delta} - \frac{\Delta}{M\theta^2\lambda_k^2}\sum_{l=1}^{M-1} l e^{-2\theta\lambda_k l\Delta} - \frac{1}{2\theta^3\lambda_k^3}\right| \\
&\leq \left|\frac{-\Delta}{2\theta^2\lambda_k^2} + \frac{\Delta}{\theta^2\lambda_k^2}\frac{\left(1-e^{-2\theta\lambda_k M\Delta}\right)}{\left(1-e^{-2\theta\lambda_k\Delta}\right)} - \frac{1}{2\theta^3\lambda_k^3}\right| + \left|\frac{\Delta}{M\theta^2\lambda_k^2}\sum_{l=1}^{M-1} l e^{-2\theta\lambda_k l\Delta}\right| \\
&=: a_{1,k,M} + b_{1,k,M}.
\end{aligned} \tag{4.3}$$

On the other hand,

$$a_{1,k,M} \leq \left|\frac{-\Delta}{2\theta^2\lambda_k^2} + \frac{\Delta}{\theta^2\lambda_k^2}\frac{1}{\left(1-e^{-2\theta\lambda_k\Delta}\right)} - \frac{1}{2\theta^3\lambda_k^3}\right| + \frac{\Delta}{\theta^2\lambda_k^2}\frac{e^{-2\theta\lambda_k M\Delta}}{\left(1-e^{-2\theta\lambda_k\Delta}\right)}$$

$$
\begin{aligned}
&= \left| \frac{\theta\lambda_k \Delta e^{-2\theta\lambda_k\Delta} + \theta\lambda_k - 1 + e^{-2\theta\lambda_k\Delta}}{2\theta^3\lambda_k^3\left(1 - e^{-2\theta\lambda_k\Delta}\right)} \right| + \frac{\Delta}{\theta^2\lambda_k^2} \frac{e^{-2\theta\lambda_k M\Delta}}{\left(1 - e^{-2\theta\lambda_k\Delta}\right)} \\
&= \frac{\Delta^2}{2\theta\lambda_k} \left| \frac{\theta\lambda_k \Delta e^{-2\theta\lambda_k\Delta} + \theta\lambda_k\Delta - 1 + e^{-2\theta\lambda_k\Delta}}{(\theta\lambda_k\Delta)^2\left(1 - e^{-2\theta\lambda_k\Delta}\right)} \right| + \frac{e^{-2\theta\lambda_k M\Delta}}{\theta^3\lambda_k^3} \frac{\theta\lambda_k\Delta}{\left(1 - e^{-2\theta\lambda_k\Delta}\right)} \\
&= C\left[\frac{\Delta^2}{2\theta\lambda_k} + \frac{e^{-2\theta\lambda_k M\Delta}}{\theta^3\lambda_k^3}(1 + \theta\lambda_k\Delta)\right], \qquad (4.4)
\end{aligned}
$$

where we used that

$$
\sup_{x>0} \frac{xe^{-2x} + x - 1 + e^{-2x}}{x^2\left(1 - e^{-2x}\right)} < C,
$$

and

$$
\sup_{x>0} \frac{x}{(1+x)(1-e^{-2x})} < C, \qquad (4.5)
$$

for some constant $C > 0$, since the functions $\frac{xe^{-2x}+x-1+e^{-2x}}{x^2(1-e^{-2x})}$ and $\frac{x}{(1+x)(1-e^{-2x})}$ are continuous on $(0, \infty)$, and

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{xe^{-2x} + x - 1 + e^{-2x}}{x^2\left(1 - e^{-2x}\right)} = 0, \\
&\lim_{x\to 0} \frac{xe^{-2x} + x - 1 + e^{-2x}}{x^2\left(1 - e^{-2x}\right)} = \lim_{x\to 0} \frac{xe^{-2x} + x - 1 + e^{-2x}}{x^3} \frac{x}{1 - e^{-2x}} = (2)(1/2) = 1,
\end{aligned}
$$

and

$$
\lim_{x\to 0} \frac{x}{(1+x)(1-e^{-2x})} = 1/2, \quad \lim_{x\to\infty} \frac{x}{(1+x)(1-e^{-2x})} = 1.
$$

Further, since $\lambda_k > 1$,

$$
\begin{aligned}
b_{1,k,M} &= \frac{\Delta}{M\theta^2\lambda_k^2} \sum_{l=1}^{M-1} l e^{-2\theta\lambda_k l\Delta} \\
&\le \frac{\Delta}{M\theta^2\lambda_k^2} \sum_{l=1}^{M-1} l e^{-2\theta l\Delta} \\
&= \frac{1}{M\theta^2\lambda_k^2\Delta} \sum_{l=1}^{M-1} (l\Delta) e^{-2\theta l\Delta}\Delta \\
&\le \frac{C_\theta}{M\lambda_k^2\Delta}, \qquad (4.6)
\end{aligned}
$$

where we used

$$
\lim_{M\to\infty} \sum_{l=1}^{M-1} (l\Delta) e^{-2\theta l\Delta}\Delta = \int_0^\infty x e^{-2\theta x} dx = \frac{1}{2\theta^2}.
$$

Combining (4.3), (4.4) and (4.6), we obtain

$$\left|B_{1,k,M} - \frac{1}{2\theta^3\lambda_k^3}\right| \le C_\theta\left[\frac{\Delta^2}{2\theta\lambda_k} + \frac{e^{-2\theta\lambda_k M\Delta}}{\theta^3\lambda_k^3}(1+\theta\lambda_k\Delta) + \frac{1}{M\lambda_k^2\Delta}\right], \tag{4.7}$$

Let us now estimate $|B_{2,k,M}|$:

$$\begin{aligned}
|B_{2,k,M}| &= \frac{\Delta}{M(\theta\lambda_k)^2}\sum_{i,j=1}^{M} e^{-2\theta\lambda_k(t_{j-1}\vee t_{i-1})} \\
&\le \frac{2\Delta}{M(\theta\lambda_k)^2}\sum_{i\le j=1}^{M} e^{-2\theta\lambda_k(t_{j-1})} \\
&= \frac{2\Delta}{M(\theta\lambda_k)^2}\sum_{i=1}^{M}\sum_{j=i}^{M} e^{-2\theta\lambda_k(j-1)\Delta} \\
&= \frac{2\Delta}{M(\theta\lambda_k)^2}\sum_{i=1}^{M} e^{-2\theta\lambda_k(i-1)\Delta}\sum_{j=i}^{M} e^{-2\theta\lambda_k(j-i)\Delta} \\
&\le \frac{2\Delta}{M(\theta\lambda_k)^2}\sum_{i=1}^{M} e^{-2\theta\lambda_k(i-1)\Delta}\frac{1}{1-e^{-2\theta\lambda_k\Delta}} \\
&\le C_\theta\frac{\Delta}{M\lambda_k^2\left(1-e^{-2\theta\lambda_k\Delta}\right)^2} \\
&\le C_\theta\left[\frac{1}{M\Delta\lambda_k^4} + \frac{\Delta}{M\lambda_k^2}\right],
\end{aligned} \tag{4.8}$$

where the last inequality comes from (4.5).

Similarly,

$$\begin{aligned}
B_{3,k,M} &= \frac{2\Delta}{M(2\theta\lambda_k)^2}\sum_{i,j=1}^{M} e^{-2\theta\lambda_k(t_{i-1}+t_{j-1})} \\
&= \frac{2\Delta}{M(2\theta\lambda_k)^2}\left(\sum_{i=1}^{M} e^{-2\theta\lambda_k(t_{i-1})}\right)^2 \\
&= \frac{2\Delta}{M(2\theta\lambda_k)^2}\left(\frac{1}{1-e^{-2\theta\lambda_k\Delta}}\right)^2 \\
&\le C_\theta\left[\frac{1}{M\Delta\lambda_k^4} + \frac{\Delta}{M\lambda_k^2}\right].
\end{aligned} \tag{4.9}$$

Combining (4.2), (4.7), (4.8), and (4.9) together with $\sup_{x\ge 0} xe^{-2x} \le C$, we get

$$\begin{aligned}
&\left|\mathbb{E}\left(F_M^2(v_k)\right) - \frac{1}{2\theta^3\lambda_k^3}\right| \\
&\le C_\theta\left[\frac{\Delta^2}{2\theta\lambda_k} + \frac{e^{-2\theta\lambda_k M\Delta}}{\theta^3\lambda_k^3}(1+\theta\lambda_k\Delta) + \frac{1}{M\lambda_k^2\Delta} + \frac{1}{M\Delta\lambda_k^4} + \frac{\Delta}{M\lambda_k^2}\right] \\
&\le C_\theta\left[\frac{\Delta^2}{\lambda_k} + \frac{1}{\lambda_k^2 M\Delta} + \frac{1}{\lambda_k^3 M} + \frac{\Delta}{\lambda_k^2 M}\right]
\end{aligned}$$

$$\leq C_\theta \left[\frac{\Delta^2}{\lambda_k} + \frac{1}{\lambda_k^2 M\Delta}\right].$$

Therefore, the desired result is obtained. □

Lemma 4.2 *Let $\Lambda_{N,M,T}$ be the process defined by* (3.6). *Then, there exists $C_\theta > 0$ that depends only on θ such that, for every $M, N \geq 1$, $T > 0$,*

$$\left| E\left[\left(\sqrt{\frac{2\theta}{T\sum_{k=1}^{N}\lambda_k}}\Lambda_{N,M,T}\right)^2\right] - 1\right| \leq \frac{C_\theta}{\sum_{k=1}^{N}\lambda_k}\left[\Delta\sum_{k=1}^{N}\lambda_k^2 + \frac{N}{M\Delta}\right]. \tag{4.10}$$

Proof Since, for every $k = 1,\dots,N$, $i = 1,\dots,M$, the random variables $\zeta_k(t_{i-1})$ and $\zeta_k(t_i) - \zeta_k(t_{i-1})$ are independent, for every $k = 1,\dots,N$, we can write

$$\begin{aligned}
&E\left[\left(\frac{1}{\sqrt{T}}\Lambda_{\lambda_k,M}\right)^2\right]\\
&= \frac{1}{T}\sum_{i,j=1}^{M} e^{-\theta\lambda_k(t_i+t_{i-1}+t_j+t_{j-1})} E\left[\zeta_k(t_{i-1})(\zeta_k(t_i) - \zeta_k(t_{i-1}))\zeta_k(t_{i-1})\left(\zeta_k(t_j) - \zeta_k(t_{j-1})\right)\right]\\
&= \frac{1}{T}\sum_{i=1}^{M} e^{-2\theta\lambda_k(t_i+t_{i-1})} E\left[\zeta_k(t_{i-1})^2(\zeta_k(t_i) - \zeta_k(t_{i-1}))^2\right]\\
&= \frac{1}{T}\sum_{i=1}^{M} e^{-2\theta\lambda_k(t_i+t_{i-1})} E\left[\zeta_k(t_{i-1})^2\right] E\left[(\zeta_k(t_i) - \zeta_k(t_{i-1}))^2\right]\\
&= \frac{1}{T}\sum_{i=1}^{M} e^{-2\theta\lambda_k(t_i+t_{i-1})} \left(\frac{e^{2\theta\lambda_k t_{i-1}} - 1}{2\theta\lambda_k}\right)\left(\frac{e^{2\theta\lambda_k t_i} - e^{2\theta\lambda_k t_{i-1}}}{2\theta\lambda_k}\right)\\
&= \frac{\left(1 - e^{-2\theta\lambda_k\Delta}\right)}{(2\theta\lambda_k)^2\Delta}\frac{1}{M}\sum_{i=1}^{M}\left(1 - e^{-2\theta\lambda_k t_{i-1}}\right)\\
&= \frac{\left(1 - e^{-2\theta\lambda_k\Delta}\right)}{(2\theta\lambda_k)^2\Delta} - \frac{\left(1 - e^{-2\theta\lambda_k\Delta}\right)}{(2\theta\lambda_k)^2\Delta}\left(\frac{1 - e^{-2\theta\lambda_k T}}{M(1 - e^{-2\theta\lambda_k\Delta})}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left| E\left[\left(\frac{1}{\sqrt{T}}\Lambda_{\lambda_k,M}\right)^2\right] - \frac{1}{2\theta\lambda_k}\right|\\
&\leq \left|\frac{\left(1 - e^{-2\theta\lambda_k\Delta}\right)}{(2\theta\lambda_k)^2\Delta} - \frac{1}{2\theta\lambda_k}\right| + \frac{\left(1 - e^{-2\theta\lambda_k\Delta}\right)}{(2\theta\lambda_k)^2\Delta}\left(\frac{1 - e^{-2\theta\lambda_k T}}{M(1 - e^{-2\theta\lambda_k\Delta})}\right)\\
&= \left|\frac{1 - e^{-2\theta\lambda_k\Delta} - 2\theta\lambda_k\Delta}{(2\theta\lambda_k\Delta)^2}\right|\Delta + \frac{1 - e^{-2\theta\lambda_k T}}{(2\theta\lambda_k)^2 M\Delta}\\
&\leq C_\theta\left[\Delta + \frac{1}{\lambda_k^2 M\Delta}\right],
\end{aligned} \tag{4.11}$$

where we used that

$$\sup_{x>0} \frac{|1-e^{-x}-x|}{x^2} < C,$$

for some constant $C > 0$, due to the continuity of $\frac{|1-e^{-x}-x|}{x^2}$ on $(0,\infty)$, and

$$\lim_{x\to\infty} \frac{|1-e^{-x}-x|}{x^2} = 0, \quad \lim_{x\to 0} \frac{|1-e^{-x}-x|}{x^2} = \frac{1}{2}.$$

Consequently, it follows from (4.11) that

$$\begin{aligned}\left| E\left[\left(\sqrt{\frac{2\theta}{T\sum\limits_{k=1}^{N}\lambda_k}}\Lambda_{N,M,T}\right)^2\right] - 1\right| &= \frac{2\theta}{\sum_{k=1}^N \lambda_k}\left|\sum_{k=1}^N \lambda_k^2 \left| E\left[\left(\frac{1}{\sqrt{T}}\Lambda_{\lambda_k,M}\right)^2\right] - \frac{1}{2\theta\lambda_k}\right]\right| \\ &\le \frac{C_\theta}{\sum_{k=1}^N \lambda_k}\sum_{k=1}^N \lambda_k^2\left[\Delta + \frac{1}{\lambda_k^2 M\Delta}\right] \\ &= \frac{C_\theta}{\sum_{k=1}^N \lambda_k}\left[\Delta\sum_{k=1}^N \lambda_k^2 + \frac{N}{M\Delta}\right],\end{aligned}$$

which implies (4.10). □

Lemma 4.3 *Let $F_M(\nu_k)$ and $\Lambda_{\lambda_k,M}$ be the processes defined by (3.5) and (3.6), respectively. Then, there exists $C_\theta > 0$ that depends only on θ such that, for every $k, M \ge 1$, $T > 0$,*

$$\left|\frac{T}{\theta\lambda_k} - 2\theta\lambda_k\sqrt{T}\mathbb{E}\left(\Lambda_{\lambda_k,M}F_M(\nu_k)\right)\right| \le C_\theta\left(\frac{M\Delta^2}{\lambda_k} + \frac{\Delta}{\lambda_k} + \frac{1}{\lambda_k^2}\right). \tag{4.12}$$

Proof Since $\mathbb{E}\Lambda_{\lambda_k,M} = 0$ and the $\zeta_k(t_{i-1})$, $k = 1,\dots,N$, $i = 1,\dots,M$ are Gaussian,

$$\begin{aligned}&\frac{T}{\theta\lambda_k} - 2\theta\lambda_k\sqrt{T}\mathbb{E}\left(\Lambda_{\lambda_k,M}F_M(\nu_k)\right) \\ &= \frac{T}{\theta\lambda_k} - 2\theta\lambda_k\mathbb{E}\left(\Lambda_{\lambda_k,M}S_M(\nu_k)\right) \\ &= \frac{T}{\theta\lambda_k} - 2\theta\lambda_k\Delta\mathbb{E}\left[\sum_{i=1}^n e^{-\theta\lambda_k(t_i+t_{i-1})}\zeta_k(t_{i-1})(\zeta_k(t_i)-\zeta_k(t_{i-1}))\sum_{j=1}^n e^{-2\theta\lambda_k t_{j-1}}\zeta_k(t_{j-1})^2\right] \\ &= \frac{T}{\theta\lambda_k} - 2\theta\lambda_k\Delta\sum_{i,j=1,i<j}^n e^{-\theta\lambda_k(t_i+t_{i-1})}e^{-2\theta\lambda_k t_{j-1}}\mathbb{E}\left[\zeta_k(t_{i-1})(\zeta_k(t_i)-\zeta_k(t_{i-1}))\zeta_k(t_{j-1})^2\right].\end{aligned}$$

Next, applying the Wick formula, we obtain

$$\begin{aligned}&\frac{T}{\theta\lambda_k} - 2\theta\lambda_k\sqrt{T}\mathbb{E}\left(\Lambda_{\lambda_k,M}F_M(\nu_k)\right) \\ &= \frac{T}{\theta\lambda_k} - 4\theta\lambda_k\Delta\sum_{i,j=1,i<j}^M e^{-\theta\lambda_k(t_i+t_{i-1})}e^{-2\theta\lambda_k t_{j-1}}\mathbb{E}\left[\zeta_k(t_{i-1})^2\right]\mathbb{E}\left[(\zeta_k(t_i)-\zeta_k(t_{i-1}))^2\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{T}{\theta\lambda_k} - 4\theta\lambda_k\Delta\sum_{j=2}^{M}\sum_{i=1}^{j-1} e^{-\theta\lambda_k(t_i+t_{i-1})}e^{-2\theta\lambda_k t_{j-1}}\left[\frac{e^{2\theta\lambda_k t_{i-1}}-1}{2\theta\lambda_k}\right]\left[\frac{e^{2\theta\lambda_k t_i}-e^{2\theta\lambda_k t_{i-1}}}{2\theta\lambda_k}\right] \\
&= \frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}\sum_{j=2}^{M} e^{-2\theta\lambda_k t_{j-1}}\sum_{i=1}^{j-1} e^{\theta\lambda_k(t_i+t_{i-1})}\left[1-e^{-2\theta\lambda_k t_{i-1}}\right]\left[1-e^{-2\theta\lambda_k\Delta}\right] \\
&= \left(\frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-2\theta\lambda_k t_{j-1}}\sum_{i=1}^{j-1} e^{\theta\lambda_k(t_i+t_{i-1})}\right) \\
&\quad + \left(\frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-2\theta\lambda_k t_{j-1}}\sum_{i=1}^{j-1} e^{\theta\lambda_k\Delta}\right) \\
&=: d_{1,k,M} + d_{2,k,M},
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
d_{1,k,M} &= \frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-2\theta\lambda_k\Delta(j-1)}\sum_{i=1}^{j-1} e^{\theta\lambda_k\Delta}e^{2\theta\lambda_k\Delta(i-1)} \\
&= \frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]e^{\theta\lambda_k\Delta}\sum_{j=2}^{M} e^{-2\theta\lambda_k\Delta(j-1)}\frac{e^{2\theta\lambda_k\Delta(j-1)}-1}{e^{2\theta\lambda_k\Delta}-1} \\
&= \frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}e^{-\theta\lambda_k\Delta}\sum_{j=2}^{M}\left(1-e^{-2\theta\lambda_k\Delta(j-1)}\right) \\
&= \frac{T}{\theta\lambda_k} - \frac{\Delta}{\theta\lambda_k}e^{-\theta\lambda_k\Delta}\left[M-1-e^{-2\theta\lambda_k\Delta}\frac{\left(1-e^{-2\theta\lambda_k\Delta(M-1)}\right)}{1-e^{-2\theta\lambda_k\Delta}}\right] \\
&= \frac{T}{\theta\lambda_k}\left(1-e^{-\theta\lambda_k\Delta}\right) + \frac{\Delta}{\theta\lambda_k}e^{-\theta\lambda_k\Delta} + \frac{\Delta}{\theta\lambda_k}e^{-3\theta\lambda_k\Delta}\frac{\left(1-e^{-2\theta\lambda_k\Delta(M-1)}\right)}{1-e^{-2\theta\lambda_k\Delta}}.
\end{aligned}
$$

Hence, using $\sup_{x>0}\frac{1-e^{-x}}{x} \le C$ and (4.5),

$$
\left|d_{1,k,M}\right| \le C_\theta\left(M\Delta^2 + \frac{\Delta}{\lambda_k} + \frac{1}{\lambda_k^2}\right).
\tag{4.14}
$$

Similarly, using $\sup_{x>0} xe^{-x} \le C$,

$$
\begin{aligned}
d_{2,k,M} &= \frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-2\theta\lambda_k t_{j-1}}(j-1)e^{\theta\lambda_k\Delta} \\
&\le C\frac{\Delta e^{\theta\lambda_k\Delta}}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-\theta\lambda_k(j-1)\Delta} \\
&= C\frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\sum_{j=2}^{M} e^{-\theta\lambda_k(j-2)\Delta} \\
&= C\frac{\Delta}{\theta\lambda_k}\left[1-e^{-2\theta\lambda_k\Delta}\right]\frac{1-e^{-\theta\lambda_k(M-2)\Delta}}{1-e^{-\theta\lambda_k\Delta}}
\end{aligned}
$$

$$\leq C\frac{\Delta}{\theta\lambda_k}\left[1+e^{-\theta\lambda_k\Delta}\right]$$
$$\leq C_\theta\frac{\Delta}{\lambda_k}. \tag{4.15}$$

Therefore, by combining (4.13), (4.14), and (4.15), we obtain the estimate (4.12). □

Lemma 4.4 *Let $A_{N,M,T}$ and $a_{N,M,T}$ be the processes given by (3.10) and (3.11), respectively. Then, there exists $C_\theta > 0$ that depends only on θ such that, for every $M,N \geq 1$, $T > 0$,*

$$\frac{\|A_{N,M,T}\|_{L^2(\Omega)} + |a_{N,M,T}|}{\sqrt{T\sum_{k=1}^{N}\lambda_k}} \leq C_\theta \max\left(N^2\Delta, N^4\Delta^2, N^3\sqrt{\frac{\Delta}{M}}, \sqrt{M\Delta^3N^7}, \sqrt{N^2\Delta}, \frac{N}{\sqrt{M\Delta}}\right). \tag{4.16}$$

If, in addition, $N,M,T \to \infty$ and $T^3N^7/M^2 \to 0$, then, for every $M,N \geq 1$, $T > 0$,

$$\frac{\|A_{N,M,T}\|_{L^2(\Omega)} + |a_{N,M,T}|}{\sqrt{T\sum_{k=1}^{N}\lambda_k}} \leq C_\theta\sqrt{\max\left(\frac{N^2T}{M}, \frac{T^3N^7}{M^2}, \frac{N^2}{T}\right)}. \tag{4.17}$$

Proof Set

$$A_{N,M,T} =: \sqrt{2\theta}\left[A_{1,N,M} + A_{2,N,M}\right],$$

where

$$A_{1,N,M} = \sum_{k=1}^{N}\lambda_k^2\left(\frac{e^{-\lambda_k\theta\Delta}-1}{\lambda_k\Delta}+\theta\right)\left(S_M(\nu_k) - \mathbb{E}S_M(\nu_k)\right),$$

and

$$A_{2,N,M} = \Lambda_{N,M,T} - \theta\sum_{k=1}^{N}\lambda_k^2\left(S_M(\nu_k) - \mathbb{E}S_M(\nu_k)\right).$$

Since the ν_k, $k = 1,\ldots,N$, are independent and $\mathbb{E}F_M(\nu_k) = 0$, we can write

$$\mathbb{E}\left(A_{1,N,M}^2\right) = T\sum_{k=1}^{N}\lambda_k^4\left(\frac{e^{-\lambda_k\theta\Delta}-1+\theta\lambda_k\Delta}{\lambda_k\Delta}\right)^2\mathbb{E}F_M^2(\nu_k).$$

Combining this with (4.1) and $\sup_{x>0}\frac{|1-e^{-x}-x|}{x^2} < C$, we get

$$\mathbb{E}\left(A_{1,N,M}^2\right) \leq T\sum_{k=1}^{N}\lambda_k^4(\lambda_k\Delta)^2\left(\frac{1}{2\theta^3\lambda_k^3}+\frac{\Delta^2}{\lambda_k}+\frac{1}{\lambda_k^2M\Delta}\right)$$
$$\leq C_\theta T\Delta^2\sum_{k=1}^{N}\left(\lambda_k^3+\lambda_k^5\Delta^2+\frac{\lambda_k^4}{M\Delta}\right),$$

which yields

$$\frac{1}{T\sum_{k=1}^{N}\lambda_k}\mathbb{E}\left(A_{1,N,M}^2\right) \le \frac{C_\theta}{\sum_{k=1}^{N}\lambda_k}\Delta^2\sum_{k=1}^{N}\left(\lambda_k^3+\lambda_k^5\Delta^2+\frac{\lambda_k^4}{M\Delta}\right)$$
$$\le C_\theta N^4\left[\Delta^2+N^4\Delta^4+\frac{\Delta N^2}{M}\right]. \tag{4.18}$$

For $A_{2,N,M}$, we have

$$A_{2,N,M}=\Lambda_{N,M,T}-\theta\sqrt{T}\sum_{k=1}^{N}\lambda_k^2F_M(\nu_k)=\sum_{k=1}^{N}\lambda_k\left(\Lambda_{\lambda_k,M}-\theta\lambda_k\sqrt{T}F_M(\nu_k)\right).$$

Since the ζ_k, $k=1,\dots,N$, are independent and $\mathbb{E}\Lambda_{\lambda_k,M}=\mathbb{E}F_M(\nu_k)=0$, we can write

$$\mathbb{E}\left(A_{2,N,M}^2\right) = \sum_{k=1}^{N}\lambda_k^2\left[\mathbb{E}\left(\Lambda_{\lambda_k,M}^2\right)+\theta^2\lambda_k^2T\mathbb{E}\left(F_M(\nu_k)^2\right)-2\theta\lambda_k\sqrt{T}\mathbb{E}\left(\Lambda_{\lambda_k,M}F_M(\nu_k)\right)\right]$$
$$\le \sum_{k=1}^{N}\lambda_k^2\left[\left|\mathbb{E}\left(\Lambda_{\lambda_k,M}^2\right)-\frac{T}{2\theta\lambda_k}\right|+\theta^2\lambda_k^2T\left|\mathbb{E}\left(F_M(\nu_k)^2\right)-\frac{1}{2\theta^3\lambda_k^3}\right|\right.$$
$$\left.+\left|\frac{T}{\theta\lambda_k}-2\theta\lambda_k\sqrt{T}\mathbb{E}\left(\Lambda_{\lambda_k,M}F_M(\nu_k)\right)\right|\right].$$

Combining this with (4.1), (4.11), and (4.12) gives

$$\frac{1}{T}\mathbb{E}\left(A_{2,N,M}^2\right) \le \frac{C_\theta}{T}$$
$$\times\sum_{k=1}^{N}\lambda_k^2\left[T\left(\Delta+\frac{1}{\lambda_k^2T}\right)+\theta\lambda_k^2T\left(\frac{\Delta^2}{\lambda_k}+\frac{1}{\lambda_k^2T}\right)+\left(\frac{T\Delta}{\lambda_k}+\frac{\Delta}{\lambda_k}+\frac{1}{\lambda_k^2}\right)\right]$$
$$\le C_\theta\sum_{k=1}^{N}\left[\left(\lambda_k^2\Delta+\frac{1}{T}\right)+\theta\left(\lambda_k^3\Delta^2+\frac{\lambda_k^2}{T}\right)+\left(\lambda_k\Delta+\frac{\lambda_k}{M}+\frac{1}{T}\right)\right]$$
$$\le C_\theta\sum_{k=1}^{N}\left[\lambda_k^2\Delta+\lambda_k^3\Delta^2+\frac{\lambda_k^2}{M\Delta}+\frac{\lambda_k}{M}\right]$$
$$\le C_\theta\sum_{k=1}^{N}\left[\lambda_k^2\Delta+\lambda_k^3\Delta^2+\frac{\lambda_k^2}{M\Delta}\right],$$

which implies

$$\frac{1}{T\sum_{k=1}^{N}\lambda_k}\mathbb{E}\left(A_{2,N,M}^2\right) \le \frac{C_\theta}{\sum_{k=1}^{N}\lambda_k}\sum_{k=1}^{N}\left[\lambda_k^2\Delta+\lambda_k^3\Delta^2+\frac{\lambda_k^2}{M\Delta}\right]$$
$$\le C_\theta N^2\left[\Delta+N^2\Delta^2+\frac{1}{M\Delta}\right]. \tag{4.19}$$

On the other hand, using $\mathbb{E}\nu_k^2(t) \leq \frac{1}{2\theta\lambda_k}$, $\sup_{x>0} \frac{|1-e^{-x}-x|}{x^2} < C$, we get

$$\frac{1}{\sqrt{2\theta}}\left|a_{N,M,T}\right| \leq \sum_{k=1}^{N} \lambda_k^2 \left|\frac{e^{-\lambda_k\theta\Delta}-1}{\lambda_k\Delta}+\theta\right| \mathbb{E}S_M(\nu_k)$$
$$\leq C_\theta T\Delta \sum_{k=1}^{N} \lambda_k^2.$$

This leads to

$$\frac{1}{\sqrt{T\sum\limits_{k=1}^{N}\lambda_k}}\left|a_{N,M,T}\right| \leq \frac{C_\theta T\Delta}{\sqrt{T\sum_{k=1}^{N}\lambda_k}} \sum_{k=1}^{N}\lambda_k^2$$
$$\leq C_\theta\sqrt{M\Delta^3 N^7}. \tag{4.20}$$

Thus, from (4.18), (4.19), and (4.20), we obtain (4.16).

Now, suppose that $N, M, T \to \infty$ and $M\Delta^3 N^7 = T^3 N^7/M^2 \to 0$. This implies that $\Delta N^2 \to 0$, since

$$(\Delta N^2)^3 = \frac{M\Delta^3 N^7}{MN} \to 0,$$

so by combining this with (4.16), we can deduce (4.17). □

Lemma 4.5 *Let $V_{N,M,T}$ be the process given by (3.13). Then, there exists $C_\theta > 0$ that depends only on θ such that, for every $M, N \geq 1$, $T > 0$,*

$$\kappa_3(V_{N,M,T}) \leq C_\theta \max\left(\frac{1}{(M\Delta)^{3/2}N^{7/2}}, \frac{\Delta^{3/2}N^{5/2}}{M^{3/2}}\right), \tag{4.21}$$
$$\kappa_4(V_{N,M,T}) \leq C_\theta \max\left(\frac{\Delta^2 N^3}{M}, \frac{1}{M\Delta N^3}\right). \tag{4.22}$$

Moreover, if, in addition, $N, M, T \to \infty$, then, for every $M, N \geq 1$, $T > 0$,

$$\max\left(\kappa_3(V_{N,M,T}), \kappa_4(V_{N,M,T})\right) \leq C_\theta \max\left(\frac{T^2N^3}{M^3}, \frac{1}{TN^3}\right). \tag{4.23}$$

Proof Let $\rho_k(r) = \mathbb{E}(Z_k(r)Z_0)$ denote the covariance of Z_k given by (3.4). It is easy to see that

$$\rho_k(t) = \mathbb{E}(Z_k(t)Z_0) = \frac{e^{-\theta|t|}}{2\theta\lambda_k}, \quad t \in \mathbb{R}.$$

In particular, $\rho_k(0) = \frac{1}{2\theta\lambda_k}$. Moreover, notice that $\rho_k(r) = \rho_k(-r)$ for all $r < 0$.

Since $\mathbb{E}[V_{N,M,T}] = 0$, we have

$$\kappa_3(V_{N,M,T}) = \mathbb{E}\left[V_{N,M,T}^3\right] \text{ and } \kappa_4(V_{N,M,T}) = \mathbb{E}\left[V_{N,M,T}^4\right] - 3\left[\mathbb{E}V_{N,M,T}^2\right]^2.$$

Furthermore, using $\mathbb{E}[F_M(Z_k)] = 0$ and the fact that Z_k, $k = 1,\dots,N$ are independent,

$$\mathbb{E}\left[V_{N,M,T}^3\right] = \frac{1}{\left(\sum_{k=1}^N \lambda_k\right)^{3/2}} \sum_{k=1}^N \lambda_k^6 \mathbb{E}\left[F_M^3(Z_k)\right] = \frac{1}{\left(\sum_{k=1}^N \lambda_k\right)^{3/2}} \sum_{k=1}^N \lambda_k^6 \kappa_3\left(F_M(Z_k)\right).$$

Further, by similar arguments as in [5], we can deduce

$$\begin{aligned} k_3(F_M(Z_k)) &\le \frac{\Delta^{3/2}}{M^{3/2}} \left(\sum_{|j|<M} \rho_k(j\Delta)\right)^3 \\ &\le \frac{\Delta^{3/2}}{M^{3/2}} \left(\frac{1-e^{-\theta\lambda_k M\Delta}}{\theta\lambda_k(1-e^{-\theta\lambda_k\Delta})}\right)^3. \end{aligned}$$

Combining this with the fact that $\sup_{x>0} \frac{x}{(1+x)(1-e^{-x})} \le C$, we obtain

$$\begin{aligned} k_3(F_M(Z_k)) &\le \frac{\Delta^{3/2}}{M^{3/2}} \left(\frac{1-e^{-\theta\lambda_k M\Delta}}{\theta\lambda_k(1-e^{-\theta\lambda_k\Delta})}\right)^3 \\ &\le C_\theta \frac{\Delta^{3/2}}{M^{3/2}} \left(\frac{1}{\Delta\lambda_k^2} + \frac{1}{\lambda_k}\right)^3 \\ &\le C_\theta \frac{\Delta^{3/2}}{M^{3/2}} \left(\frac{1}{\Delta^3\lambda_k^6} + \frac{1}{\lambda_k^3}\right), \end{aligned}$$

which leads to

$$\begin{aligned} \kappa_3(V_{N,M,T}) &\le C_\theta \frac{\Delta^{3/2}}{\left(M\sum_{k=1}^N \lambda_k\right)^{3/2}} \sum_{k=1}^N \left(\frac{1}{\Delta^3} + \lambda_k^3\right) \\ &= C_\theta \frac{\Delta^{3/2}}{\left(M\sum_{k=1}^N \lambda_k\right)^{3/2}} \left(\frac{N}{\Delta^3} + \sum_{k=1}^N \lambda_k^3\right) \\ &\le C_\theta \left(\frac{1}{(M\Delta)^{3/2} N^{7/2}} + \frac{\Delta^{3/2} N^{5/2}}{M^{3/2}}\right), \end{aligned}$$

which implies (4.21). On the other hand, using $\mathbb{E}[F_M(Z_k)] = 0$ and the fact that Z_k, $k = 1,\dots,N$ are independent, we get

$$\begin{aligned} \mathbb{E}\left[V_{N,M,T}^4\right] &= \frac{1}{\left(\sum_{k=1}^N \lambda_k\right)^2} \sum_{k_1,k_2,k_3,k_4=1}^N \lambda_{k_1}^2\lambda_{k_2}^2\lambda_{k_3}^2\lambda_{k_4}^2 \mathbb{E}\left[F_M(Z_{k_1})F_M(Z_{k_2})F_M(Z_{k_3})F_M(Z_{k_4})\right] \\ &= \frac{1}{\left(\sum_{k=1}^N \lambda_k\right)^2} \left(\sum_{k=1}^N \lambda_k^8 \mathbb{E}\left[F_M^4(Z_k)\right] + 3\sum_{j\neq k=1}^N \lambda_j^4\lambda_k^4 \mathbb{E}\left[F_M^2(Z_j)\right]\mathbb{E}\left[F_M^2(Z_k)\right]\right) \end{aligned}$$

$$= \frac{1}{\left(\sum_{k=1}^{N}\lambda_k\right)^2}\left(\sum_{k=1}^{N}\lambda_k^8\mathbb{E}\left[F_M^4(Z_k)\right] + 3\left[\sum_{k=1}^{N}\lambda_k^4\mathbb{E}\left[F_M^2(Z_k)\right]\right]^2 - 3\sum_{k=1}^{N}\lambda_k^8\mathbb{E}\left[\mathbb{E}\left[F_M^2(Z_k)\right]\right]^2\right).$$

Moreover,

$$3\left[\mathbb{E}V_{N,M,T}^2\right]^2 = \frac{3}{\left(\sum_{k=1}^{N}\lambda_k\right)^2}\left[\sum_{k=1}^{N}\lambda_k^4\mathbb{E}\left[F_M^2(Z_k)\right]\right]^2.$$

Thus,

$$\begin{aligned}\kappa_4(V_{N,M,T}) &= \mathbb{E}\left[V_{N,M,T}^4\right] - 3\left[\mathbb{E}V_{N,M,T}^2\right]^2\\ &= \frac{1}{\left(\sum_{k=1}^{N}\lambda_k\right)^2}\left(\sum_{k=1}^{N}\lambda_k^8\mathbb{E}\left[F_M^4(Z_k)\right] - 3\sum_{k=1}^{N}\lambda_k^8\mathbb{E}\left[\mathbb{E}\left[F_M^2(Z_k)\right]\right]^2\right)\\ &= \frac{1}{\left(\sum_{k=1}^{N}\lambda_k\right)^2}\sum_{k=1}^{N}\lambda_k^8\kappa_4(F_M(Z_k)).\end{aligned}$$

Furthermore, using similar arguments as in [5], we have

$$\begin{aligned}k_4(F_M(Z_k)) &\le C\frac{\Delta^2}{M}\left(\sum_{|j|<M}|\rho_k(j\Delta)|^{\frac{4}{3}}\right)^3\\ &\le C\frac{\Delta^2}{M}\left(\frac{1-e^{-\frac{4}{3}\theta\lambda_k M\Delta}}{(\theta\lambda_k)^{\frac{4}{3}}(1-e^{-\frac{4}{3}\theta\lambda_k\Delta})}\right)^3.\end{aligned}$$

Combining this with the fact that $\sup_{x>0}\frac{x}{(1+x)(1-e^{-\frac{4}{3}x})} \le C$, we obtain

$$\begin{aligned}k_4(F_M(Z_k)) &\le C\frac{\Delta^2}{M}\left(\frac{1+\frac{1}{\theta\lambda_k\Delta}}{(\theta\lambda_k)^{\frac{4}{3}}}\right)^3\\ &\le C_\theta\frac{\Delta^2}{M}\left(\frac{1}{\lambda_k^4}+\frac{1}{\lambda_k^7\Delta^3}\right).\end{aligned}$$

Therefore,

$$\kappa_4(V_{N,M,T}) \le C_\theta\frac{1}{\left(\sum_{k=1}^{N}\lambda_k\right)^2}\sum_{k=1}^{N}\lambda_k^8\frac{\Delta^2}{M}\left(\frac{1}{\lambda_k^4}+\frac{1}{\lambda_k^7\Delta^3}\right)$$

$$= C_\theta \frac{\Delta^2}{M\left(\sum_{k=1}^{N}\lambda_k\right)^2}\sum_{k=1}^{N}\left(\lambda_k^4 + \frac{\lambda_k}{\Delta^3}\right)$$

$$\leq C_\theta\left(\frac{\Delta^2 N^3}{M} + \frac{1}{M\Delta N^3}\right),$$

which proves (4.22). Now, if we suppose $N, T = M\Delta \to \infty$, then by straightforward calculations,

$$\frac{1}{(M\Delta)^{3/2}N^{7/2}} \leq C_\theta \frac{1}{M\Delta N^3}, \quad \frac{\Delta^{3/2}N^{5/2}}{M^{3/2}} \leq C_\theta \frac{\Delta^2 N^3}{M}.$$

Thus, (4.23) is obtained. □

Now, we are ready to state the main result of this paper.

Theorem 4.6 *Let $\widetilde{\theta}_{N,M,T}$ be the estimator defined by* (3.3). *Suppose, as $N, M, T \to \infty$,*

$$T^3N^7/M^2 \to 0, \text{ and } N^2/T \to 0.$$

Then, there exists a positive constant C_θ that depends only on θ such that, for all $N, M \geq 1$, $T > 0$,

$$d_{\mathrm{Kol}}\left(\sqrt{\frac{T}{2\theta}\sum_{k=1}^{N}\lambda_k}\left(\theta - \widetilde{\theta}_{N,M,T}\right), \mathcal{N}(0,1)\right) \leq C_\theta\sqrt{\max\left(\frac{N^2T}{M}, \frac{T^3N^7}{M^2}, \frac{N^2}{T}\right)},$$

In particular, since $\sum_{k=1}^{N}\lambda_k \sim \pi^2N^3/3$ as $N \to \infty$, then, as $N, M, T \to \infty$,

$$\sqrt{T}N^{\frac{3}{2}}\left(\theta - \widetilde{\theta}_{N,M,T}\right) \xrightarrow{law} \mathcal{N}\left(0, \frac{6\theta}{\pi^2}\right).$$

Proof According to (3.12), we have

$$\sqrt{\frac{T}{2\theta}\sum_{k=1}^{N}\lambda_k}\left(\theta - \widetilde{\theta}_{N,M,T}\right) = \frac{\frac{1}{\sigma}\left(G_{N,M,T} - \mathbb{E}G_{N,M,T}\right) + \frac{1}{\sqrt{T\sum_{k=1}^{N}\lambda_k}}\left(A_{N,M,T} + a_{N,M,T}\right)}{\frac{1}{\rho\sqrt{T\sum_{k=1}^{N}\lambda_k}}G_{N,M,T}},$$

where $G_{N,M,T}$, $A_{N,M,T}$, and $a_{N,M,T}$ are given by (3.9), (3.10), and (3.11), respectively. Moreover, $\rho = \frac{1}{2\theta}$, $\sigma^2 = \frac{1}{2\theta^3}$ and

$$G_{N,M,T} - \mathbb{E}G_{N,M,T} = V_{N,M,T} + \frac{1}{\sqrt{\varphi_{N,M,T}}}R_{N,M,T}, \quad N, M \geq 1, T > 0,$$

where $V_{N,M,T}$ and $R_{N,M,T}$ are given by (3.13) and (3.14), respectively.

Using (3.4) and the fact that Z_k is a Gaussian stationary process, we can show that there exists C_θ that depends only on θ such that

$$\left|\mathbb{E}\left(v_k^2(t)\right) - \frac{1}{2\theta\lambda_k}\right| \leq C_\theta \frac{e^{-\theta\lambda_k t}}{\lambda_k}, \quad k \geq 1, \ t \geq 0.$$

This implies

$$
\begin{aligned}
\left|\frac{1}{\rho\sqrt{T\sum_{k=1}^{N}\lambda_k}}\mathbb{E}G_{N,M,T}-1\right| &\leq \frac{C_\theta}{M\sum_{k=1}^{N}\lambda_k}\sum_{k=1}^{N}\lambda_k^2\sum_{i=1}^{M}\frac{e^{-\theta\lambda_k t_{i-1}}}{\lambda_k} \\
&\leq \frac{C_\theta}{M\sum_{k=1}^{N}\lambda_k}\sum_{k=1}^{N}\frac{\lambda_k}{1-e^{-\theta\lambda_k\Delta}} \\
&\leq C_\theta\left(\frac{1}{TN^2}+\frac{1}{T}\right) \\
&\leq \frac{C_\theta}{T},
\end{aligned}
\tag{4.24}
$$

where we used (4.5). On the other hand, according to (4.1)

$$
\begin{aligned}
\left|\mathbb{E}[(G_{N,M,T}-\mathbb{E}G_{N,M,T})^2]-\sigma^2\right| &= \left|\frac{1}{\sum_{k=1}^{N}\lambda_k}\sum_{k=1}^{N}\lambda_k^4\left(\mathbb{E}\left(F_M^2(\nu_k)\right)-\frac{1}{2\theta^3\lambda_k^3}\right)\right| \\
&\leq \frac{C_\theta}{\sum_{k=1}^{N}\lambda_k}\sum_{k=1}^{N}\lambda_k^4\left[\frac{\Delta^2}{\lambda_k}+\frac{1}{\lambda_k^2 T}\right] \\
&\leq C_\theta\left(\Delta^2 N^4+\frac{N^2}{T}\right) \\
&= C_\theta\left(\frac{T^2N^4}{M^2}+\frac{N^2}{T}\right).
\end{aligned}
\tag{4.25}
$$

Furthermore, notice that $R_{N,M,T}\in\mathcal{H}_2$. By combining (3.4) and the fact that Z_k is a Gaussian stationary process, we can show that there exists C_θ that depends only on θ such that

$$
\frac{1}{\sqrt{T\sum_{k=1}^{N}\lambda_k}}\|R_{N,M,T}\|_{L^2(\Omega)}\leq C_\theta\left(\frac{1}{\sqrt{T^3N^3}}+\sqrt{\frac{N}{M^2T}}\right).
\tag{4.26}
$$

Using $T^3N^7/M^2\to 0$ and $N^2/T\to 0$ as $N,M,T\to\infty$, we see that the upper bounds in (4.17), (4.24), (4.25), and (4.26) all converge to zero, so $\{G_{N,M,T},N,M\geq 1,T>0\}$ and $\{(A_{N,M,T},a_{N,M,T}),N,M\geq 1,T>0\}$ satisfy the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, respectively.

Therefore, applying Theorem 3.1, we obtain

$$
\begin{aligned}
& d_{\mathrm{Kol}}\left(\sqrt{\frac{T}{2\theta}\sum_{k=1}^{N}\lambda_k}\left(\theta-\widetilde{\theta}_{N,M,T}\right),\mathcal{N}(0,1)\right) \\
&\leq \max\left(\left|\kappa_3\left(\frac{V_{N,M,T}}{\sigma}\right)\right|,\kappa_4\left(\frac{V_{N,M,T}}{\sigma}\right)\right)+C\left(T\sum_{k=1}^{N}\lambda_k\right)^{\frac{1}{4}}\left|\frac{1}{\rho\sqrt{\varphi_{N,M,T}}}\mathbb{E}G_{N,M,T}-1\right|
\end{aligned}
$$

$$+C\left|\mathbb{E}[(G_{N,M,T}-\mathbb{E}G_{N,M,T})^2]-\sigma^2\right|$$
$$+\frac{C}{\sqrt{T\sum_{k=1}^{N}\lambda_k}}\left(\|R_{N,M,T}\|_{L^2(\Omega)}+\|A_{N,M,T}\|_{L^2(\Omega)}+|a_{N,M,T}|\right).$$

Combining this result with (4.23), (4.17), (4.24), (4.25), and (4.26), we have

$$d_{\text{Kol}}\left(\sqrt{\frac{T}{2\theta}\sum_{k=1}^{N}\lambda_k}\left(\theta-\widetilde{\theta}_{N,M,T}\right),\mathcal{N}(0,1)\right)$$
$$\le C_\theta\left[\max\left(\frac{T^2N^3}{M^3},\frac{1}{TN^3}\right)+\frac{N^{3/4}}{T^{3/4}}+\frac{T^2N^4}{M^2}+\frac{N^2}{T}\right.$$
$$\left.+\frac{1}{\sqrt{T^3N^3}}+\sqrt{\frac{N}{M^2T}}+\sqrt{\max\left(\frac{N^2T}{M},\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)}\right].$$

Using $N,M,T\to\infty$, $T^3N^7/M^2\to 0$, $N^2/T\to 0$, we deduce

$$d_{\text{Kol}}\left(\sqrt{\frac{T}{2\theta}\sum_{k=1}^{N}\lambda_k}\left(\theta-\widetilde{\theta}_{N,M,T}\right),\mathcal{N}(0,1)\right)\le C_\theta\sqrt{\max\left(\frac{N^2T}{M},\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)}$$

due to

$$\max\left(\frac{T^2N^3}{M^3},\frac{1}{TN^3}\right)\le C\max\left(\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)\le C\sqrt{\max\left(\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)},$$
$$\frac{N^{3/4}}{T^{3/4}}=\sqrt{\frac{N^2}{T}}\frac{1}{(TN)^{1/4}}\le C\sqrt{\frac{N^2}{T}},$$
$$\frac{T^2N^4}{M^2}+\frac{N^2}{T}=\frac{T^3N^7}{M^2}\frac{1}{TN^3}+\frac{N^2}{T}\le C\max\left(\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)\le C\sqrt{\max\left(\frac{T^3N^7}{M^2},\frac{N^2}{T}\right)},$$
$$\frac{1}{\sqrt{T^3N^3}}+\sqrt{\frac{N}{M^2T}}\le C\sqrt{\frac{N^2}{T}}.$$

Thus, the proof is complete. □

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