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Optimal control analysis in a reaction-diffusion SIRC model with cross-immune class

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Abstract

In this paper, we propose a reaction-diffusion SIRC model with cross-immunization. The model includes the implementation of vaccination measures for susceptible, and treatment and quarantine measures for infected in order to control the spread of the disease. The optimal control strategies for the spread of disease are mainly investigated. Our research focuses on four main aspects. Firstly, we prove the existence and uniqueness of the global positive strong solution to the control system by employing the theories of semigroups of operators. Secondly, we demonstrate the existence of optimal control through the utilization of functional analysis techniques. Thirdly, we establish the first-order necessary optimality conditions that the optimal control must satisfy, employing the methods of convex perturbation. Lastly, we provide numerical examples and simulations to verify the feasibility of optimal control.

Keywords: Reaction-diffusion SIRC model; Cross-immunization; Optimal control; First-order necessary condition; Numerical simulation

1 Introduction

As we all know, infectious diseases have always been the great enemy threatening human health and social economy. To effectively prevent and control the spread of infectious diseases, human beings urgently need to do in-depth research on infectious disease patterns, development trends, and prevention and control strategies. However, mathematical models of infectious diseases play a crucial role in the study of infectious diseases, which helps to understand the process and pattern of the evolution of infectious diseases in time and space, to predict the peaks and trends of disease development, and to assess the effectiveness of preventive and control measures [1, 2]. In recent years, mathematical models of infectious disease dynamics have gained popularity as a common tool for studying infectious diseases. In the last 20 years, based on the traditional SIR model proposed by Kermack and Mckendrick [3], many epidemiological models for forecasting disease transmission through populations has been presented.

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The class of cross-immunized individuals (C), which is an intermediate state between the fully susceptible state (S) and the fully protected state (R), was introduced in the population by Casagrandi et al. [4]. As a result, they proposed and investigated the following SIRC epidemic model of four-dimensional ordinary differential equation

$$\begin{cases} \dfrac{\partial S(t)}{\partial t} =A - \mu S(t) - \beta S(t)I(t) + \gamma C(t), \\ \dfrac{\partial I(t)}{\partial t} =\beta S(t)I(t) + \sigma\beta C(t)I(t) - (\mu + \alpha)I(t), \\ \dfrac{\partial R(t)}{\partial t} =(1 - \sigma)\beta C(t)I(t) + \alpha I(t) - (\mu + \delta)R(t), \\ \dfrac{\partial C(t)}{\partial t} =\delta R(t) - \beta C(t)I(t) - (\mu + \gamma)C(t). \end{cases} \tag{1.1}$$

This model (1.1) takes into account temporary partial immunity and may well describe, for example, influenza A. Many scholars have undertaken an extensive study on this model, however, the majority of these works are based on ordinary differential equations [5–8].

In many years of research on infectious diseases, researchers have found that space has a certain impact on the spread of diseases. For example, the spread rate of the COVID in Asia differs from that in North America. Therefore, in the process of mathematical modeling needs to be taken into account the spatial heterogeneity and diffusion behaviors, studying the optimal control of reaction-diffusion epidemic models has more important biological and mathematical significance. However, to the best of our knowledge, there is no literature that studies the optimal control problem of reaction-diffusion SIRC model. In order to bridge this gap and make a contribution to the optimal control problems, in this paper, we extend the model (1.1) to the reaction-diffusion SIRC epidemic model with cross-immunization and spatial heterogeneity and are mainly interested in investigating the optimal control problem for this model. The model is given as follows

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} =d_1\Delta S(t,x) + A(x) - \beta_1(x)S(t,x)I(t,x) + \gamma(x)C(t,x) \\ \qquad\qquad - \mu(x)S(t,x),\ t > 0, x \in \Omega, \\ \dfrac{\partial I(t,x)}{\partial t} =d_2\Delta I(t,x) + \beta_1(x)S(t,x)I(t,x) + \sigma(x)\beta_2(x)C(t,x)I(t,x) \\ \qquad\qquad - (\mu(x) + \alpha(x))I(t,x),\ t > 0, x \in \Omega, \\ \dfrac{\partial R(t,x)}{\partial t} =d_3\Delta R(t,x) + (1 - \sigma(x))\beta_2(x)C(t,x)I(t,x) + \alpha(x)I(t,x) \\ \qquad\qquad - (\mu(x) + \delta(x))R(t,x),\ t > 0, x \in \Omega, \\ \dfrac{\partial C(t,x)}{\partial t} =d_4\Delta C(t,x) + \delta(x)R(t,x) - \beta_2(x)C(t,x)I(t,x) \\ \qquad\qquad - (\mu(x) + \gamma(x))C(t,x),\ t > 0, x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $S(t,x)$, $I(t,x)$, $R(t,x)$ and $C(t,x)$ stand for the densities of susceptible, infected, recovered, and cross-immune individuals at time t and spatial location x, respectively. d_i $(1 \le i \le 4)$ are the diffusion rate coefficient of susceptible, infected, recovered, and cross-immune individuals, respectively; $A(x)$ denotes the rate of immigrant and newborn of the population in the location x; $\mu(x)$ denotes the mortality rates of S, I, R and C in the location x, respectively; $\beta_1(x)$ and $\beta_2(x)$ are the infection rates from susceptible to infected and cross-immunizer to infected in the location x, respectively; $\alpha(x)$ is the removal rate from infected to recovered in the location x; $\delta(x)$ is the conversion rate from recovered to cross-immunizer in the location x; $\gamma(x)$ is the lose rate of immunity for cross-immunizer in the location x; $\sigma(x)$ is the recruitment rate of cross-immune into the infective in the location x, and $0 \le \sigma \le 1$.

The development of maximal principle and dynamic programming theory in the 1950s marked a turning point in the field of optimal control theory research. Since then, as control theories have developed rapidly, some researchers have started to think about using these theories to control infectious diseases. It is noteworthy that there has been an increasing interest lately in the optimal control of epidemic models. Partial differential equation optimal control theory was systematically developed by Lions[9]. Wang et al. investigated a semilinear elliptic equation for optimal control issues (see[10, 11]). In their study of the best the optimal treatment strategies to cure tuberculosis, Jun et al. [12] used optimal control theories to a class of two species tuberculosis model represented by ordinary differential equations. Arino et al. [13] studied an influenza model incorporating antiviral therapy and vaccination, then the numerical computation outcomes of the deterministic model and the random model of influenza were contrasted. Iacoviello and Stasio [14] presented the controls for influenza to both the susceptible and infected classes, suggested an appropriate cost index, and evaluated the existence outcome. They talked about the numerical implementation of the optimal controls and gave the analytical formulas for them. Using optimal control theories and methods, Xiang and Liu [15] examined the parameter estimation problem of a class of reaction-diffusion SIS infectious disease model. Li et al. [16] provided a delayed SIRC model of influenza A and used mathematical analysis and optimal control theory to examine the SIRC epidemic model. Zhou et al. [17] studied the optimal control problem for a reaction-diffusion SIR model and incorporated two control strategies into this epidemic system. In order to prevent the death of infected individuals in a specific area, Laaroussi et al. [18] developed an optimal regional control and a spatiotemporal epidemic transmission model of the Ebola disease. First-order necessary conditions for the optimal control were established by Dai and Liu [19] after studying an optimal control problem of a generic reaction-diffusion eco-epidemiological the model with disease in the prey. They then used comparable techniques to obtain the first-order necessary condition for optimal control in a general reaction-diffusion tumor-immune system using chemotherapy (see [20]). In order to reduce the weighted tumor burden, side effects, and treatment costs, Dai and Liu [21] addressed an optimal control issue for a generic reaction-diffusion tumor-immune interaction system under immunotherapy and chemotherapy. To investigate the implications of vaccination rates, protection rates, and contact counts across age groups on the management of influenza transmission, Chen et al. [22] developed a mixed cross-infection influenza model by age group.

In order to establish the optimal control problems of the reaction-diffusion SIRC epidemic model (1.2). We study two control strategies in the context of a disease epidemic: the first control strategy is to vaccinate susceptible individuals, which we represent by the control variable v_1, which is proportional to the density of existing susceptible individuals, and the second control strategy is to treat or quarantine infected individuals, which we denote by the control variable v_2, which is proportional to the density of existing infected individuals. Hence, in the first equation of model (1.2) we should subtract a term $v_1 S$, in the second equation of model (1.2) we should subtract a term $v_2 I$, and in the third equation of model (1.2) we should add a term $v_1 S + v_2 I$. Considering that the rates of vaccination, treatment and quarantine vary from region to region and from time to time, i.e., $v_1 = v_1(t,x)$, $v_2 = v_2(t,x)$ (v_1 and v_2 not only depend on $t \in [0,T]$, but also on $x \in \Omega$). Thus, a controlled system corresponding to the reaction-diffusion SIRC model (1.2) is given as follows

$$
\begin{cases}
\dfrac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) + A(x) - \beta_1(x) S(t,x) I(t,x) + \gamma(x) C(t,x) \\
\qquad - \mu(x) S(t,x) - v_1(t,x) S(t,x), \\
\dfrac{\partial I(t,x)}{\partial t} = d_2 \Delta I(t,x) + \beta_1(x) S(t,x) I(t,x) + \sigma(x) \beta_2(x) C(t,x) I(t,x) \\
\qquad - (\mu(x) + \alpha(x)) I(t,x) - v_2(t,x) I(t,x), \\
\dfrac{\partial R(t,x)}{\partial t} = d_3 \Delta R(t,x) + (1 - \sigma(x)) \beta_2(x) C(t,x) I(t,x) + \alpha(x) I(t,x) \\
\qquad - (\mu(x) + \delta(x)) R(t,x) + v_1(t,x) S(t,x) + v_2(t,x) I(t,x), \\
\dfrac{\partial C(t,x)}{\partial t} = d_4 \Delta C(t,x) + \delta(x) R(t,x) - \beta_2(x) C(t,x) I(t,x) \\
\qquad - (\mu(x) + \gamma(x)) C(t,x),
\end{cases}
\tag{1.3}
$$

where $(t,x) \in \Omega_T = (0,T) \times \Omega$. We assume that the solution of system (1.3) satisfies the following Neumann boundary condition and initial conditions

$$
\frac{\partial S(t,x)}{\partial n} = \frac{\partial I(t,x)}{\partial n} = \frac{\partial R(t,x)}{\partial n} = \frac{\partial C(t,x)}{\partial n} = 0, \ (t,x) \in \Sigma_T = (0,T) \times \partial\Omega \tag{1.4}
$$

and

$$
S(0,x) = \psi_1(x), \ I(0,x) = \psi_2(x), \ R(0,x) = \psi_3(x), \ C(0,x) = \psi_4(x), \ x \in \Omega, \tag{1.5}
$$

where n is the outward with normal vector on $\partial\Omega$ and $\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \in \mathbb{X}^+$ is the initial function, where the space $\mathbb{X}^+$ will be defined below. The boundary condition (1.4) implies that system (1.3) is self-contained and there are no emigrations across $\partial\Omega$.

The admissible control set $\mathcal{U}$ is defined by

$$
\mathcal{U} = \{ v = (v_1, v_2) \in (L^2(\Omega_T))^2, \ 0 \le v_i(t,x) \le 1, \ i = 1,2, \ a.e. \ in \ \Omega_T \}, \tag{1.6}
$$

where the space $L^2(\Omega_T)$ is defined below. As a natural goal, we attempt to minimize the total number of susceptible and infected individuals and reduce the total cost of vaccines and treatment in the time interval $[0,T]$, while minimizing the number of susceptible and infected individuals and the cost of vaccines and treatment at terminal time T. To do this, we set the following objectives functional

$$J(S,I,v)=\int_0^T\int_\Omega L(S,I,v)(t,x)dxdt+\int_\Omega \varphi(S,I,v)(T,x)dx, \tag{1.7}$$

where

$$\begin{aligned} L(S,I,v)(t,x)&=\lambda_1(t,x)S(t,x)+\lambda_2(t,x)I(t,x)+\kappa_1(t,x)v_1(t,x)+\kappa_2(t,x)v_2(t,x),\\ \varphi(S,I,v)(T,x)&=\omega_1(x)S(T,x)+\omega_2(x)I(T,x)+\rho_1(x)v_1(T,x)+\rho_2(x)v_2(T,x). \end{aligned} \tag{1.8}$$

In (1.7), the first integral expresses the total number of susceptible and infected individuals and the total cost of vaccines and treatment on the time interval $[0,T]$, and the second integral indicates the number of susceptible and infected individuals and the cost of vaccines and treatment at terminal time T. In (1.8), the nonnegative functions $\lambda_i(t,x)\in L^\infty(\Omega_T)$ $(i=1,2)$ and $\omega_i(x)\in L^\infty(\Omega)$ $(i=1,2)$ represent the weights of S and I, respectively, and the nonnegative functions $\kappa_i(t,x)\in L^\infty(\Omega_T)$ $(i=1,2)$ and $\rho_i(x)\in L^\infty(\Omega)$ $(i=1,2)$ are the measure of the cost of interventions associated with the control for vaccination and treatment in $(t,x)\in\Omega_T$, where the spaces $L^\infty(\Omega_T)$ and $L^\infty(\Omega)$ are defined below.

Let (S,I,R,C) is the solution to system (1.3)–(1.5). Then the optimal control problem is to minimize the objective functional $J(S,I,v)$. That is, to find the control function $\overline{v}=(\overline{v}_1,\overline{v}_2)\in\mathcal{U}$ such that

$$J(\overline{S},\overline{I},\overline{v})=\inf_{v=(v_1,v_2)\in\mathcal{U}}J(S,I,v), \tag{1.9}$$

where $(\overline{S},\overline{I},\overline{R},\overline{C})$ is the solution to system (1.3)–(1.5) with $v=\overline{v}$.

The main purpose of this paper is to investigate the necessary conditions satisfied by an optimal control system (1.3)–(1.5). This research proposes a model that takes into account two control strategies: vaccination and treatment/quarantine, which is more in line with the actual scenario of infectious disease prevention and control, and takes into account the spatial heterogeneity and diffusion behavior. While this work has implications for the prevention and control of infectious diseases, our primary focus is on the mathematical analysis of the optimal control problem. We do not give details of the actual implementation of prevention and control of specific diseases, but we give here a theoretical foundation and numerical simulations that one can develop. The main contributions and innovations are summarized as follows:

(1) The existing research on optimal control of infectious diseases, such as influenza A, HIV, tuberculosis, and so on, mainly focused on the epidemic models described by ordinary differential equations, while the control system considered in this paper is a reaction-diffusion equation.

(2) The global existence and uniqueness of positive solutions to the control system (1.3)–(1.5) is established by using truncation function techniques, the results of Theorem 2.1 (see Sect. 2 below) and C_0-semigroup theory.

(3) We establish the existence of the optimal pair of control systems (1.3)–(1.5) by using the compactness and convergence methods, and the minimal sequence techniques.

(4) The first-order necessary condition for the optimal control system (1.3)–(1.5) is established by using dual techniques.

(5) We provide numerical examples and numerical simulations for validating the rationality of the theoretical results.

This paper is structured as follows. We provide some necessary notions, preliminary information, and helpful presumptions for the research in Sect. 2. The focus of Sect. 3 is demonstrating the global existence and uniqueness of positive solutions to control system (1.3)–(1.5). In Sect. 4, we establish the existence of optimal control. In Sect. 5, we state and prove the first-order necessary condition for optimal control. In Sect. 6, the numerical examples and numerical simulations are provided for the existence of optimal control. Lastly, in Sect. 7, a short conclusion is given and some new problems are proposed that can be studied in the future.

2 Notations and preliminaries

We initially provide some notations and recall some initial results that will be frequently used going forward and important to solving our problem before beginning the mathematical study of our optimal control problem.

In this paper, for the convenience of understanding, we first introduce the following notations and concepts. Let $D \subset R^m$ be a bounded domain with smooth boundary ∂D, and $\overline{D}$ is the closure of D. For any bounded function $k(x)$ defined on $\overline{D}$ we denote $k^* = \sup_{x\in\overline{D}} k(x)$ and $k_* = \inf_{x\in\overline{D}} k(x)$. Denote by $C(\overline{D})$ the Banach space of all continuous functions $\phi : \overline{D} \to \mathbb{R}$ with the supremum norm $\|\phi\|_{C(\overline{D})} = \sup_{x\in\overline{D}} |\phi(x)|$. Let $C_+(\overline{D}) = \{\phi \in C(\overline{D}) : \phi(x) \geq 0,\ \phi(x) \not\equiv 0,\ x \in \overline{D}\}$ be the positive cone of $C(\overline{D})$. For any constant $p \geq 1$, denote by $L^p(D)$ the Banach space of all Lebesgue measurable functions $\phi : D \to \mathbb{R}$ with the norm $\|\phi\|_{L^p(D)} = (\int_D |\phi(x)|^p dx)^{\frac{1}{p}} < \infty$. Particularly, when $p = \infty$ we have the space $L^\infty(D)$ of all Lebesgue measurable functions $\phi : D \to \mathbb{R}$ with the norm $\|\phi\|_{L^\infty(D)} = \text{esssup}_{x\in D}|\phi(x)| < \infty$. For the convenience, we denote $\mathbb{X} = (C(\overline{D}))^4$, $\mathbb{X}^+ = (C_+(\overline{D}))^4$ and $\mathbb{Y} = (L^2(D))^4$. For any $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{Y}$ we have the norm $\|\phi\|_{\mathbb{Y}} = (\int_D \sum_{i=1}^4 |\phi_i(x)|^2 dx)^{\frac{1}{2}} < \infty$. For any two points $\phi_i = (\phi_{i1}, \phi_{i2}, \phi_{i3}, \phi_{i4}) \in \mathbb{Y}$ $(i = 1, 2)$, we define the scalar product by $\langle\phi_1, \phi_2\rangle = \sum_{i=1}^4 \int_D \phi_{1i}(x)\phi_{2i}(x)dx$. Particularly, we have $\langle\phi_1, \phi_1\rangle = \sum_{i=1}^4 \int_D \phi_{1i}(x)^2 dx = \|\phi_1\|^2_{\mathbb{Y}}$. Thus, $\mathbb{Y}$ is also a Hilbert space.

We further denote $H^1(D) = W^{1,2}(D)$ and $H^2(D) = W^{2,2}(D)$, where $W^{1,2}(D)$ and $W^{2,2}(D)$ are the Sobolev spaces defined on D. The detail on the Sobolev space can be referred to [23]. Let $\mathbb{B}$ is a Banach space with the norm $\|\cdot\|_{\mathbb{B}}$. For any constant $p \geq 1$ we define by $L^p(0, T; \mathbb{B})$ the Banach space of all absolutely Lebesgue integrable functions $\phi : [0, T] \to \mathbb{B}$ with the norm $\|\phi\|_{L^p(0,T;\mathbb{B})} = (\int_0^T \|\phi(t)\|^p_{\mathbb{B}} dt)^{\frac{1}{p}} < \infty$. Particularly, for $p = 2, \infty$ and $\mathbb{B} = L^2(D)$, $H^1(D)$, $H^2(D)$, we have the Banach spaces as follows.

The space $L^2(0, T; L^2(D))$ with the norm $\|\phi\|_{L^2(0,T;L^2(D))} = (\int_0^T \|\phi(t)\|^2_{L^2(D)} dt)^{\frac{1}{2}} < \infty$;

The space $L^2(0, T; H^2(D))$ with the norm $\|\phi\|_{L^2(0,T;H^2(D))} = (\int_0^T \|\phi(t)\|^2_{H^2(D)} dt)^{\frac{1}{2}} < \infty$;

The space $L^\infty(0, T; H^1(D))$ with the norm $\|\phi\|_{L^\infty(0,T;H^1(D))} = \text{esssup}_{t\in[0,T]}\|\phi(t)\|_{H^1(D)} < \infty$.

Lastly, we define by $W^{1,2}(0, T; \mathbb{B})$ the space of all absolutely continuous functions $\phi : [0, T] \to \mathbb{B}$ satisfying the property that the derivative $\frac{d\phi}{dt} \in L^2(0, T; \mathbb{B})$. Particularly, we have the space $W^{1,2}(0, T; L^2(D))$ of all absolutely continuous functions $\phi : [0, T] \to L^2(D)$ satisfying the derivative $\frac{d\phi}{dt} \in L^2(0, T; L^2(D))$. Furthermore, defined by $C(0, T; L^2(D))$ the Banach space of all continuous functions $\phi : [0, T] \to L^2(D)$ with the supremum norm $\|\phi\|_{C(0,T;L^2(D))} = \sup_{t\in[0,T]} \|\phi(t)\|_{L^2(D)}$.

We define the linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq \mathbb{Y} \to \mathbb{Y}$ as follows

$$\mathcal{L} = \begin{pmatrix} d_1\Delta & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 \\ 0 & 0 & d_3\Delta & 0 \\ 0 & 0 & 0 & d_4\Delta \end{pmatrix},$$

where $\mathbb{Y} = (L^2(\Omega))^4$ and $D(\mathcal{L})$ is the domain of definition of $\mathcal{L}$, which is expressed as

$$D(\mathcal{L}) = \left\{(S, I, R, C) \in \mathbb{Y} : \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = \frac{\partial C}{\partial n} = 0 \text{ } on \text{ } \partial\Omega\right\}.$$

Further, we define by $P = (S, I, R, C)$ and $F(t, P) = (F_1(t, P), F_2(t, P), F_3(t, P), F_4(t, P))$ with

$$\begin{cases} F_1(t,P) = A(x) - \beta_1(x)S(t,x)I(t,x) + \gamma(x)C(t,x) - \mu(x)S(t,x) - \nu_1(t,x)S(t,x), \\ F_2(t,P) = \beta_1(x)S(t,x)I(t,x) + \sigma(x)\beta_2(x)C(t,x)I(t,x) - (\mu(x) + \alpha(x))I(t,x) \\ \qquad\qquad - \nu_2(t,x)I(t,x), \\ F_3(t,P) = (1 - \sigma(x))\beta_2(x)C(t,x)I(t,x) + \alpha(x)I(t,x) - (\mu(x) + \delta(x))R(t,x) \\ \qquad\qquad + \nu_1(t,x)S(t,x) + \nu_2(t,x)I(t,x), \\ F_4(t,P) = \delta(x)R(t,x) - \beta_2(x)C(t,x)I(t,x) - (\mu(x) + \gamma(x))C(t,x). \end{cases}$$

Then, system (1.3)–(1.5) can be expressed as the Cauchy problem

$$\begin{cases} \dfrac{\partial P}{\partial t} = \mathcal{L}P + F(t,P), \quad t \in [0,T], \\ P(0,x) = \psi(x), \end{cases} \tag{2.1}$$

where $\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \in \mathbb{X}^+$. In order to establish the existence and uniqueness of the strong solution to the system (2.1), we introduce the following well-known results (Chap. 11 in [24] and Proposition 1.2 in [25], see also [26]) as a lemma.

Lemma 2.1 (*See* [24–26]) *Let $\mathcal{X}$ be a real Banach space and the operator $\mathcal{L} : D(\mathcal{L}) \subseteq \mathcal{X} \to \mathcal{X}$ be an infinitesimal generator of a C_0-semigroup of contractions $\{\widetilde{S}(t), t \geq 0\}$ on $\mathcal{X}$. Moreover, assume that $\Phi : [0,T] \times \mathcal{X} \to \mathcal{X}$ is a function, which is measurable in t and Lipschitz in $P \in \mathcal{X}$, uniformly for $t \in [0,T]$. Then, the Cauchy problem*

$$\begin{cases} \dfrac{\partial P}{\partial t} = \mathcal{L}P + \Phi(t,P), \ t \in [0,T], \\ P(0,x) = P_0(x) \in \mathcal{X} \end{cases}$$

admits a unique mild solution $P \in C([0,T]; \mathcal{X})$, which can be expressed by

$$P(t) = \widetilde{S}(t)P_0 + \int_0^t \widetilde{S}(t-s)\Phi(s, P(s))ds, \quad t \in [0,T].$$

Furthermore, if $\mathcal{X}$ is a Hilbert space and $\mathcal{L}$ is self-adjoint and dissipative on $\mathcal{X}$, then the mild solution is also a strong solution satisfying $P \in W^{1,2}(0,T; \mathcal{X}) \cap L^2(0,T; D(\mathcal{L}))$.

Throughout this paper, if not specifically stated, we always assume that the parameters $\mu(x)$, $\beta_1(x)$, $\beta_2(x)$, $\alpha(x)$, $\delta(x)$, $\sigma(x)$ and $\gamma(x)$ are positive continuous functions for $x \in \overline{\Omega}$.

3 Well-posedness of solutions

In this section, we focus on the well-posedness of solutions for system (1.3)–(1.5). We first obtain that operator $d_i\Delta - \rho_i(\cdot)$ ($i = 1,2,3,4$) subjects to Neumann boundary condition (1.4) generates a C_0-semigroup $T_i(t) : C(\overline{\Omega}) \to C(\overline{\Omega})$, where $\rho_1(x) = \mu(x)$, $\rho_2(x) = \mu(x) + \alpha(x)$, $\rho_3(x) = \mu(x) + \delta(x)$ and $\rho_4(x) = \mu(x) + \gamma(x)$, respectively. Then, we have the expression

$$(T_i(t)\psi)(x) = \int_\Omega \Gamma_i(t,x,y)\psi(y)dy, \quad t > 0, \ \psi \in C(\overline{\Omega}), \ i = 1,2,3,4, \tag{3.1}$$

where $\Gamma_i(t,x,y)$ is the Green function of operator $d_i\Delta - \rho_i(\cdot)$ subjects to the Neumann boundary condition. Furthermore, we can obtain that $T_i(t)$ ($i = 1,2,3,4$) are strongly positive and compact for every $t > 0$, and there exist constants $N_i > 0$ ($i = 1,2,3,4$) such that $\|T_i(t)\| \le N_i e^{w_i t}$ for $t \ge 0$, where $w_i < 0$ is the principal eigenvalue of operator $d_i\Delta - \rho_i(\cdot)$ subjects to the Neumann boundary condition.

Define $F = (F_1, F_2, F_3, F_4)$ as follows

$$\begin{cases} F_1(\psi)(x) = A(x) - \beta_1(x)\psi_1(x)\psi_2(x) + \gamma(x)\psi_4(x) - \nu_1(t,x)\psi_1(x), \\ F_2(\psi)(x) = \beta_1(x)\psi_1(x)\psi_2(x) + \sigma(x)\beta_2(x)\psi_4(x)\psi_2(x) - \nu_2(t,x)\psi_2(x), \\ F_3(\psi)(x) = (1-\sigma(x))\beta_2(x)\psi_4(x)\psi_2(x) + \alpha(x)\psi_2(x) + \nu_1(t,x)\psi_1(x) + \nu_2(t,x)\psi_2(x), \\ F_4(\psi)(x) = \delta(x)\psi_3(x) - \beta_2(x)\psi_4(x)\psi_2(x), \end{cases} \tag{3.2}$$

where $x \in \Omega$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$, where $\mathbb{X}^+ = (C_+(\overline{\Omega}))^4$. Let $P(t,\cdot,\psi) = (S(t,\cdot,\psi), I(t,\cdot,\psi), R(t,\cdot,\psi), C(t,\cdot,\psi))$ be the solution of system (1.3) with initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$ at start time $t = 0$. Then, system (1.3)–(1.5) can be rewritten as the following integral equations

$$\begin{cases} S(t,\cdot,\psi) = T_1(t)\psi_1 + \int_0^t T_1(t-s)F_1(P(s,\cdot,\psi))ds, \\ I(t,\cdot,\psi) = T_2(t)\psi_2 + \int_0^t T_2(t-s)F_2(P(s,\cdot,\psi))ds, \\ R(t,\cdot,\psi) = T_3(t)\psi_3 + \int_0^t T_3(t-s)F_3(P(s,\cdot,\psi))ds, \\ C(t,\cdot,\psi) = T_4(t)\psi_4 + \int_0^t T_4(t-s)F_4(P(s,\cdot,\psi))ds. \end{cases} \tag{3.3}$$

With the use of basic calculation, we can prove that the following subtangential conditions hold

$$\lim_{h \to 0^+} \operatorname{dist}(\psi + hF(\psi), \mathbb{X}^+) = 0, \quad \psi \in \mathbb{X}^+.$$

From Corollary 4 in [27], we can directly obtain

Lemma 3.1 *For any initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$, system (1.3)–(1.5) has a unique nonnegative mild solution $P(t,\cdot,\psi) = (S(t,\cdot,\psi), I(t,\cdot,\psi), R(t,\cdot,\psi), C(t,\cdot,\psi)) \in \mathbb{X}^+$ on the existence interval $[0,\tau_\psi)$ with $\tau_\psi \leq \infty$. Furthermore, this solution also is a classical solution.*

Next, the following results are established regarding the existence and ultimate boundedness of the global solution for system (1.3)–(1.5).

Theorem 3.1 *For any initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$, model (1.3)–(1.5) has a unique nonnegative solution $P(t,\cdot,\psi) = (S(t,\cdot,\psi), I(t,\cdot,\psi), R(t,\cdot,\psi), C(t,\cdot,\psi))$ defined on $[0,\infty) \times \overline{\Omega}$ and is ultimately bounded.*

Proof According to Lemma 3.1, system (1.3) has a unique classical solution $P(t,x) = (S(t,x), I(t,x), R(t,x), C(t,x))$ with the initial value $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$ defined for $t \in [0,\tau_\psi)$ and $x \in \overline{\Omega}$, where $\tau_\psi \leq +\infty$. Then, we prove that the global existence of the solution, i.e., $\tau_\psi = +\infty$. On the contrary, suppose that $\tau_\psi < \infty$, then we have $\|P(t,x)\|_{\mathbb{X}} \to \infty$ as $t \to \tau_\psi$ from Theorem 2 in [27]. Hence, it suffices to prove that the solution is bounded in $[0,\tau_\psi) \times \Omega$. To this end, we define

$$W(t) = \int_\Omega (S(t,x) + I(t,x) + R(t,x) + C(t,x))dx.$$

Calculating the derivative of $W(t)$, and using the Divergence theorem [28] and the homogeneous Neumann boundary conditions, we can obtain

$$\begin{aligned}\frac{dW(t)}{dt} &= \int_\Omega \big(A(x) - \mu(x)S - \mu(x)I - \mu(x)R - \mu(x)C\big)dx \\ &\leq \int_\Omega \big(A^* - \mu_*(S + I + R + C)\big)dx \\ &\leq A^*|\Omega| - \mu_* W(t),\end{aligned}$$

where $|\Omega|$ is the measure of Ω. The comparison principle implies

$$W(t) \leq W(0)e^{-\mu_* t} + \frac{A^*|\Omega|}{\mu_*}(1 - e^{-\mu_* t}) \tag{3.4}$$

for all $t \in [0,\tau_\psi)$. This shows that $W(t)$ is bounded on $[0,\tau_\psi)$. Consequently, there is a constant $H_0 > 0$ such that

$$\int_\Omega S(t,x)dx \leq H_0, \int_\Omega I(t,x)dx \leq H_0, \int_\Omega R(t,x)dx \leq H_0, \int_\Omega C(t,x)dx \leq H_0, \tag{3.5}$$

for all $t \in [0,\tau_\psi)$.

It is clear from [29] that

$$\Gamma_i(t,x,y) = \sum_{n\geq 1} e^{\pi_n^i t}\varphi_n^i(x)\varphi_n^i(y), \quad t > 0,\ x,y \in \overline{\Omega},$$

where π_n^i ($n = 1,2,\ldots, i = 1,2,3,4$) are the eigenvalue of $d_1\Delta - \mu(x)$, $d_2\Delta - (\mu(x) + \alpha(x))$, $d_3\Delta - (\mu(x) + \delta(x))$, $d_4\Delta - (\mu(x) + \gamma(x))$ subjects to the Neumann boundary condition corresponding to the eigenfunction $\varphi_n^i(x)$, and satisfies $\pi_1^i > \pi_2^i > \pi_3^i > \cdots > \pi_n^i > \cdots$, respectively. Since $\{\varphi_n^i(x)\}$ is uniformly bounded on $\overline{\Omega}$, there exists a constant $k_1 > 0$ such that

$$\Gamma_i(t,x,y) \le k_1 \sum_{n\ge 1} e^{\pi_n^i t}, \quad t > 0,\ x,y \in \overline{\Omega}.$$

Furthermore, let τ_n^i $(n = 1,2,\ldots)$ be the eigenvalues of $d_1\Delta - \mu_*$, $d_2\Delta - (\mu_* + \alpha_*)$, $d_3\Delta - (\mu_* + \delta_*)$, $d_4\Delta - (\mu_* + \gamma_*)$ subjects to the Neumann boundary condition satisfying $\tau_1^i > \tau_2^i > \tau_3^i > \cdots > \tau_n^i > \cdots$. We have $\tau_1^1 = -\mu_*$, $\tau_1^2 = -(\mu_* + \alpha_*)$, $\tau_1^3 = -(\mu_* + \delta_*)$, $\tau_1^4 = -(\mu_* + \gamma_*)$, and by Theorem 2.4.7 in [30], $\pi_j^i \le \tau_j^i$ for all $j \in \mathbb{N}_+$. Due to the fact that τ_j^i decreases like $-i^2$, then there is a constant $k_2 > 0$ such that

$$\Gamma_i(t,x,y) \le k_1 \sum_{n\ge 1} e^{\tau_n^i t} \le k_2 e^{\tau_1^i t}, \quad t > 0,\ x,y \in \overline{\Omega}.$$

From (3.1)–(3.3) and (3.5), for any $t \in [0, \tau_\psi)$ and $x \in \Omega$, we have that

$$\begin{aligned}
S(t,x) =& T_1(t)\psi_1(x) + \int_0^t T_1(t-s)[A(x) - \beta_1(x)S(s,x)I(s,x) + \gamma(x)C(s,x) \\
&- \nu_1(s,x)S(s,x)]ds \\
=& T_1(t)\psi_1(x) + \int_0^t \int_\Omega \Gamma_1(t-s,x,y)[A(y) - \beta_1(y)S(s,y)I(s,y) \\
&+ \gamma(y)C(s,y) - \nu_1(s,y)S(s,y)]dyds \\
\le& N_1 e^{w_1 t}\|\psi_1\|_{\mathbb{X}} + k_2 \int_0^t e^{-\mu_*(t-s)} \int_\Omega (A^* + \gamma^* C(s,y))dyds \\
=& N_1 e^{w_1 t}\|\psi_1\|_{\mathbb{X}} + k_2(A^* + \gamma^* H_0)\frac{1 - e^{-\mu_* t}}{\mu_*} \\
\le& N_1\|\psi_1\|_{\mathbb{X}} + \frac{k_2(A^* + \gamma^* H_0)}{\mu_*} := H_1.
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
C(t,x) =& T_4(t)\psi_4(x) + \int_0^t T_4(t-s)[\delta(x)R(s,x) - \beta_2(x)C(s,x)I(s,x)]ds \\
=& T_4(t)\psi_4(x) + \int_0^t \int_\Omega \Gamma_4(t-s,x,y)[\delta(y)R(s,y) - \beta_2(y)C(s,y)I(s,y)]dyds \\
\le& N_4 e^{w_4 t}\|\psi_4\|_{\mathbb{X}} + k_2 \int_0^t e^{-(\mu_* + \gamma_*)(t-s)}\delta^* \int_\Omega R(s,y)dyds \\
=& N_4 e^{w_4 t}\|\psi_4\|_{\mathbb{X}} + k_2\delta^* H_0 \frac{1 - e^{-(\mu_* + \gamma_*)t}}{\mu_* + \gamma_*} \\
\le& N_4\|\psi_4\|_{\mathbb{X}} + \frac{k_2\delta^* H_0}{\mu_* + \gamma_*} := H_2.
\end{aligned} \tag{3.7}$$

From (3.1)–(3.3) and (3.5)–(3.7), for any $t \in [0, \tau_\psi)$ and $x \in \Omega$, we obtain

$$\begin{aligned} I(t,x) =& T_2(t)\psi_2(x) + \int_0^t T_2(t-s)[\beta_1(x)S(s,x)I(s,x) + \sigma(x)\beta_2(x)C(s,x)I(s,x) \\ & - \nu_2(s,x)I(s,x)]ds \\ =& T_2(t)\psi_2(x) + \int_0^t \int_\Omega \Gamma_2(t-s,x,y)[\beta_1(y)S(s,y)I(s,y) \\ & + \sigma(y)\beta_2(y)C(s,y)I(s,y) - \nu_2(s,y)I(s,y)]dyds \\ \leq& N_2 e^{w_2 t}\|\psi_2\|_{\mathbb{X}} + k_2 \int_0^t e^{-(\mu_*+\alpha_*)(t-s)}(\beta_1^* H_1 + \sigma^* \beta_2^* H_2) \int_\Omega I(s,y)dyds \\ =& N_2 e^{w_2 t}\|\psi_2\|_{\mathbb{X}} + k_2(\beta_1^* H_1 + \sigma^* \beta_2^* H_2)H_0 \frac{1-e^{-(\mu_*+\alpha_*)t}}{\mu_*+\alpha_*} \\ \leq& N_2\|\psi_2\|_{\mathbb{X}} + \frac{k_2(\beta_1^* H_1 + \sigma^* \beta_2^* H_2)H_0}{\mu_*+\alpha_*} := H_3. \end{aligned} \tag{3.8}$$

From (3.1)–(3.3) and (3.5)–(3.8), for any $t \in [0, \tau_\psi)$ and $x \in \Omega$, we have

$$\begin{aligned} R(t,x) =& T_3(t)\psi_3(x) + \int_0^t T_3(t-s)[(1-\sigma(x))\beta_2(x)C(s,x)I(s,x) + \alpha(x)I(s,x) \\ & + \nu_1(s,x)S(s,x) + \nu_2(s,x)I(s,x)]ds \\ =& T_3(t)\psi_3(x) \\ & + \int_0^t \int_\Omega \Gamma_3(t-s,x,y)[(1-\sigma(y))\beta_2(y)C(s,y)I(s,y) + \alpha(y)I(s,y) \\ & + \nu_1(s,y)S(s,y) + \nu_2(s,y)I(s,y)]dyds \\ \leq& N_3 e^{w_3 t}\|\psi_3\|_{\mathbb{X}} + k_2 \int_0^t e^{-(\mu_*+\delta_*)(t-s)}(\beta_2^* H_2 H_3 + \alpha^* H_3 + H_1 + H_3)ds \\ =& N_3 e^{w_3 t}\|\psi_3\|_{\mathbb{X}} + k_2(\beta_2^* H_2 H_3 + \alpha^* H_3 + H_1 + H_3)\frac{1-e^{-(\mu_*+\delta_*)t}}{\mu_*+\delta_*} \\ \leq& N_3\|\psi_3\|_{\mathbb{X}} + \frac{k_2(\beta_2^* H_2 H_3 + \alpha^* H_3 + H_1 + H_3)}{\mu_*+\delta_*} := H_4. \end{aligned} \tag{3.9}$$

The above discussions show that the solution $(S(t,x), I(t,x), R(t,x), C(t,x))$ of system (1.3)-(1.5) is bounded for $t \in [0, \tau_\psi)$ and $x \in \overline{\Omega}$. This leads to a contradiction. Therefore, we have $\tau_\psi = \infty$. That is, solution $(S(t,x), I(t,x), R(t,x), C(t,x))$ is defined for all $t \geq 0$ and $x \in \overline{\Omega}$.

Now, we prove that the solution also is ultimately bounded. In fact, from (3.4) we can obtain $\limsup_{t\to\infty} W(t) \leq \frac{A^*|\Omega|}{\mu_*}$. Hence, without loss of generality, we can choose $H_0 = \frac{A^*|\Omega|}{\mu_*}$ and a $T_0 > 0$ such that (3.5) holds for all $t \geq T_0$. Then, similar to (3.6) we can obtain

$$\begin{aligned} S(t,x) =& T_1(t)S(T_0,x) + \int_{T_0}^t T_1(t-s)[A(x) - \beta_1(x)S(s,x)I(s,x) + \gamma(x)C(s,x) \\ & - \nu_1(s,x)S(s,x)]ds \\ \leq& N_1 e^{w_1(t-T_0)}\|S(T_0,x)\|_{\mathbb{X}} + \frac{k_2(A^* + \gamma^* H_0)}{\mu_*}. \end{aligned}$$

Therefore, we further have $\limsup_{t\to\infty} S(t,x) \le \frac{k_2(A^*+\gamma^* H_0)}{\mu_*} := H_1^*$ uniformly for $x \in \overline{\Omega}$.

Using the same method, similar to (3.7)–(3.9), we can obtain

$$\limsup_{t\to\infty} C(t,x) \le \frac{k_2\delta^* H_0}{\mu_* + \gamma_*} := H_2^*,$$
$$\limsup_{t\to\infty} I(t,x) \le \frac{k_2(\beta_1^* H_1^* + \sigma^*\beta_2^* H_2^*)H_0}{\mu_* + \alpha_*} := H_3^*,$$
$$\limsup_{t\to\infty} R(t,x) \le \frac{k_2(\beta_2^* H_2^* H_3^* + \alpha^* H_3^* + H_1^* + H_3^*)}{\mu_* + \delta_*} := H_4^*$$

uniformly for $x \in \overline{\Omega}$. This implies that the solution $(S(t,x), I(t,x), R(t,x), C(t,x))$ is ultimately bounded. This completes the proof. □

Theorem 3.2 *For given $v = (v_1, v_2) \in \mathcal{U}$, $\mathcal{U}$ is defined as in (1.6), control system (1.3)–(1.5) admits a unique positive strong solution $P = (S, I, R, C)$ satisfying*

$$S \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{3.10}$$
$$I \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{3.11}$$
$$R \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{3.12}$$
$$C \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). \tag{3.13}$$

Furthermore, there exists a positive constant C_1, independent of $v \in \mathcal{U}$ and P, such that

$$\|\frac{\partial S}{\partial t}\|_{L^2(\Omega_T)} + \|S\|_{L^2(0,T;H^2(\Omega))} + \|S\|_{L^\infty(0,T;H^1(\Omega))} + \|S\|_{L^\infty(\Omega_T)} \le C_1, \tag{3.14}$$
$$\|\frac{\partial I}{\partial t}\|_{L^2(\Omega_T)} + \|I\|_{L^2(0,T;H^2(\Omega))} + \|I\|_{L^\infty(0,T;H^1(\Omega))} + \|I\|_{L^\infty(\Omega_T)} \le C_1, \tag{3.15}$$
$$\|\frac{\partial R}{\partial t}\|_{L^2(\Omega_T)} + \|R\|_{L^2(0,T;H^2(\Omega))} + \|R\|_{L^\infty(0,T;H^1(\Omega))} + \|R\|_{L^\infty(\Omega_T)} \le C_1, \tag{3.16}$$
$$\|\frac{\partial C}{\partial t}\|_{L^2(\Omega_T)} + \|C\|_{L^2(0,T;H^2(\Omega))} + \|C\|_{L^\infty(0,T;H^1(\Omega))} + \|C\|_{L^\infty(\Omega_T)} \le C_1. \tag{3.17}$$

Proof To prove this theorem we will use the method introduced in [31]. From Theorem 3.1, we know that system (1.3)–(1.5) has a unique nonnegative classical solution $P(t,x) = (S(t,x), I(t,x), R(t,x), C(t,x))$ defined on $[0,\infty) \times \Omega$ which is also ultimately bounded. Therefore, there is a constant $M_0 > 0$ such that $0 \le S(t,x) \le M_0$, $0 \le I(t,x) \le M_0$, $0 \le R(t,x) \le M_0$ and $0 \le C(t,x) \le M_0$ for all $(t,x) \in [0,\infty) \times \Omega$. It is clear that we only need to prove that the solution $P(t,x) = (S(t,x), I(t,x), R(t,x), C(t,x))$ is a positive strong solution and satisfies (3.10)–(3.13) and (3.14)–(3.17).

For the positive constant M_0, we define the truncation form of a function $G(t,x)$ defined for $(t,x) \in \Omega_T$ as follows

$$G^{M_0}(t,x) = \begin{cases} M_0, & \text{if } (t,x) \in \{(t,x) \in \Omega_T : G(t,x) > M_0\}, \\ G(t,x), & \text{if } (t,x) \in \{(t,x) \in \Omega_T : |G(t,x)| \le M_0\}, \\ -M_0, & \text{if } (t,x) \in \{(t,x) \in \Omega_T : G(t,x) < -M_0\}. \end{cases}$$

Consider the following truncated Cauchy problem

$$\begin{cases} \dfrac{\partial P}{\partial t} = \mathcal{L}P + F^{M_0}(t,P),\ t \in [0,T], \\ P(0,x) = \psi(x), \end{cases} \tag{3.18}$$

where $P = (S,I,R,C)$, $\psi(x) = (\psi_1(x),\psi_2(x),\psi_3(x),\psi_4(x))$, and $F^{M_0}(t,P)$ is the truncation form of $F(t,P)$ and it is defined as

$$F^{M_0}(t,P) := (F_1^{M_0}(t,P),F_2^{M_0}(t,P),F_3^{M_0}(t,P),F_4^{M_0}(t,P)) \tag{3.19}$$

with

$$\begin{cases} F_1^{M_0}(t,P) = A(x) - \beta_1(x)S^{M_0}(t,x)I^{M_0}(t,x) + \gamma(x)C^{M_0}(t,x) \\ \qquad - \mu(x)S^{M_0}(t,x) - \nu_1(t,x)S^{M_0}(t,x), \\ F_2^{M_0}(t,P) = \beta_1(x)S^{M_0}(t,x)I^{M_0}(t,x) + \sigma(x)\beta_2(x)C^{M_0}(t,x)I^{M_0}(t,x) \\ \qquad - (\mu(x)+\alpha(x))I^{M_0}(t,x) - \nu_2(t,x)I^{M_0}(t,x), \\ F_3^{M_0}(t,P) = (1-\sigma(x))\beta_2(x)C^{M_0}(t,x)I^{M_0}(t,x) + \alpha(x)I^{M_0}(t,x) \\ \qquad - (\mu(x)+\delta(x))R^{M_0}(t,x) + \nu_1(t,x)S^{M_0}(t,x) + \nu_2(t,x)I^{M_0}(t,x), \\ F_4^{M_0}(t,P) = \delta(x)R^{M_0}(t,x) - \beta_2(x)C^{M_0}(t,x)I^{M_0}(t,x) - (\mu(x)+\gamma(x))C^{M_0}(t,x), \end{cases} \tag{3.20}$$

Obviously, $P(t,x) = (S(t,x),I(t,x),R(t,x),C(t,x))$ is also the solution of truncated Cauchy problem (3.18). It is easy to see that the function $F^{M_0}(t,P)$ defined in (3.19) is global Lipschitz continuous for P uniformly with respect to $t \in [0,T]$. Therefore, using Lemma 2.1 to system (1.3)–(1.5), we can obtain that the solution $P = (S,I,R,C) \in (W^{1,2}(0,T;L^2(\Omega)))^4 \cap (L^2(0,T;H^2(\Omega)))^4$ and is also a strong solution.

We further show that the solution $P = (S,I,R,C) \in (L^\infty(\Omega_T))^4$. It is clear that

$$\begin{aligned} S(t,x) &= \widetilde{T}_1(t)\psi_1(x) + \int_0^t \widetilde{T}_1(t-s)F_1^{M_0}(s,P)ds, \\ I(t,x) &= \widetilde{T}_2(t)\psi_2(x) + \int_0^t \widetilde{T}_2(t-s)F_2^{M_0}(s,P)ds, \\ R(t,x) &= \widetilde{T}_3(t)\psi_3(x) + \int_0^t \widetilde{T}_3(t-s)F_3^{M_0}(s,P)ds, \\ C(t,x) &= \widetilde{T}_4(t)\psi_4(x) + \int_0^t \widetilde{T}_4(t-s)F_4^{M_0}(s,P)ds, \end{aligned} \tag{3.21}$$

where $\{\widetilde{T}_i(t), t \geq 0\}$ $(i = 1,2,3,4)$ is the C_0-semigroup of contractions generated by the operator $\mathcal{L}_i : D(\mathcal{L}_i) \subseteq L^2(\Omega) \to L^2(\Omega)$ defined as follows

$$\mathcal{L}_i U = d_i \Delta U,\ \ D(\mathcal{L}_i) = \{U : U \in H^2(\Omega), \frac{\partial U}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

From the expressions of $F_i^{M_0}$ $(i = 1,2,3,4)$ defined in (3.20) and the control function $v = (v_1, v_2) \in U$, we can choose a constant H^{M_0}, which is independent of v such that

$$H^{M_0} \geq \max\{\|F_i^{M_0}(t,P)\|_{L^\infty(\Omega_T)},\ \|\psi_i(x)\|_{L^\infty(\Omega)},\ i = 1,2,3,4\}. \tag{3.22}$$

We introduce the following auxiliary Cauchy problem

$$\begin{cases} \dfrac{\partial \overline{S}(t,x)}{\partial t} = d_1 \Delta \overline{S}(t,x) + F_1^{M_0}(t, \overline{P}(t,x)) - H^{M_0}, & (t,x) \in \Omega_T, \\ \overline{S}(0,x) = \psi_1(x) - \|\psi_1(x)\|_{L^\infty(\Omega)}, & x \in \Omega, \end{cases} \tag{3.23}$$

where $\overline{P}(t,x) = (\overline{S}(t,x), \overline{I}(t,x), \overline{R}(t,x), \overline{C}(t,x))$. By virtue of Theorem 2.1 and (3.21), the problem (3.23) has a unique strong solution that can be expressed as

$$\overline{S}(t,x) = \widetilde{T}_1(t)(\psi_1(x) - \|\psi_1(x)\|_{L^\infty(\Omega)}) + \int_0^t \widetilde{T}_1(t-s)(F_1^{M_0}(s,\overline{P}) - H^{M_0})ds. \tag{3.24}$$

Let $Y_1(t,x) = \overline{S}(t,x) + \|\psi_1(x)\|_{L^\infty(\Omega)}$, then problem (3.23) can be rewritten as follows

$$\begin{cases} \dfrac{\partial Y_1(t,x)}{\partial t} = d_1 \Delta Y_1(t,x) + F_1^{M_0}(t, \overline{P}(t,x)) - H^{M_0}, & in\ \Omega_T, \\ Y_1(0,x) = \psi_1(x), & in\ \Omega. \end{cases} \tag{3.25}$$

By virtue of Lemma 2.1, the problem (3.25) has a unique strong solution that can be expressed as

$$Y_1(t,x) = \widetilde{T}_1(t)\psi_1(x) + \int_0^t \widetilde{T}_1(t-s)(F_1^{M_0}(s,\overline{P}) - H^{M_0})ds.$$

Namely,

$$\overline{S}(t,x) + \|\psi_1(x)\|_{L^\infty(\Omega)} = \widetilde{T}_1(t)\psi_1(x) + \int_0^t \widetilde{T}_1(t-s)(F_1^{M_0}(s,\overline{P}) - H^{M_0})ds. \tag{3.26}$$

Moreover, it follows from the solution of (3.24) and (3.26) is equivalent to

$$\overline{S}(t,x) = S(t,x) - \|\psi_1(x)\|_{L^\infty(\Omega)} - \int_0^t \widetilde{T}_1(t-s)H^{M_0}ds, \quad (t,x) \in \Omega_T. \tag{3.27}$$

We further consider another initial value problem

$$\begin{cases} \dfrac{\partial Z_1(t,x)}{\partial t} = d_1 \Delta Z_1(t,x), & (t,x) \in \Omega_T, \\ Z_1(0,x) = 0, & x \in \Omega. \end{cases} \tag{3.28}$$

It is obvious that $Z_1(t,x) = 0$ is the unique solution to problem (3.28). Meanwhile, by the definition of H^{M_0} in (3.22), we can see $\psi_1(x) - \|\psi_1(x)\|_{L^\infty(\Omega)} \leq 0$ and $F_1^{M_0}(t,P) - H^{M_0} \leq 0$.

Then the comparison principle of parabolic equation yields that $\overline{S}(t,x) \leq Z_1(t,x) = 0$ for all $(t,x) \in \Omega_T$, Further, from (3.27), we have

$$S(t,x) \leq \|\psi_1(x)\|_{L^\infty(\Omega)} + \int_0^t \widetilde{T}_1(t-s)H^{M_0}ds, \ \ (t,x) \in \Omega_T.$$

This implies that $S \in L^\infty(\Omega_T)$, and there is a constant $C_{11} > 0$, independent of $v \in \mathcal{U}$ and P, such that $\|S\|_{L^\infty(\Omega_T)} \leq C_{11}$. We can employ similar methods to prove that $I \in L^\infty(\Omega_T)$, $R \in L^\infty(\Omega_T)$ and $C \in L^\infty(\Omega_T)$, and there are the constants $C_{1i} > 0$ $(i = 2,3,4)$, independent of $v \in \mathcal{U}$ and P, such that $\|I\|_{L^\infty(\Omega_T)} \leq C_{12}$, $\|R\|_{L^\infty(\Omega_T)} \leq C_{13}$ and $\|C\|_{L^\infty(\Omega_T)} \leq C_{14}$.

Now, we further show that the solution $P = (S,I,R,C) \in (L^\infty(0,T;H^1(\Omega)))^4$. From the first equation of system (3.18), we can get

$$\begin{aligned} &\int_0^t \int_\Omega (\frac{\partial S(s,x)}{\partial s})^2 dxds - 2d_1 \int_0^t \int_\Omega \frac{\partial S(s,x)}{\partial s} \Delta S(s,x) dxds \\ &+ d_1^2 \int_0^t \int_\Omega (\Delta S(s,x))^2 dxds = \int_0^t \int_\Omega (F_1^{M_0}(s,P(s,x)))^2 dxds. \end{aligned} \tag{3.29}$$

Notice that it is given by Green's formula that

$$-2d_1 \int_0^t \int_\Omega \frac{\partial S(s,x)}{\partial s} \Delta S(s,x) dxds = d_1 \int_\Omega |\nabla S(t,x)|^2 dx - d_1 \int_\Omega |\nabla \psi_1(x)|^2 dx. \tag{3.30}$$

Substituting (3.30) into (3.29), we can obtain

$$\begin{aligned} &\int_0^t \int_\Omega (\frac{\partial S(s,x)}{\partial s})^2 dxds + d_1 \int_\Omega |\nabla S(t,x)|^2 dx + d_1^2 \int_0^t \int_\Omega (\Delta S(s,x))^2 dxds \\ &= \int_0^t \int_\Omega (F_1^{M_0}(s,P(s,x)))^2 dxds + d_1 \int_\Omega |\nabla \psi_1(x)|^2 dx. \end{aligned} \tag{3.31}$$

Noting that $\psi_1(x) \in H^2(\Omega)$, $S \in L^\infty(\Omega_T) \cap L^2(0,T;H^2(\Omega))$ and (3.31), we can conclude $S \in L^\infty(0,T;H^1(\Omega))$, and there is a constant $C_{21} > 0$, independent of $v \in \mathcal{U}$ and P, such that

$$\|\frac{\partial S}{\partial t}\|_{L^2(\Omega_T)} + \|S\|_{L^\infty(0,T;H^1(\Omega))} \leq C_{21}.$$

Similarly, by using the same methods, we also can prove $I \in L^\infty(0,T;H^1(\Omega))$, $R \in L^\infty(0,T;H^1(\Omega))$ and $C \in L^\infty(0,T;H^1(\Omega))$. Meanwhile, there are constants $C_{2i} > 0$ $(i = 2,3,4)$, independent of $v \in \mathcal{U}$ and P, such that

$$\|\frac{\partial I}{\partial t}\|_{L^2(\Omega_T)} + \|I\|_{L^\infty(0,T;H^1(\Omega))} \leq C_{21},$$
$$\|\frac{\partial R}{\partial t}\|_{L^2(\Omega_T)} + \|R\|_{L^\infty(0,T;H^1(\Omega))} \leq C_{21}$$

and

$$\|\frac{\partial C}{\partial t}\|_{L^2(\Omega_T)} + \|C\|_{L^\infty(0,T;H^1(\Omega))} \leq C_{21}.$$

Furthermore, based on (3.31), we also can obtain that there is a constant $C_{31} > 0$, independent of $\nu \in \mathcal{U}$ and P, such that $\|S\|_{L^2(0,T;H^2(\Omega))} \le C_{31}$. Similarly, by using the same method, we further can obtain that there are constants $C_{3i} > 0$ ($i = 2,3,4$), independent of $\nu \in \mathcal{U}$ and P, such that $\|I\|_{L^2(0,T;H^2(\Omega))} \le C_{32}$, $\|R\|_{L^2(0,T;H^2(\Omega))} \le C_{33}$ and $\|C\|_{L^2(0,T;H^2(\Omega))} \le C_{34}$. Lastly, summarize the above discussions we finally obtain that the inequalities (3.14)–(3.17) hold.

We further prove the positivity of the solution $P = (S,I,R,C)$. That is, $S(t,x) > 0$, $I(t,x) > 0$, $R(t,x) > 0$ and $C(t,x) > 0$. We first prove that $S(t,x)$ is positive. To do this, we first define

$$S(t,x) = (S(t,x))^+ - (S(t,x))^-, \tag{3.32}$$

with $(S(t,x))^+ = \max\{S(t,x),0\}$, $(S(t,x))^- = -\min\{S(t,x),0\}$. Multiplying the first equation of (3.18) by $(S(t,x))^-$, we obtain that

$$\begin{aligned}\frac{\partial S(t,x)}{\partial t}(S(t,x))^- =& d_1\Delta S(t,x)(S(t,x))^- - \beta_1(x)S^{M_0}(t,x)I^{M_0}(t,x)(S(t,x))^- \\ &+ \gamma(x)C^{M_0}(t,x)(S(t,x))^- + A(x)(S(t,x))^- \\ &- \mu(x)S^{M_0}(t,x)(S(t,x))^- - \nu_1(t,x)S^{M_0}(t,x)(S(t,x))^-.\end{aligned} \tag{3.33}$$

By calculating, we can obtain that

$$S^{M_0}(t,x)(S(t,x))^- = \left\{\begin{array}{ll} M_0(S(t,x))^-, & \text{if } S(t,x) > M_0, \\ S(t,x)(S(t,x))^-, & \text{if } |S(t,x)| \le M_0, \\ -M_0(S(t,x))^-, & \text{if } S(t,x) < -M_0 \end{array}\right\} \le -((S(t,x))^-)^2.$$

In addition, we easily obtain that $\frac{\partial S(t,x)}{\partial t} = \frac{\partial (S(t,x))^+}{\partial t} - \frac{\partial (S(t,x))^-}{\partial t}$, $\Delta S(t,x) = \Delta(S(t,x))^+ - \Delta(S(t,x))^-$, $\frac{\partial (S(t,x))^+}{\partial t}(S(t,x))^- \equiv 0$ and $\Delta(S(t,x))^+(S(t,x))^- \equiv 0$. Thus, from (3.33), we further obtain

$$\begin{aligned}-\frac{1}{2}\frac{\partial}{\partial t}((S(t,x))^-)^2 \ge& -d_1(S(t,x))^-\Delta(S(t,x))^- \\ &+ ((S(t,x))^-)^2(\mu(x) + \beta_1(x)I^{M_0}(t,x) + \nu_1(t,x)).\end{aligned} \tag{3.34}$$

Integrating both sides of (3.34) over Ω, one can obtain

$$\begin{aligned}\int_\Omega \frac{\partial}{\partial t}((S(t,x))^-)^2dx \le& 2d_1\int_\Omega (S(t,x))^-\Delta(S(t,x))^-dx \\ &- 2\int_\Omega ((S(t,x))^-)^2(\beta_1(x)I^{M_0}(t,x) + \mu(x) + \nu_1(t,x))dx.\end{aligned} \tag{3.35}$$

Applying the Green formula, one obtains

$$\int_\Omega (S(t,x))^-\Delta(S(t,x))^-dx = -\int_\Omega |\nabla(S(t,x))^-|^2dx. \tag{3.36}$$

Therefore, we substitute (3.36) into (3.35) to get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}((S(t,x))^{-})^{2}dx \leq & -2d_{1}\int_{\Omega}|\nabla(S(t,x))^{-}|^{2}dx \\ & -2\int_{\Omega}((S(t,x))^{-})^{2}(\beta_{1}(x)I^{M_{0}}(t,x)+\mu(x)+\nu_{1}(t,x))dx.\end{aligned} \tag{3.37}$$

According to the boundedness of $I(t,x)$, we can obtain that

$$\beta_{1}(x)I^{M_{0}}(t,x)+\mu(x)+\nu_{1}(t,x)\in L^{\infty}(\Omega_{T}). \tag{3.38}$$

Combining (3.37) and (3.38), one acquires

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega}((S(t,x))^{-})^{2}dx \\ &\leq -2\|\beta_{1}(x)I^{M_{0}}(t,x)+\mu(x)+\nu_{1}(t,x)\|_{L^{\infty}(\Omega_{T})}\int_{\Omega}((S(t,x))^{-})^{2}dx.\end{aligned} \tag{3.39}$$

Set $\eta=\|\beta_{1}(x)I^{M_{0}}(t,x)+\mu(x)+\nu_{1}(t,x)\|_{L^{\infty}(\Omega_{T})}$, then integrating (3.39) from 0 to t for any $t\in[0,T]$, we gain

$$\int_{\Omega}((S(t,x))^{-})^{2}dx\leq -2\eta\int_{0}^{t}\int_{\Omega}((S(t,x))^{-})^{2}ds\leq 0. \tag{3.40}$$

From (3.40), we further obtain that $(S(t,x))^{-}\equiv 0$ in Ω_{T}, which implies by (3.32) that $S(t,x)\geq 0$ in Ω_{T}. And then, it follows from $\psi_{1}(x)>0$ that $S(t,x)>0$ for all $(t,x)\in\Omega_{T}$. Similarly, we also can show that $I(t,x)>0$, $R(t,x)>0$ and $C(t,x)>0$ in Ω_{T}. This completes the proof. □

4 The existence of optimal solution

This section is devoted to the existence of the optimal pair for the optimal control problem (1.3)–(1.5). We have the following result.

Theorem 4.1 *The control system* (1.3)–(1.5) *admits an optimal solution* $(\overline{S},\overline{I},\overline{R},\overline{C},\overline{\nu})$ *with* $\overline{\nu}=(\overline{\nu}_{1},\overline{\nu}_{2})\in\mathcal{U}$ *such that*

$$J(\overline{S},\overline{I},\overline{\nu})=\inf_{\nu=(\nu_{1},\nu_{2})\in\mathcal{U}}J(S,I,\nu).$$

Proof From Theorem 3.2, for any $\nu=(\nu_{1},\nu_{2})\in\mathcal{U}$ the control system (1.3)–(1.5) has a unique positive strong solution $P=(S,I,R,C)$. Hence, we know that $J(S,I,\nu_{1},\nu_{2})$ is bounded below on $\mathcal{U}$. Thus, there exists a constant $\varrho=\inf_{(\nu_{1},\nu_{2})\in\mathcal{U}}J(S,I,\nu_{1},\nu_{2})$ and a minimizing sequence $\{S^{n},I^{n},\nu_{1}^{n},\nu_{2}^{n}:n\geq 1\}$ such that

$$\lim_{n\to\infty}J(S^{n},I^{n},\nu_{1}^{n},\nu_{2}^{n})=\inf_{(\nu_{1},\nu_{2})\in\mathcal{U}}J(S,I,\nu_{1},\nu_{2})=\varrho, \tag{4.1}$$

where (S^n, I^n, R^n, C^n) satisfies the following system with (v_1^n, v_2^n) for each $n = 1, 2, \dots,$

$$\begin{cases} \dfrac{\partial S^n(t,x)}{\partial t} = d_1\Delta S^n(t,x) + A(x) - \beta_1(x)S^n(t,x)I^n(t,x) + \gamma(x)C^n(t,x) \\ \qquad\qquad - \mu(x)S^n(t,x) - v_1^n(t,x)S^n(t,x), \\ \dfrac{\partial I^n(t,x)}{\partial t} = d_2\Delta I^n(t,x) + \beta_1(x)S^n(t,x)I^n(t,x) + \sigma(x)\beta_2(x)C^n(t,x)I^n(t,x) \\ \qquad\qquad - (\mu(x) + \alpha(x))I^n(t,x) - v_2^n(t,x)I^n(t,x), \\ \dfrac{\partial R^n(t,x)}{\partial t} = d_3\Delta R^n(t,x) + (1-\sigma(x))\beta_2(x)C^n(t,x)I^n(t,x) + \alpha(x)I^n(t,x) \\ \qquad\qquad - (\mu(x) + \delta(x))R^n(t,x) + v_1^n(t,x)S^n(t,x) + v_2^n(t,x)I^n(t,x), \\ \dfrac{\partial C^n(t,x)}{\partial t} = d_4\Delta C^n(t,x) + \delta(x)R^n(t,x) - \beta_2(x)C^n(t,x)I^n(t,x) - (\mu(x) + \gamma(x))C^n(t,x), \\ \dfrac{\partial S^n(t,x)}{\partial n} = \dfrac{\partial I^n(t,x)}{\partial n} = \dfrac{\partial R^n(t,x)}{\partial n} = \dfrac{\partial C^n(t,x)}{\partial n} = 0, \ (t,x) \in \Sigma_T = (0,T) \times \partial\Omega, \\ S^n(0,x) = \psi_1(x), I^n(0,x) = \psi_2(x), R^n(0,x) = \psi_3(x), C^n(0,x) = \psi_4(x), \ \ x \in \Omega. \end{cases} \tag{4.2}$$

Furthermore, in (4.1), without loss of generality, we can assume that for $n = 1, 2, \dots,$

$$\varrho \le J(S^n, I^n, v_1^n, v_2^n) \le \varrho + \frac{1}{n}, \tag{4.3}$$

Firstly, since $\{v_1^n\}$ and $\{v_2^n\}$ are uniformly bounded in $L^2(\Omega_T)$, there exist functions $\overline{v}_1$ and $\overline{v}_2$ and subsequence of $\{v_1^n\}$ and $\{v_2^n\}$, still denoted by themselves, such that

$$v_1^n \to \overline{v}_1, \ \ v_2^n \to \overline{v}_2 \ \ \textit{weakly in} \ \ L^2(\Omega_T), \ \textit{as}\ n \to \infty. \tag{4.4}$$

Since $\mathcal{U}$ is a closed and convex set in $L^2(\Omega_T)$, it also is weakly closed. Hence, (4.4) implies that $\overline{v} = (\overline{v}_1, \overline{v}_2) \in \mathcal{U}$.

From Theorem 3.2, we further obtain that for each $n = 1, 2, \dots,$

$$S^n \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{4.5}$$

$$I^n \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{4.6}$$

$$R^n \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{4.7}$$

$$C^n \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). \tag{4.8}$$

Moreover, there exists a constant $C > 0$, independent of any n, such that

$$\|\frac{\partial S^n}{\partial t}\|_{L^2(\Omega_T)} + \|S^n\|_{L^2(0,T;H^2(\Omega))} + \|S^n\|_{L^\infty(0,T;H^1(\Omega))} + \|S^n\|_{L^\infty(\Omega_T)} \le C, \tag{4.9}$$

$$\|\frac{\partial I^n}{\partial t}\|_{L^2(\Omega_T)} + \|I^n\|_{L^2(0,T;H^2(\Omega))} + \|I^n\|_{L^\infty(0,T;H^1(\Omega))} + \|I^n\|_{L^\infty(\Omega_T)} \le C, \tag{4.10}$$

$$\|\frac{\partial R^n}{\partial t}\|_{L^2(\Omega_T)} + \|R^n\|_{L^2(0,T;H^2(\Omega))} + \|R^n\|_{L^\infty(0,T;H^1(\Omega))} + \|R^n\|_{L^\infty(\Omega_T)} \le C, \tag{4.11}$$

$$\|\frac{\partial C^n}{\partial t}\|_{L^2(\Omega_T)} + \|C^n\|_{L^2(0,T;H^2(\Omega))} + \|C^n\|_{L^\infty(0,T;H^1(\Omega))} + \|C^n\|_{L^\infty(\Omega_T)} \le C. \tag{4.12}$$

It can be seen from (4.5)–(4.8) that $S^n, I^n, R^n, C^n \in W^{1,2}(0,T;L^2(\Omega))$, which implies that $S^n, I^n, R^n, C^n \in C(0,T;L^2(\Omega))$. Since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ (see [32]), by (4.9)–(4.12) we can deduce that the sequence $\{(S^n, I^n, R^n, C^n)\}$ is relatively compact in $(L^2(\Omega))^4$. Furthermore, from (4.9)–(4.12), we can deduce that $\{(S^n, I^n, R^n, C^n)\}$ is also uniformly bounded in $(L^2(\Omega))^4$, and the sequence $\{(\frac{\partial S^n}{\partial t}, \frac{\partial I^n}{\partial t}, \frac{\partial R^n}{\partial t}, \frac{\partial C^n}{\partial t})\}$ is bounded in $(L^2(\Omega_T))^4$. This implies that $\{(S^n, I^n, R^n, C^n)\}$ is equicontinuous in $(C(0,T;L^2(\Omega)))^4$. Thus, the Ascoliz–Arzela theorem implies that the sequence $\{(S^n, I^n, R^n, C^n)\}$ is also compact in $(C(0,T;L^2(\Omega)))^4$. Hence, there exists a $(\overline{S}, \overline{I}, \overline{R}, \overline{C}) \in (C(0,T;L^2(\Omega)))^4$ and a subsequence of $\{(S^n, I^n, R^n, C^n)\}$, still denoted by itself, such that

$$\begin{cases} \lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|S^n(t) - \overline{S}(t)\|_{L^2(\Omega)} = 0, \ \lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|I^n(t) - \overline{I}(t)\|_{L^2(\Omega)} = 0, \\ \lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|R^n(t) - \overline{R}(t)\|_{L^2(\Omega)} = 0, \ \lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|C^n(t) - \overline{C}(t)\|_{L^2(\Omega)} = 0. \end{cases}$$

That is,

$$(S^n, I^n, R^n, C^n) \to (\overline{S}, \overline{I}, \overline{R}, \overline{C}) \ \textit{in} \ (L^2(\Omega))^4 \ \textit{uniformly for} \ t \in [0,T]. \tag{4.13}$$

Next, we further prove that $(\overline{S}, \overline{I}, \overline{R}, \overline{C}, \overline{v}_1, \overline{v}_2)$ is an optimal pair of the control system (1.3)–(1.5). From the first equation of (4.2), we have

$$\begin{aligned} d_1 \Delta S^n(t,x) = {} & \frac{\partial S^n(t,x)}{\partial t} - (A(x) - \beta_1(x) S^n(t,x) I^n(t,x) \\ & + \gamma(x) C^n(t,x) - \mu(x) S^n(t,x) - v_1^n(t,x) S^n(t,x)). \end{aligned} \tag{4.14}$$

According to (4.9)–(4.12), from (4.14) we easily obtain that the sequence $\{\Delta S^n\}$ is bounded in $L^2(\Omega_T)$. Similarly, we can get that the sequences $\{\Delta I^n\}$, $\{\Delta R^n\}$ and $\{\Delta C^n\}$ also are bounded in $L^2(\Omega_T)$. Namely, $\{(\Delta S^n, \Delta I^n, \Delta R^n, \Delta C^n)\}$ is bounded in $(L^2(\Omega_T))^4$. Hence, there exists a subsequence of $\{(\Delta S^n, \Delta I^n, \Delta R^n, \Delta C^n)\}$, still denoted by itself, which is weakly convergent. This implies that $(\Delta\overline{S}, \Delta\overline{I}, \Delta\overline{R}, \Delta\overline{C})$ exists on Ω_T and $\{(\Delta S^n, \Delta I^n, \Delta R^n, \Delta C^n)\}$ weakly converges to $(\Delta\overline{S}, \Delta\overline{I}, \Delta\overline{R}, \Delta\overline{C})$ in $(L^2(\Omega_T))^4$ as $n \to \infty$. This is,

$$\lim_{n\to\infty} (\Delta S^n, \Delta I^n, \Delta R^n, \Delta C^n) = (\Delta\overline{S}, \Delta\overline{I}, \Delta\overline{R}, \Delta\overline{C}) \ \textit{weakly in} \ (L^2(0,T;L^2(\Omega)))^4. \tag{4.15}$$

Furthermore, from $\frac{\partial S^n}{\partial n} = \frac{\partial I^n}{\partial n} = \frac{\partial R^n}{\partial n} = \frac{\partial C^n}{\partial n} \equiv 0$ for all $(t,x) \in \Sigma_T$ and $n = 1, 2, \dots$, we also have $\frac{\partial \overline{S}}{\partial n} = \frac{\partial \overline{I}}{\partial n} = \frac{\partial \overline{R}}{\partial n} = \frac{\partial \overline{C}}{\partial n} \equiv 0$ on Σ_T.

In addition, from the estimates (4.9)–(4.12), we also can obtain that $(\frac{\partial \overline{S}}{\partial t}, \frac{\partial \overline{I}}{\partial t}, \frac{\partial \overline{R}}{\partial t}, \frac{\partial \overline{C}}{\partial t})$ exists on Ω_T and when $n \to \infty$

$$(\frac{\partial S^n}{\partial t}, \frac{\partial I^n}{\partial t}, \frac{\partial R^n}{\partial t}, \frac{\partial C^n}{\partial t}) \to (\frac{\partial \overline{S}}{\partial t}, \frac{\partial \overline{I}}{\partial t}, \frac{\partial \overline{R}}{\partial t}, \frac{\partial \overline{C}}{\partial t}) \ \textit{weakly in} \ (L^2(0,T;L^2(\Omega)))^4. \tag{4.16}$$

Now, we show that $S^n I^n \to \overline{SI}$, $C^n I^n \to \overline{CI}$ strongly in $L^2(\Omega_T)$ as $n \to \infty$. In fact, we first write $S^n I^n - \overline{SI} = S^n(I^n - \overline{I}) + \overline{I}(S^n - \overline{S})$ and $C^n I^n - \overline{CI} = C^n(I^n - \overline{I}) + \overline{I}(C^n - \overline{C})$. From the convergence $S^n \to \overline{S}$, $I^n \to \overline{I}$, $C^n \to \overline{C}$ strongly in $L^2(\Omega_T)$ as $n \to \infty$, and the boundedness of $\{(S^n, I^n, R^n, C^n)\}$ in $L^\infty(\Omega_T)$, we directly have

$$S^n I^n \to \overline{SI},\ C^n I^n \to \overline{CI}\ \ strongly\ in\ \ L^2(\Omega_T)\ \ as\ \ n \to \infty. \tag{4.17}$$

Next, we prove that $v_1^n S^n \to \overline{v}_1\overline{S}$, $v_2^n I^n \to \overline{v}_2\overline{I}$ weakly in $L^2(\Omega_T)$. In fact, we can write $v_1^n S^n - \overline{v}_1\overline{S} = v_1^n(S^n - \overline{S}) + \overline{S}(v_1^n - \overline{v}_1)$, $v_2^n I^n - \overline{v}_2\overline{I} = v_2^n(I^n - \overline{I}) + \overline{I}(v_2^n - \overline{v}_2)$. Using the convergences $S^n \to \overline{S}$, $I^n \to \overline{I}$ strongly in $L^2(\Omega_T)$, and (4.4), we obtain that

$$v_1^n S^n \to \overline{v}_1\overline{S},\ v_2^n I^n \to \overline{v}_2\overline{I}\ \ weakly\ in\ \ L^2(\Omega_T)\ \ as\ \ n \to \infty. \tag{4.18}$$

Therefore, from the above convergence (4.13) and (4.15)–(4.18), by taking $n \to \infty$ in (4.2), we finally obtain that $(\overline{S}, \overline{I}, \overline{R}, \overline{C})$ is a solution of control problem (1.3)–(1.5) corresponding to $(\overline{v}_1, \overline{v}_2) \in \mathcal{U}$.

By using the weak sequentially lower semi-continuity of objective functional $J(S, I, v_1, v_2)$, from (4.3), we further have

$$\begin{aligned}
J(\overline{S}, \overline{I}, \overline{v}_1, \overline{v}_2) = &\int_0^T \int_\Omega (\lambda_1(t,x)\overline{S}(t,x) + \lambda_2(t,x)\overline{I}(t,x) + \kappa_1(t,x)\overline{v}_1(t,x) \\
&+ \kappa_2(t,x)\overline{v}_2(t,x))dxdt + \int_\Omega (\omega_1(x)\overline{S}(T,x) + \omega_2(x)\overline{I}^n(T,x) \\
&+ \rho_1(x)\overline{v}_1(T,x) + \rho_2\overline{v}_2(T,x))dx \\
\le &\liminf_{n\to\infty}\Big(\int_0^T \int_\Omega (\lambda_1(t,x)S^n(t,x) + \lambda_2(t,x)I^n(t,x) \\
&+ \kappa_1(t,x)v_1^n(t,x) + \kappa_2(t,x)v_2^n(t,x))dxdt + \int_\Omega (\omega_1(x)S^n(T,x) \\
&+ \omega_2(x)I^n(T,x) + \rho_1(x)v_1^n(T,x) + \rho_2 v_2^n(T,x))dx\Big) \\
\le &\lim_{n\to\infty} J(S^n, I^n, v_1^n, v_2^n) \\
= &\inf_{(v_1,v_2)\in\mathcal{U}} J(S, I, v_1, v_2).
\end{aligned}$$

This shows that J attains its minimum at $(\overline{S}, \overline{I}, \overline{v}_1, \overline{v}_2)$. Therefore, $(\overline{S}, \overline{I}, \overline{R}, \overline{C}, \overline{v}_1, \overline{v}_2)$ is an optimal pair of the control system (1.3)–(1.5). This completes the proof. □

5 Necessary optimality condition

In this section, to further establish the first-order necessary condition for optimal control, we consider the adjoint system of the state system (2.1), which can be expressed as follows

$$\begin{cases} \dfrac{\partial Q}{\partial t} = -\mathcal{L}^* Q - F_{\overline{P}}^* Q + L_{\overline{P}}, & t \in (0, T), \\ Q(T, x) = -\varphi_{\overline{P}}(T, x), \end{cases} \tag{5.1}$$

where $\mathcal{L}^*$ is the adjoint operator associated to $\mathcal{L}$, and $F_{\overline{P}}^*$ is the adjoint matrix of the Jacobian matrix $F_{\overline{P}}$, that is, $F_{\overline{P}}^*$ is the transposition of $\frac{\partial F}{\partial \overline{P}}$, and $L_{\overline{P}} := \frac{\partial L(\overline{P},\overline{v})}{\partial \overline{P}}$, $\varphi_{\overline{P}} := \frac{\partial \varphi(\overline{P},\overline{v})}{\partial \overline{P}}$. Let

$(\overline{P},\overline{v}) = (\overline{S},\overline{I},\overline{R},\overline{C},\overline{v}_1,\overline{v}_2)$ be an optimal pair of control system (2.1) and $Q = (S^*, I^*, R^*, C^*)$ be the adjoint variable [25]. Then, system (5.1) can be written in detail as

$$\begin{cases} \dfrac{\partial S^*(t,x)}{\partial t} = - d_1 \Delta S^*(t,x) + (\mu(x) + \beta_1(x)\overline{I}(t,x) + \overline{v}_1(t,x))S^*(t,x) \\ \qquad - \beta_1(x)\overline{I}(t,x)I^*(t,x) - \overline{v}_1(t,x)R^*(t,x) + \lambda_1(t,x), \\ \dfrac{\partial I^*(t,x)}{\partial t} = - d_2 \Delta I^*(t,x) + \beta_1(x)\overline{S}(t,x)S^*(t,x) - [\beta_1(x)\overline{S}(t,x) + \sigma(x)\beta_2(x)\overline{C}(t,x) \\ \qquad - (\mu(x) + \alpha(x) + \overline{v}_2(t,x))]I^*(t,x) - [(1-\sigma(x))\beta_2(x)\overline{C}(t,x) + \alpha(x) \\ \qquad + \overline{v}_2(t,x)]R^*(t,x) + \beta_2(x)\overline{C}(t,x)C^*(t,x) + \lambda_2(t,x), \\ \dfrac{\partial R^*(t,x)}{\partial t} = - d_3 \Delta R^*(t,x) + (\mu(x) + \delta(x))R^*(t,x) - \delta(x)C^*(t,x), \\ \dfrac{\partial C^*(t,x)}{\partial t} = - d_4 \Delta C^*(t,x) - \gamma(x)S^*(t,x) - \sigma(x)\beta_2(x)\overline{I}(t,x)I^*(t,x) - (1-\sigma(x))\beta_2(x) \\ \qquad \overline{I}(t,x)R^*(t,x) + (\beta_2(x)\overline{I}(t,x) + \mu(x) + \gamma(x))C^*(t,x), \\ \dfrac{\partial S^*(t,x)}{\partial n} = \dfrac{\partial I^*(t,x)}{\partial n} = \dfrac{\partial R^*(t,x)}{\partial n} = \dfrac{\partial C^*(t,x)}{\partial n} = 0, \\ S^*(T,x) = - \omega_1, I^*(T,x) = -\omega_2, R^*(T,x) = 0, C^*(T,x) = 0. \end{cases} \tag{5.2}$$

By introducing a change of variable $s = T - t$ and using the same methods in Theorem 3.2, we easily establish the following results.

Lemma 5.1 *Assume that* $(\overline{S},\overline{I},\overline{R},\overline{C},\overline{v}_1,\overline{v}_2)$ *is an optimal pair of control system* (1.3)–(1.5). *Then the adjoint system* (5.2) *admits a unique strong solution* (S^*, I^*, R^*, C^*) *such that*

$$S^* \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{5.3}$$

$$I^* \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{5.4}$$

$$R^* \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{5.5}$$

$$C^* \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). \tag{5.6}$$

To establish the first-order necessary condition for optimal control, the key point of our analysis is proving the differentiability of the control-to-state mapping.

For any $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon) \in \mathcal{U}$, let $(S^\varepsilon, I^\varepsilon, R^\varepsilon, C^\varepsilon)$ be the solution of control system (1.3)–(1.5) with $v_1(t,x) = v_1^\varepsilon(t,x)$ and $v_1(t,x) = v_2^\varepsilon(t,x)$. We define a mapping as follows

$$\Phi : \mathcal{U} \subset (L^2(\Omega_T))^2 \to (L^2(\Omega_T))^4 \tag{5.7}$$

by $\Phi(v^\varepsilon) = (S^\varepsilon, I^\varepsilon, R^\varepsilon, C^\varepsilon)$. The mapping Φ is obviously well defined due to Theorem 3.2, we have the following result for the control-to-state mapping Φ.

Lemma 5.2 *The mapping* Φ *defined in* (5.7) *is Gâteaux differentiable at* $\overline{v} = (\overline{v}_1, \overline{v}_2)$. *That is, there exists a bounded linear operator* $\Phi'(\overline{v}) : (L^2(\Omega_T))^2 \to (L^2(\Omega_T))^4$ *such that for any*

$\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in L^2(\Omega_T)$ and positive number ε, satisfying $v^\varepsilon = \overline{v} + \varepsilon\widetilde{v} \in \mathcal{U}$, one has

$$\lim_{\varepsilon\to 0} \| \frac{\Phi(v^\varepsilon) - \Phi(\overline{v})}{\varepsilon} - \Phi'(\overline{v})\widetilde{v}\| = 0, \ \ \forall\, \widetilde{v} \in L^2(\Omega_T), \tag{5.8}$$

where $\Phi(v^\varepsilon) = (S^\varepsilon, I^\varepsilon, R^\varepsilon, C^\varepsilon)$, $\Phi(\overline{v}) := (\overline{S}, \overline{I}, \overline{R}, \overline{C})$ are the solutions of control system (1.3)-(1.5) with $(v_1, v_2) = (v_1^\varepsilon, v_2^\varepsilon)$ and $(v_1, v_2) = (\overline{v}_1, \overline{v}_2)$, respectively. Moreover, $\Phi'(\overline{v})\widetilde{v} := Z = (Z_S, Z_I, Z_R, Z_C)$ is the solution of the following linear system:

$$\begin{cases} \dfrac{\partial Z_S(t,x)}{\partial t} = d_1 \Delta Z_S(t,x) - (\mu(x) + \beta_1(x)\overline{I}(t,x) + \overline{v}_1(t,x))Z_S(t,x) \\ \qquad - \beta_1(x)\overline{S}(t,x)Z_I(t,x) + \gamma(x)Z_C(t,x) - v_1(t,x)\overline{S}(t,x), \\ \dfrac{\partial Z_I(t,x)}{\partial t} = d_2 \Delta Z_I(t,x) + \beta_1(x)\overline{I}(t,x)Z_S(t,x) + [\beta_1(x)\overline{S}(t,x) \\ \qquad + \sigma(x)\beta_2(x)\overline{C}(t,x) - (\mu(x) + \alpha(x) + \overline{v}_2(t,x))]Z_I(t,x) \\ \qquad + \sigma(x)\beta_2(x)\overline{I}(t,x)Z_C(t,x) - v_2(t,x)\overline{I}(t,x), \\ \dfrac{\partial Z_R(t,x)}{\partial t} = d_3 \Delta Z_R(t,x) + \overline{v}_1(t,x)Z_S(t,x) + [(1-\sigma(x))\beta_2(x)\overline{C}(t,x) \\ \qquad + \alpha(x) + \overline{v}_2(t,x)]Z_I(t,x) - (\mu(x) + \delta(x))Z_R(t,x) \\ \qquad + (1-\sigma(x))\beta_2(x)\overline{I}(t,x)Z_C(t,x) + v_1(t,x)\overline{S}(t,x) + v_2(t,x)\overline{I}(t,x), \\ \dfrac{\partial Z_C(t,x)}{\partial t} = d_4 \Delta Z_C(t,x) - \beta_2(x)\overline{C}(t,x)Z_I(t,x) + \delta(x)Z_R(t,x) \\ \qquad - (\beta_2(x)\overline{I}(t,x) + \mu(x) + \gamma(x))Z_C(t,x), \\ \dfrac{\partial Z_S(t,x)}{\partial n} = \dfrac{\partial Z_I(t,x)}{\partial n} = \dfrac{\partial Z_R(t,x)}{\partial n} = \dfrac{\partial Z_C(t,x)}{\partial n} = 0, \\ Z_S(0,x) = Z_I(0,x) = Z_R(0,x) = Z_C(0,x) = 0, \end{cases} \tag{5.9}$$

with $(v_1(t,x), v_2(t,x)) = (\widetilde{v}_1(t,x), \widetilde{v}_2(t,x))$.

Proof Firstly, based on the fundamental theory of linear reaction-diffusion equations we easily know that system (5.9) with $(v_1, v_2) = (\widetilde{v}_1, \widetilde{v}_2)$ has a unique solution $Z = (Z_S, Z_I, Z_R, Z_C)$ defined for $(t,x) \in \Omega_T$. From this, we can define a mapping $\Phi'(\overline{v}) : (L^2(\Omega_T))^2 \to (L^2(\Omega_T))^4$ as follows

$$\Phi'(\overline{v})\widetilde{v} = Z = (Z_S, Z_I, Z_R, Z_C), \ \ \widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in (L^2(\Omega_T))^2.$$

Furthermore, we easily prove that the mapping $\Phi'(\overline{v})$ is a linear operator. That is, if $\Phi'(v)\widetilde{v}$ and $\Phi'(v)\widehat{v}$ are the solutions of system (5.9) with $v = \widetilde{v}$ and $v = \widehat{v}$, respectively, then $\Phi'(v)(\alpha\widetilde{v} + \beta\widehat{v})$ also is the solution of system (5.9) with $v = \alpha\widetilde{v} + \beta\widehat{v}$, and $\Phi'(v)(\alpha\widetilde{v} + \beta\widehat{v}) = \alpha\Phi'(v)\widetilde{v} + \beta\Phi'(v)\widehat{v}$, where α and β are two positive constants.

We set

$$Z^\varepsilon = (Z_S^\varepsilon, Z_I^\varepsilon, Z_R^\varepsilon, Z_C^\varepsilon) = \frac{\Phi(v^\varepsilon) - \Phi(\overline{v})}{\varepsilon} = (\frac{S^\varepsilon - \overline{S}}{\varepsilon}, \frac{I^\varepsilon - \overline{I}}{\varepsilon}, \frac{R^\varepsilon - \overline{R}}{\varepsilon}, \frac{C^\varepsilon - \overline{C}}{\varepsilon}).$$

It is easily verified that $(Z_S^\varepsilon, Z_I^\varepsilon, Z_R^\varepsilon, Z_C^\varepsilon)$ satisfies the following system

$$\begin{cases} \dfrac{\partial Z_S^\varepsilon(t,x)}{\partial t} = d_1\Delta Z_S^\varepsilon(t,x) - (\mu(x) + \beta_1(x)I^\varepsilon(t,x) + \overline{\nu}_1(t,x))Z_S^\varepsilon(t,x) \\ \qquad - \beta_1(x)\overline{S}(t,x)Z_I^\varepsilon(t,x) + \gamma(x)Z_C^\varepsilon(t,x) - \widetilde{\nu}_1(t,x)S^\varepsilon(t,x), \\ \dfrac{\partial Z_I^\varepsilon(t,x)}{\partial t} = d_2\Delta Z_I^\varepsilon(t,x) + \beta_1(x)I^\varepsilon(t,x)Z_S^\varepsilon(t,x) + [\beta_1(x)\overline{S}(t,x) + \sigma(x)\beta_2(x)\overline{C}(t,x) \\ \qquad - (\mu(x) + \alpha(x) + \overline{\nu}_2(t,x))]Z_I^\varepsilon(t,x) + \sigma(x)\beta_2(x)I^\varepsilon(t,x)Z_C^\varepsilon(t,x) \\ \qquad - \widetilde{\nu}_2(t,x)I^\varepsilon(t,x), \\ \dfrac{\partial Z_R^\varepsilon(t,x)}{\partial t} = d_3\Delta Z_R^\varepsilon(t,x) + \nu_1^\varepsilon(t,x)Z_S^\varepsilon(t,x) + [(1-\sigma(x))\beta_2(x)\overline{C}(t,x) \\ \qquad + \alpha(x) + \nu_2^\varepsilon(t,x)]Z_I^\varepsilon(t,x) - (\mu(x) + \delta(x))Z_R^\varepsilon(t,x) \\ \qquad + (1-\sigma(x))\beta_2(x)I^\varepsilon(t,x)Z_C^\varepsilon(t,x) + \widetilde{\nu}_1(t,x)\overline{S}(t,x) + \widetilde{\nu}_2(t,x)\overline{I}(t,x) \\ \dfrac{\partial Z_C^\varepsilon(t,x)}{\partial t} = d_4\Delta Z_C^\varepsilon(t,x) - \beta_2(x)C^\varepsilon(t,x)Z_I^\varepsilon(t,x) + \delta(x)Z_R^\varepsilon(t,x) \\ \qquad - (\beta_2(x)\overline{I}(t,x) + \mu(x) + \gamma(x))Z_C^\varepsilon(t,x), \\ \dfrac{\partial Z_S^\varepsilon(t,x)}{\partial n} = \dfrac{\partial Z_I^\varepsilon(t,x)}{\partial n} = \dfrac{\partial Z_R^\varepsilon(t,x)}{\partial n} = \dfrac{\partial Z_C^\varepsilon(t,x)}{\partial n} = 0, \\ Z_S^\varepsilon(0,x) = Z_I^\varepsilon(0,x) = Z_R^\varepsilon(0,x) = Z_C^\varepsilon(0,x) = 0. \end{cases} \tag{5.10}$$

In order to prove that (5.8) holds, it is sufficient to show that the following conclusion holds

$$\begin{aligned} &\lim_{\varepsilon\to 0}\|Z_S^\varepsilon - Z_S\|_{L^2(\Omega_T)} = 0, \quad \lim_{\varepsilon\to 0}\|Z_I^\varepsilon - Z_I\|_{L^2(\Omega_T)} = 0, \\ &\lim_{\varepsilon\to 0}\|Z_R^\varepsilon - Z_R\|_{L^2(\Omega_T)} = 0, \quad \lim_{\varepsilon\to 0}\|Z_C^\varepsilon - Z_C\|_{L^2(\Omega_T)} = 0. \end{aligned} \tag{5.11}$$

For this purpose, we first prove that Z^ε is bounded in $L^2(\Omega_T)$ uniformly with respect to ε. To this end, denoting

$$H^\varepsilon(t) = \begin{pmatrix} -(\beta_1 I^\varepsilon + \mu + \overline{\nu}_1) & -\beta_1\overline{S} & 0 & \gamma \\ \beta_1 I^\varepsilon & \beta_1\overline{S} + \sigma\beta_2\overline{C} - (\mu + \alpha + \overline{\nu}_2) & 0 & \sigma\beta_2 I^\varepsilon \\ \nu_1^\varepsilon & (1-\sigma)\beta_2\overline{C} + \alpha + \nu_2^\varepsilon & -\mu - \delta & (1-\sigma)\beta_2 I^\varepsilon \\ 0 & -\beta_2 C^\varepsilon & \delta & -(\beta_2\overline{I} + \mu + \gamma) \end{pmatrix}, \tag{5.12}$$

and

$$L^\varepsilon(t) = \begin{pmatrix} -S^\varepsilon & 0 \\ 0 & -I^\varepsilon \\ \overline{S} & \overline{I} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{\nu} = \begin{pmatrix} \widetilde{\nu}_1 \\ \widetilde{\nu}_2 \end{pmatrix}. \tag{5.13}$$

Then system (5.10) is rewritten by

$$\begin{cases} \dfrac{\partial Z^\varepsilon}{\partial t} = \mathcal{L}Z^\varepsilon(t) + H^\varepsilon(t)Z^\varepsilon(t) + L^\varepsilon(t)\widetilde{v}, & (t,x) \in \Omega_T, \\ Z^\varepsilon(0) = 0, & x \in \Omega. \end{cases} \tag{5.14}$$

It follows from Theorem 3.2 that the Cauchy problem (5.14) possesses a unique strong solution that can be expressed as

$$Z^\varepsilon(t) = \int_0^t \Lambda_1(t-s)H^\varepsilon(s)Z^\varepsilon(s)ds + \int_0^t \Lambda_1(t-s)L^\varepsilon(s)\widetilde{v}ds, \quad t \in [0,T], \tag{5.15}$$

where $\{\Lambda_1(t), t \geq 0\}$ is the C_0-semigroup of contractions generated by the operator $\mathcal{L}$. In addition, according to Lemma 5.1 and Theorem 3.2, all elements of matrices H^ε and L^ε are bounded uniformly with respect to ε. Therefore, from (5.15) there exists a positive constant K_3 such that

$$\|Z_S^\varepsilon\|_{L^2(\Omega_T)} \leq K_3, \ \|Z_I^\varepsilon\|_{L^2(\Omega_T)} \leq K_3, \ \|Z_R^\varepsilon\|_{L^2(\Omega_T)} \leq K_3, \ \|Z_C^\varepsilon\|_{L^2(\Omega_T)} \leq K_3. \tag{5.16}$$

This shows that Z^ε is bounded in $L^2(\Omega_T)$ uniformly with respect to ε. Moreover, from (5.16), we further obtain that

$$S^\varepsilon \to \overline{S}, \ I^\varepsilon \to \overline{I}, \ R^\varepsilon \to \overline{R}, \ C^\varepsilon \to \overline{C} \ \ in\, L^2(\Omega_T) \ \ as \ \ \varepsilon \to 0. \tag{5.17}$$

We now prove that (5.11) holds. Let

$$H(t) = \begin{pmatrix} -(\beta_1\overline{I} + \mu + \overline{v}_1) & -\beta_1\overline{S} & 0 & \gamma \\ \beta_1\overline{I} & \beta_1\overline{S} + \sigma\beta_2\overline{C} - (\mu + \alpha + \overline{v}_2) & 0 & \sigma\beta_2\overline{I} \\ \overline{v}_1 & (1-\sigma)\beta_2\overline{C} + \alpha + \overline{v}_2 & -\mu - \delta & (1-\sigma)\beta_2\overline{I} \\ 0 & -\beta_2\overline{C} & \delta & -(\beta_2\overline{I} + \mu + \gamma) \end{pmatrix}, \tag{5.18}$$

and

$$L(t) = \begin{pmatrix} -\overline{S} & 0 \\ 0 & -\overline{I} \\ \overline{S} & \overline{I} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{v} = \begin{pmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \end{pmatrix}. \tag{5.19}$$

Then system (5.9) with $(v_1, v_2) = (\widetilde{v}_1, \widetilde{v}_2)$ can be abbreviated as

$$\begin{cases} \dfrac{\partial Z}{\partial t} = \mathcal{L}Z(t) + H(t)Z(t) + L(t)\widetilde{v}, & (t,x) \in \Omega_T, \\ Z(0) = 0, & x \in \Omega. \end{cases} \tag{5.20}$$

The solution of (5.20) can be expressed as

$$Z(t) = \int_0^t \Lambda_1(t-s)H(s)Z(s)ds + \int_0^t \Lambda_1(t-s)L(s)\widetilde{v}ds, \quad t \in [0,T]. \tag{5.21}$$

Then the combination of (5.15) and (5.21) arrives at

$$\begin{aligned} Z^\varepsilon(t) - Z(t) = &\int_0^t \Lambda_1(t-s)(H^\varepsilon(s) - H(s))Z(s)ds \\ &+ \int_0^t \Lambda_1(t-s)H^\varepsilon(s)(Z^\varepsilon(s) - Z(s))ds \\ &+ \int_0^t \Lambda_1(t-s)(L^\varepsilon(s) - L(s))\widetilde{v}ds. \end{aligned} \tag{5.22}$$

From the boundedness of the semigroup $\{\Lambda_1(t), t \geq 0\}$, there is a constant $L_1^* > 0$ such that $\|\Lambda_1(t)\| \leq L_1^*$ for all $t \geq 0$. From Theorem 3.2, (5.12), (5.13) and (5.17)–(5.19), we can obtain that there is a constant $M_0 > 0$ such that $\|H^\varepsilon\|_{L^2(\Omega_T)} \leq M_0$ for $0 \leq \varepsilon \leq 1$, and $\|H^\varepsilon - H\|_{L^2(\Omega_T)} \to 0$, $\|L^\varepsilon - L\|_{L^2(\Omega_T)} \to 0$ and $\|\widetilde{v}\|_{L^2(\Omega_T)} \to 0$ as $\varepsilon \to 0$. From (5.22), we further obtain

$$\begin{aligned} |Z^\varepsilon(t) - Z(t)| \leq &L_1^*\|Z\|_{L^2(\Omega_T)}\|H^\varepsilon - H\|_{L^2(\Omega_T)}T + L_1^* M_0 \int_0^t |Z^\varepsilon(s) - Z(s)|ds \\ &+ L_1^*\|L^\varepsilon - L\|_{L^2(\Omega_T)}\|\widetilde{v}\|_{L^2(\Omega_T)}T. \end{aligned} \tag{5.23}$$

Then, applying the Gronwall inequality to (5.23), we can obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\|Z_S^\varepsilon - Z_S\|_{L^2(\Omega_T)} = 0, \quad \lim_{\varepsilon\to 0}\|Z_I^\varepsilon - Z_I\|_{L^2(\Omega_T)} = 0, \\ &\lim_{\varepsilon\to 0}\|Z_R^\varepsilon - Z_R\|_{L^2(\Omega_T)} = 0, \quad \lim_{\varepsilon\to 0}\|Z_C^\varepsilon - Z_C\|_{L^2(\Omega_T)} = 0. \end{aligned}$$

That is, (5.11) holds.

Furthermore, applying the L^2 estimation methods of parabolic equation (see Theorem 2 in [23]), we can obtain

$$\|\Phi'(\overline{v})\widetilde{v}\|_{L^2(\Omega_T)} \leq K_4\|\widetilde{v}\|_{L^2(\Omega_T)},$$

where K_4 is a positive constant. It then follows that $\Phi'(\overline{v})$ is a bounded linear operator. This completes the proof. □

Theorem 5.1 (*First-order necessary optimality condition*) *Let* $(\overline{S}, \overline{I}, \overline{R}, \overline{C}, \overline{v}_1, \overline{v}_2)$ *be an optimal pair of control problem* (1.3)–(1.9) *such that*

$$J(\overline{S}, \overline{I}, \overline{v}_1, \overline{v}_2) = \inf_{v=(v_1,v_2)\in\mathcal{U}} J(S, I, v),$$

and (S^*, I^*, R^*, C^*) *be the solution of adjoint system* (5.2). *Then, for any* $v = (v_1, v_2) \in \mathcal{U}$, *one has*

$$\begin{aligned} &\int_0^T\int_\Omega (\overline{S}S^* - \overline{S}R^* + \kappa_1)(v_1 - \overline{v}_1)(t,x)dxdt + \int_0^T\int_\Omega (\overline{I}I^* - \overline{I}R^* + \kappa_2)(v_2 - \overline{v}_2)(t,x)dxdt \\ &\geq -\int_\Omega \rho_1(v_1 - \overline{v}_1)(T,x)dx - \int_\Omega \rho_2(v_2 - \overline{v}_2)(T,x)dx. \end{aligned} \tag{5.24}$$

Furthermore, if $\rho_1(x) = \rho_2(x) \equiv 0$ *in* Ω, *then the optimal control* $\overline{v} = (\overline{v}_1, \overline{v}_2)$ *can be characterized as*

$$\overline{v}_1(t,x) = \begin{cases} 1, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\overline{S}S^* - \overline{S}R^* + \kappa_1)(t,x) \leq 0\}, \\ 0, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\overline{S}S^* - \overline{S}R^* + \kappa_1)(t,x) > 0\}, \end{cases}$$

and

$$\overline{v}_2(t,x) = \begin{cases} 1, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\overline{I}I^* - \overline{I}R^* + \kappa_2)(t,x) \leq 0\}, \\ 0, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\overline{I}I^* - \overline{I}R^* + \kappa_2)(t,x) > 0\}. \end{cases}$$

Proof Assume that $(\overline{S}, \overline{I}, \overline{R}, \overline{C}, \overline{v}_1, \overline{v}_2)$ is an optimal pair and the objective functional $J(S, I, v_1, v_2)$ is defined as in (1.7). Then we have

$$J(\overline{S}, \overline{I}, \overline{v}_1, \overline{v}_2) \leq J(S^\varepsilon, I^\varepsilon, v_1^\varepsilon, v_2^\varepsilon), \quad \forall\, \varepsilon > 0,$$

where $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ and $(S^\varepsilon, I^\varepsilon, R^\varepsilon, C^\varepsilon)$ are defined in Lemma 5.2. Therefore, we have for any $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in (L^2(\Omega_T))^2$

$$\begin{aligned} 0 \leq & \frac{J(S^\varepsilon, I^\varepsilon, v_1^\varepsilon, v_2^\varepsilon) - J(\overline{S}, \overline{I}, \overline{v}_1, \overline{v}_2)}{\varepsilon} \\ = & \frac{1}{\varepsilon}\{\int_0^T \int_\Omega (\lambda_1 S^\varepsilon + \lambda_2 I^\varepsilon + \kappa_1 v_1^\varepsilon + \kappa_2 v_2^\varepsilon) dxdt \\ & + \int_\Omega (\omega_1 S^\varepsilon(T,x) + \omega_2 I^\varepsilon(T,x) + \rho_1 v_1^\varepsilon(T,x) + \rho_2 v_2^\varepsilon(T,x)) dx \\ & - \int_0^T \int_\Omega (\lambda_1 \overline{S} + \lambda_2 \overline{I} + \kappa_1 \overline{v}_1 + \kappa_2 \overline{v}_2) dxdt \\ & - \int_\Omega (\omega_1 \overline{S}(T,x) + \omega_2 \overline{I}(T,x) + \rho_1 \overline{v}_1(T,x) + \rho_2 \overline{v}_2(T,x)) dx\} \\ = & \int_0^T \int_\Omega (\lambda_1 Z_S^\varepsilon + \lambda_2 Z_I^\varepsilon + \kappa_1 \widetilde{v}_1 + \kappa_2 \widetilde{v}_2) dxdt \\ & + \int_\Omega (\omega_1 Z_S^\varepsilon(T,x) + \omega_2 Z_I^\varepsilon(T,x) + \rho_1 \widetilde{v}_1(T,x) + \rho_2 \widetilde{v}_2(T,x)) dx. \end{aligned} \tag{5.25}$$

On the one hand, it follows from Lemma 5.2 that

$$Z_S^\varepsilon \to Z_S, \; Z_I^\varepsilon \to Z_I \quad in \;\; L^2(\Omega_T) \quad as\; \varepsilon \to 0. \tag{5.26}$$

Since $L^2(\Omega_T) \subset L^1(\Omega_T)$, from (5.26), we further also have

$$Z_S^\varepsilon \to Z_S, \; Z_I^\varepsilon \to Z_I \quad in \;\; L^1(\Omega_T) \quad as\; \varepsilon \to 0. \tag{5.27}$$

On the other hand, since the coefficients in system (5.10) are uniformly bounded, by applying the same argument in Theorem 3.2 to the linear parabolic system (5.10), there exists a constant $C > 0$, independent of ε, such that

$$\begin{aligned}
&\|\frac{\partial Z_S^\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|Z_S^\varepsilon\|_{L^2(0,T;H^2(\Omega))} + \|Z_S^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|Z_S^\varepsilon\|_{L^\infty(\Omega_T)} \le C,\\
&\|\frac{\partial Z_I^\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|Z_I^\varepsilon\|_{L^2(0,T;H^2(\Omega))} + \|Z_I^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|Z_I^\varepsilon\|_{L^\infty(\Omega_T)} \le C,\\
&\|\frac{\partial Z_R^\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|Z_R^\varepsilon\|_{L^2(0,T;H^2(\Omega))} + \|Z_R^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|Z_R^\varepsilon\|_{L^\infty(\Omega_T)} \le C,\\
&\|\frac{\partial Z_C^\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|Z_C^\varepsilon\|_{L^2(0,T;H^2(\Omega))} + \|Z_C^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|Z_C^\varepsilon\|_{L^\infty(\Omega_T)} \le C.
\end{aligned} \tag{5.28}$$

This implies that the function family $\{(Z_S^\varepsilon, Z_I^\varepsilon, Z_R^\varepsilon, Z_C^\varepsilon)\}_\varepsilon$ is equicontinuous in $(L^2(\Omega_T))^4$ for $t \in [0,T]$ with respect to the arbitrary positive number ε. Notice that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ (See [32]), then from (5.28) there is a constant $C_2 > 0$ such that $\|Z_S^\varepsilon\|_{L^2(\Omega)} \le C_2$ and $\|Z_I^\varepsilon\|_{L^2(\Omega)} \le C_2$ for any $t \in [0,T]$ and arbitrary positive number ε. Since $Z_S, Z_I \in W^{1,2}(0,T;L^2(\Omega)) \subset C(0,T;L^2(\Omega))$, by using the Ascoli–Arzela theorem and the uniqueness of limits, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \|Z_S^\varepsilon(T) - Z_S(T)\|_{L^2(\Omega)} \le \lim_{\varepsilon\to 0} \max_{t\in[0,T]} \|Z_S^\varepsilon(t) - Z_S(t)\|_{L^2(\Omega)} = 0,\\
&\lim_{\varepsilon\to 0} \|Z_I^\varepsilon(T) - Z_I(T)\|_{L^2(\Omega)} \le \lim_{\varepsilon\to 0} \max_{t\in[0,T]} \|Z_I^\varepsilon(t) - Z_I(t)\|_{L^2(\Omega)} = 0,
\end{aligned}$$

which means that

$$Z_S^\varepsilon(T,x) \to Z_S(T,x), \quad Z_I^\varepsilon(T,x) \to Z_I(T,x) \quad in\ L^1(\Omega) \quad as\ \varepsilon \to 0. \tag{5.29}$$

Now, combine (5.27) and (5.29), sending $\varepsilon \to 0$ in (5.25), it turns out that for any $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in L^2(\Omega_T)$

$$\begin{aligned}
&\int_0^T \int_\Omega (\lambda_1 Z_S + \lambda_2 Z_I + \kappa_1 \widetilde{v}_1 + \kappa_2 \widetilde{v}_2) dxdt\\
&+ \int_\Omega (\omega_1 Z_S(T,x) + \omega_2 Z_I(T,x) + \rho_1 \widetilde{v}_1(T,x) + \rho_2 \widetilde{v}_2(T,x)) dx \ge 0.
\end{aligned} \tag{5.30}$$

Multiplying the first four equations of the system (5.2) by Z_S, Z_I, Z_R and Z_C, respectively, we obtain

$$\begin{aligned}
&\frac{\partial S^*}{\partial t} Z_S = -d_1 \Delta S^* Z_S + (\mu + \beta_1 \overline{I} + \overline{v}_1) S^* Z_S - \beta_1 \overline{I} I^* Z_S - \overline{v}_1 R^* Z_S + \lambda_1 Z_S,\\
&\frac{\partial I^*}{\partial t} Z_I = -d_2 \Delta I^* Z_I + \beta_1 \overline{S} S^* Z_I - [\beta_1 \overline{S} + \sigma \beta_2 \overline{C} - (\mu + \alpha + \overline{v}_2)] I^* Z_I\\
&\qquad - [(1-\sigma)\beta_2 \overline{C} + \alpha + \overline{v}_2] R^* Z_I + \beta_2 \overline{C} C^* Z_I + \lambda_2 Z_I,\\
&\frac{\partial R^*}{\partial t} Z_R = -d_3 \Delta R^* Z_R + (\mu + \delta) R^* Z_R - \delta C^* Z_R,\\
&\frac{\partial C^*}{\partial t} Z_C = -d_4 \Delta C^* Z_C - \gamma S^* Z_C - \sigma \beta_2 \overline{I} I^* Z_C - (1-\sigma)\beta_2 \overline{I} R^* Z_C\\
&\qquad + (\beta_2 \overline{I} + \mu + \gamma) C^* Z_C.
\end{aligned} \tag{5.31}$$

And then, multiplying the first four equations of the system (5.9) by S^*, I^*, R^* and C^*, respectively, we also obtain

$$\begin{aligned}
\frac{\partial Z_S}{\partial t}S^* =& d_1\Delta Z_S S^* - (\mu + \beta_1\bar{I} + \bar{v}_1)Z_S S^* - \beta_1\bar{S}Z_I S^* + \gamma Z_C S^* - \widetilde{v}_1\bar{S}S^*,\\
\frac{\partial Z_I}{\partial t}I^* =& d_2\Delta Z_I I^* + \beta_1\bar{I}Z_S I^* + [\beta_1\bar{S} + \sigma\beta_2\bar{C} - (\mu + \alpha + \bar{v}_2)]Z_I I^*\\
&+ \sigma\beta_2\bar{I}Z_C I^* - \widetilde{v}_2\bar{I}I^*,\\
\frac{\partial Z_R}{\partial t}R^* =& d_3\Delta Z_R R^* + \bar{v}_1 Z_S R^* + [(1-\sigma)\beta_2\bar{C} + \alpha + \bar{v}_2]Z_I R^* - (\mu + \delta)Z_R R^*\\
&+ (1-\sigma)\beta_2\bar{I}Z_C R^* + \widetilde{v}_1\bar{S}R^* + \widetilde{v}_2\bar{I}R^*,\\
\frac{\partial Z_C}{\partial t}C^* =& d_4\Delta Z_C C^* - \beta_2\bar{C}Z_I C^* + \delta Z_R C^* - (\beta_2\bar{I} + \mu + \gamma)Z_C C^*.
\end{aligned} \tag{5.32}$$

Adding (5.31) and (5.32), we further obtain

$$\begin{aligned}
&\frac{\partial S^*}{\partial t}Z_S + \frac{\partial I^*}{\partial t}Z_I + \frac{\partial R^*}{\partial t}Z_R + \frac{\partial C^*}{\partial t}Z_C + \frac{\partial Z_S}{\partial t}S^* + \frac{\partial Z_I}{\partial t}I^* + \frac{\partial Z_R}{\partial t}R^* + \frac{\partial Z_C}{\partial t}C^*\\
=& - d_1\Delta S^* Z_S - d_2\Delta I^* Z_I - d_3\Delta R^* Z_R - d_4\Delta C^* Z_C\\
&+ d_1\Delta Z_S S^* + d_2\Delta Z_I I^* + d_3\Delta Z_R R^* + d_4\Delta Z_C C^*\\
&+ \lambda_1 Z_S + \lambda_2 Z_I - \widetilde{v}_1\bar{S}S^* - \widetilde{v}_2\bar{I}I^* + \widetilde{v}_1\bar{S}R^* + \widetilde{v}_2\bar{I}R^*.
\end{aligned} \tag{5.33}$$

Integrating both side of (5.33) on Ω_T, we have

$$\begin{aligned}
&\int_0^T\int_\Omega\Big\{\frac{\partial S^*}{\partial t}Z_S + \frac{\partial I^*}{\partial t}Z_I + \frac{\partial R^*}{\partial t}Z_R + \frac{\partial C^*}{\partial t}Z_C + \frac{\partial Z_S}{\partial t}S^* + \frac{\partial Z_I}{\partial t}I^*\\
&+ \frac{\partial Z_R}{\partial t}R^* + \frac{\partial Z_C}{\partial t}C^*\Big\}dxdt\\
=& \int_0^T\int_\Omega(-d_1\Delta S^* Z_S - d_2\Delta I^* Z_I - d_3\Delta R^* Z_R - d_4\Delta C^* Z_C)dxdt\\
&+ \int_0^T\int_\Omega(d_1\Delta Z_S S^* + d_2\Delta Z_I I^* + d_3\Delta Z_R R^* + d_4\Delta Z_C C^*)dxdt\\
&+ \int_0^T\int_\Omega(\lambda_1 Z_S + \lambda_2 Z_I - \widetilde{v}_1\bar{S}S^* - \widetilde{v}_2\bar{I}I^* + \widetilde{v}_1\bar{S}R^* + \widetilde{v}_2\bar{I}R^*)dxdt.
\end{aligned} \tag{5.34}$$

Combining the initial boundary conditions of S^*, I^*, R^*, C^* in (5.2) and Z_S, Z_I, Z_R, Z_C in (5.9), we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega\Big\{\frac{\partial S^*}{\partial t}Z_S + \frac{\partial I^*}{\partial t}Z_I + \frac{\partial R^*}{\partial t}Z_R + \frac{\partial C^*}{\partial t}Z_C + \frac{\partial Z_S}{\partial t}S^* + \frac{\partial Z_I}{\partial t}I^*\\
&+ \frac{\partial Z_R}{\partial t}R^* + \frac{\partial Z_C}{\partial t}C^*\Big\}dxdt\\
=& \int_\Omega\int_0^T\Big\{\frac{\partial S^*}{\partial t}Z_S + \frac{\partial Z_S}{\partial t}S^*\Big\}dtdx + \int_\Omega\int_0^T\Big\{\frac{\partial I^*}{\partial t}Z_I + \frac{\partial Z_I}{\partial t}I^*\Big\}dtdx\\
&+ \int_\Omega\int_0^T\Big\{\frac{\partial R^*}{\partial t}Z_R + \frac{\partial Z_R}{\partial t}R^*\Big\}dtdx + \int_\Omega\int_0^T\Big\{\frac{\partial C^*}{\partial t}Z_C + \frac{\partial Z_C}{\partial t}C^*\Big\}dtdx
\end{aligned} \tag{5.35}$$

$$= \int_{\Omega} (Z_S S^*\big|_0^T) dx + \int_{\Omega} (Z_I I^*\big|_0^T) dx + \int_{\Omega} (Z_R R^*\big|_0^T) dx + \int_{\Omega} (Z_C C^*\big|_0^T) dx$$
$$= \int_{\Omega} (-\omega_1 Z_S(T,x) - \omega_2 Z_I(T,x)) dx$$

By the Divergence theorem and the homogeneous Neumann boundary conditions, we have

$$\begin{aligned} &\int_0^T \int_{\Omega} (-d_1 \Delta S^* Z_S - d_2 \Delta I^* Z_I - d_3 \Delta R^* Z_R - d_4 \Delta C^* Z_C) dx dt \\ &+ \int_0^T \int_{\Omega} (d_1 \Delta Z_S S^* + d_2 \Delta Z_I I^* + d_3 \Delta Z_R R^* + d_4 \Delta Z_C C^*) dx dt = 0. \end{aligned} \tag{5.36}$$

Substituting (5.35) and (5.36) in (5.34), one obtains

$$\begin{aligned} &\int_{\Omega} (\omega_1 Z_S(T,x) + \omega_2 Z_I(T,x)) dx \\ &= -\int_0^T \int_{\Omega} (\lambda_1 Z_S + \lambda_2 Z_I - \widetilde{v}_1 \overline{S} S^* - \widetilde{v}_2 \overline{I} I^* + \widetilde{v}_1 \overline{S} R^* + \widetilde{v}_2 \overline{I} R^*) dx dt. \end{aligned} \tag{5.37}$$

Substituting (5.37) in (5.30), we get

$$\begin{aligned} &\int_0^T \int_{\Omega} (\overline{S} S^* - \overline{S} R^* + \kappa_1) \widetilde{v}_1(t,x) dx dt + \int_0^T \int_{\Omega} (\overline{I} I^* - \overline{I} R^* + \kappa_2) \widetilde{v}_2(t,x) dx dt \\ &\geq -\int_{\Omega} \rho_1 \widetilde{v}_1(T,x) dx - \int_{\Omega} \rho_2 \widetilde{v}_2(T,x) dx. \end{aligned} \tag{5.38}$$

Since $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in L^2(\Omega_T)$ is arbitrary, we can take $\widetilde{v}_1 = v_1 - \overline{v}_1$, $\widetilde{v}_2 = v_2 - \overline{v}_2$ in (5.38), for any $v = (v_1, v_2) \in \mathcal{U}$, then

$$\begin{aligned} &\int_0^T \int_{\Omega} (\overline{S} S^* - \overline{S} R^* + \kappa_1)(v_1 - \overline{v}_1)(t,x) dx dt \\ &+ \int_0^T \int_{\Omega} (\overline{I} I^* - \overline{I} R^* + \kappa_2)(v_2 - \overline{v}_2)(t,x) dx dt \\ &\geq -\int_{\Omega} \rho_1 (v_1 - \overline{v}_1)(T,x) dx - \int_{\Omega} \rho_2 (v_2 - \overline{v}_2)(T,x) dx. \end{aligned} \tag{5.39}$$

Therefore, the inequality (5.24) is proved.

Particularly, if $\rho_1(x) = \rho_2(x) \equiv 0$ in Ω, then from (5.39) we have for any $v = (v_1, v_2) \in \mathcal{U}$

$$\begin{aligned} &\int_0^T \int_{\Omega} (\overline{S} S^* - \overline{S} R^* + \kappa_1)(v_1 - \overline{v}_1)(t,x) dx dt \\ &+ \int_0^T \int_{\Omega} (\overline{I} I^* - \overline{I} R^* + \kappa_2)(v_2 - \overline{v}_2)(t,x) dx dt \geq 0. \end{aligned} \tag{5.40}$$

Obviously, the inequality (5.40) holds for any $v = (v_1, v_2) \in \mathcal{U}$ is equivalent to

$$\int_0^T \int_{\Omega} (\overline{S} S^* - \overline{S} R^* + \kappa_1)(v_1 - \overline{v}_1)(t,x) dx dt \geq 0 \tag{5.41}$$

and

$$\int_0^T \int_\Omega (\bar{I}I^* - \bar{I}R^* + \kappa_2)(\nu_2 - \bar{\nu}_2)(t,x)dxdt \geq 0 \tag{5.42}$$

for any $\nu = (\nu_1, \nu_2) \in \mathcal{U}$. We consider the inequality (5.41). Let $\Omega_1 = \{(t,x) \in \Omega_T : (\bar{S}S^* - \bar{S}R^* + \kappa_1)(t,x) \leq 0\}$ and $\Omega_2 = \{(t,x) \in \Omega_T : (\bar{S}S^* - \bar{S}R^* + \kappa_1)(t,x) > 0\}$. Then, we have

$$\begin{aligned}&\int_0^T \int_\Omega (\bar{S}S^* - \bar{S}R^* + \kappa_1)(\nu_1 - \bar{\nu}_1)(t,x)dxdt \\ &= \int_{\Omega_1} (\bar{S}S^* - \bar{S}R^* + \kappa_1)(\nu_1 - \bar{\nu}_1)(t,x)dxdt + \int_{\Omega_2} (\bar{S}S^* - \bar{S}R^* + \kappa_1)(\nu_1 - \bar{\nu}_1)(t,x)dxdt.\end{aligned}$$

We easily prove that the inequality (5.41) holds for any $\nu = (\nu_1, \nu_2) \in \mathcal{U}$ is equivalent to

$$\int_{\Omega_1} (\bar{S}S^* - \bar{S}R^* + \kappa_1)(\nu_1 - \bar{\nu}_1)(t,x)dxdt \geq 0$$

and

$$\int_{\Omega_2} (\bar{S}S^* - \bar{S}R^* + \kappa_1)(\nu_1 - \bar{\nu}_1)(t,x)dxdt \geq 0$$

for any $\nu = (\nu_1, \nu_2) \in \mathcal{U}$, which further is equivalent to $\nu_1 - \bar{\nu}_1 \leq 0$ for all $(t,x) \in \Omega_1$ and $\nu_1 - \bar{\nu}_1 \geq 0$ for all $(t,x) \in \Omega_2$ for any $\nu = (\nu_1, \nu_2) \in \mathcal{U}$. And then, we can only obtain that $\bar{\nu}_1 \equiv 1$ for all $(t,x) \in \Omega_1$ and $\bar{\nu}_1 \equiv 0$ for all $(t,x) \in \Omega_2$. Therefore, we finally can obtain that

$$\bar{\nu}_1(t,x) = \begin{cases} 1, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\bar{S}S^* - \bar{S}R^* + \kappa_1)(t,x) \leq 0\}, \\ 0, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\bar{S}S^* - \bar{S}R^* + \kappa_1)(t,x) > 0\}. \end{cases}$$

Similarly, based on (5.42), we also can obtain that

$$\bar{\nu}_2(t,x) = \begin{cases} 1, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\bar{I}I^* - \bar{I}R^* + \kappa_2)(t,x) \leq 0\}, \\ 0, & \textit{if } (t,x) \in \{(t,x) \in \Omega_T : (\bar{I}I^* - \bar{I}R^* + \kappa_2)(t,x) > 0\}. \end{cases}$$

This completes the proof. □

6 Numerical simulations

This section begins with a numerical example of influenza transmission with the cross-immune class that demonstrates the applications of the result established in Theorem 5.1 by using MATLAB. The solving the optimality system, which consists of eight PDEs from the state and adjoint equations, yields the optimal approach. The optimality system is solved using an iterative technique. The trapezoidal rule was used to calculate the objective function (1.7).

In order to discuss the influence of control strategies on the number of susceptible populations and infected populations. We start to solve the system (1.3)–(1.5) for control ν using the Crank–Nicolson finite difference method. It is well known that this scheme is numerically stable, with second-order accuracy in space and first-order accuracy in time.

Table 1 The values of parameters in control system (1.3)

Parameter	Value	Source
d_1	0.002	Assumed
d_2	0.0125	Assumed
d_3	0.009	Assumed
d_4	0.006	Assumed
A	$\frac{10000}{75\times52\times7}$	[33]
μ	$\frac{1}{75\times52\times7}$	[33]
β_1	$0.8\times10^4(1.1+0.5\cos(2\pi x))$	[33]
β_2	$0.5\times10^4(1.1+0.5\cos(2\pi x))$	Assumed
α	0.25	[5]
γ	$\frac{1}{1.5\times52\times7}$	[33]
δ	0.625	[34]
σ	0.078	[34]

The adjoint system (5.1) is solved by the Crank–Nicolson scheme backward in time using the current iteration solution of the system (1.3)–(1.5). Then the control $v = (v_1, v_2)$ are updated by using the relation given in Theorem 5.1. The iterative process continues until the difference in the current and previous values for the states, adjoint variables, and control variables are within an acceptable error range.

Example 1 For simplicity, we take all parameters in the control system (1.3) and the adjoint system (5.1) to be positive constants, except for the parameters β_1 and β_2, which are positive continuous functions, and the spatiotemporal domain $\Omega_T = (0,T)\times\Omega = (0,1)\times(0,1)$. All parameters of the model (1.3) are shown in Table 1.

Besides, we take the following initial levels for the control system (1.3)–(1.5) in our numerical simulations: $S(0,x) = 0.03 + 0.1|0.1 + 0.32\sin(10x)|$, $I(0,x) = x(1-x)e^{4x}$, $R(0,x) = 0.05(1 + x(1-x)^2(2-x)^3)$ and $C(0,x) = 0.02(1 + x(1-x)^2(2-x)^3)$ for all $x \in [0,1]$. We also take the following weight parameters in the objective function (1.7)–(1.8) in our numerical simulations: $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.02$, $\omega_1 = 0.15$, $\omega_2 = 0.02$, $\rho_1 = 0$ and $\rho_2 = 0$.

The computational results of the variation profiles of control system, adjoint system and the optimal control of interactive process are presented in Figs. 1–3, respectively. It is not difficult to see from Fig. 3 that the optimal control $\bar{v}_1$ and $\bar{v}_2$ are all of the Bang-Bang form.

Example 2 To illustrate our results, we choose the parameters of model (1.3) are positive continuous function. For simplicity, we take diffusive coefficients in the control system (1.3) and adjoint system (5.1) to be positive constants. All parameters in model (1.3) are shown in Table 2.

We consider the spatiotemporal domain $\Omega_T = (0,T)\times\Omega = (0,1)\times(0,1)$. Besides, we have used the following initial levels for the control system (1.3)–(1.5) in our numerical simulations: $S(0,x) = 0.1 + 0.1|0.1 + 0.32sin(10x)|$, $I(0,x) = x(1-x)e^{2x}$, $R(0,x) = 0.05(1 + x(1-x)^2(2-x)^3)$, $C(0,x) = 0.02(1+x(1-x)^2(2-x)^3)$, $x \in [0,1]$. We have used the following weight parameter in the objective function (1.7)–(1.8) in our numerical simulations: $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\kappa_1 = 0.02$, $\kappa_2 = 0.03$, $\omega_1 = 0.2$, $\omega_2 = 0.04$, $\rho_1 = 0$, $\rho_2 = 0$.

For this example, the computational results of the variation profiles of control system, adjoint system and the optimal control of interactive process are presented in Figs. 4–6, respectively. It is not difficult to see from Fig. 6 that the optimal control $\bar{v}_1$ and $\bar{v}_2$ are all of the Bang-Bang form.

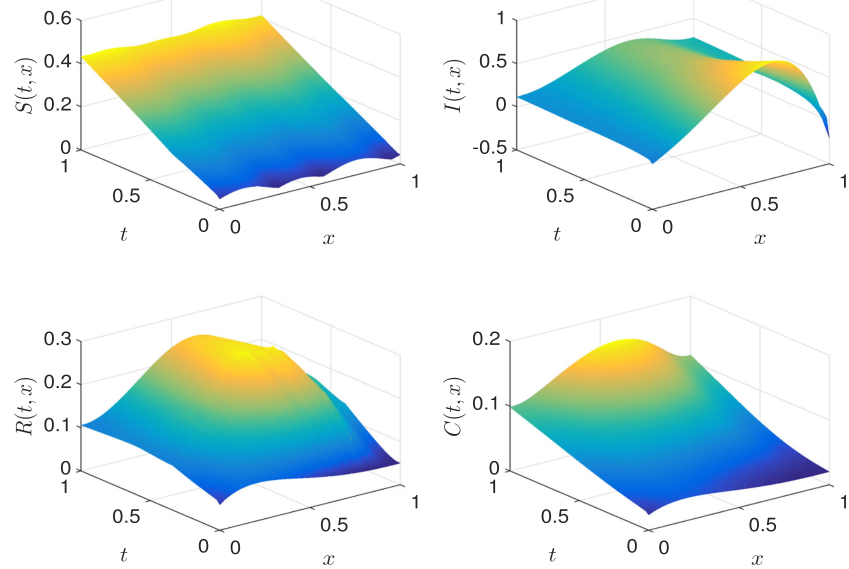

Figure 1 Time evolution of solutions $(S(t,x), I(t,x), R(t,x), C(t,x))$

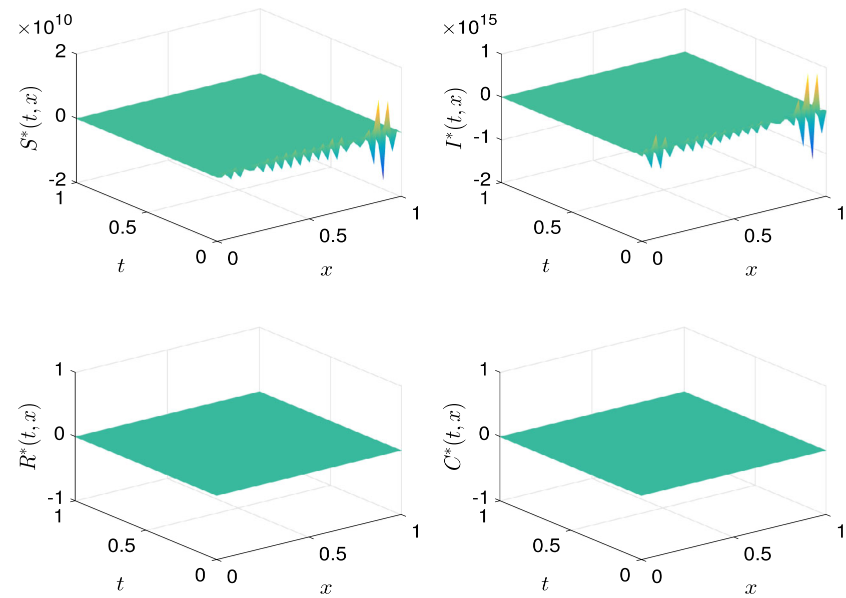

Figure 2 Time evolution of solutions $(S^*(t,x), I^*(t,x), R^*(t,x), C^*(t,x))$

Example 3 We consider the spatiotemporal domain $\Omega_T = (0,T) \times \Omega = (0,1) \times (0,1)$, and all parameter of model (1.3) are shown in Table 3.

Besides, we have used the following initial levels for the control system (1.3)–(1.5) in our numerical simulations: $S(0,x) = 0.03 + 0.1|0.1 + 0.3sin(10x)|$, $I(0,x) = x(1-x)e^{2x}$, $R(0,x) = 0.05(1 + x(1-x)^2(2-x)^3)$, $C(0,x) = 0.02(1 + x(1-x)^2(2-x)^3)$, $x \in [0,1]$. We have

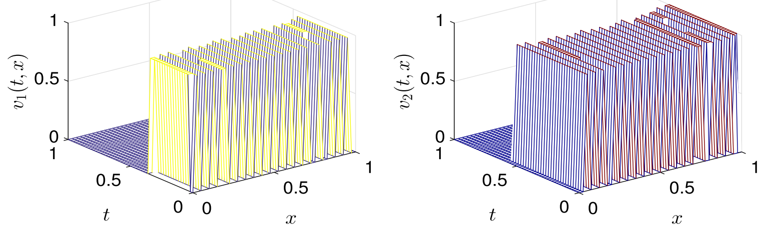

Figure 3 The optimal controls on $\Omega = [0,1] \times [0,1]$

Table 2 The values of parameters in control system (1.3)

Parameter	Value	Parameter	Value
d_1	0.003	d_2	0.0155
d_3	0.009	d_4	0.008
A(x)	$0.4 + 2sin(2\pi x)$	$\mu(x)$	0.02
$\beta_1(x)$	$0.03 + +0.0002sin(2\pi x)$	$\beta_2(x)$	$0.02 + 0.0001sin(2\pi x)$
$\alpha(x)$	$0.005 + 0.25sin(2\pi x)$	$\gamma(x)$	$0.35 + 0.005sin(2\pi x)$
$\delta(x)$	$0.625 + 0.045sin(2\pi x)$	$\sigma(x)$	$0.07 + 0.02sin(2\pi x)$

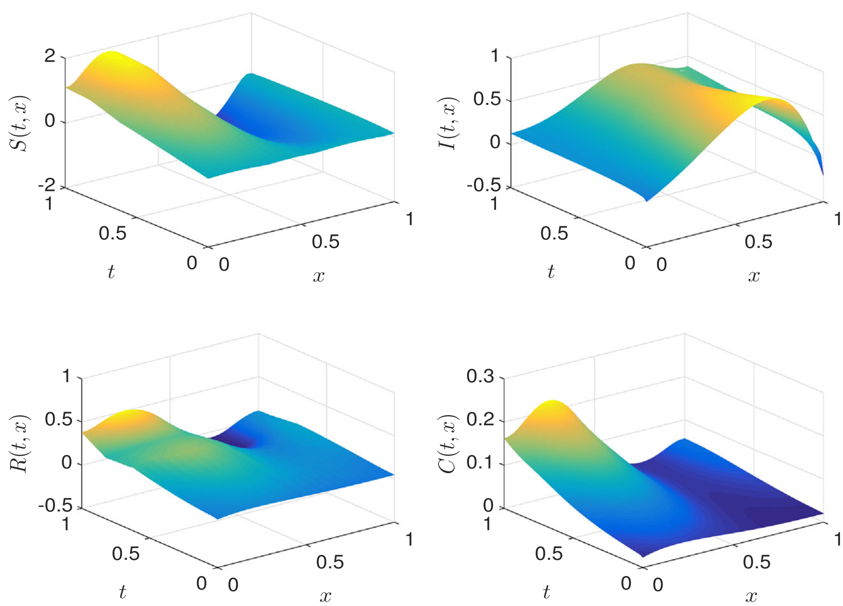

Figure 4 Time evolution of solutions $(S(t,x), I(t,x), R(t,x), C(t,x))$

used the following weight parameter in the objective function (1.7)–(1.8) in our numerical simulations: $\lambda_1 = 1$, $\lambda_2 = 1.2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.02$, $\omega_1 = 0.15$, $\omega_2 = 0.02$, $\rho_1 = 0$, $\rho_2 = 0$.

For this example, the computational results of the variation profiles of control system, adjoint system, and the optimal control of interactive process are presented in Figs. 7–9, respectively. It is not difficult to see from Fig. 9 that the optimal control $\bar{v}_1$ and $\bar{v}_2$ are all of the Bang-Bang form.

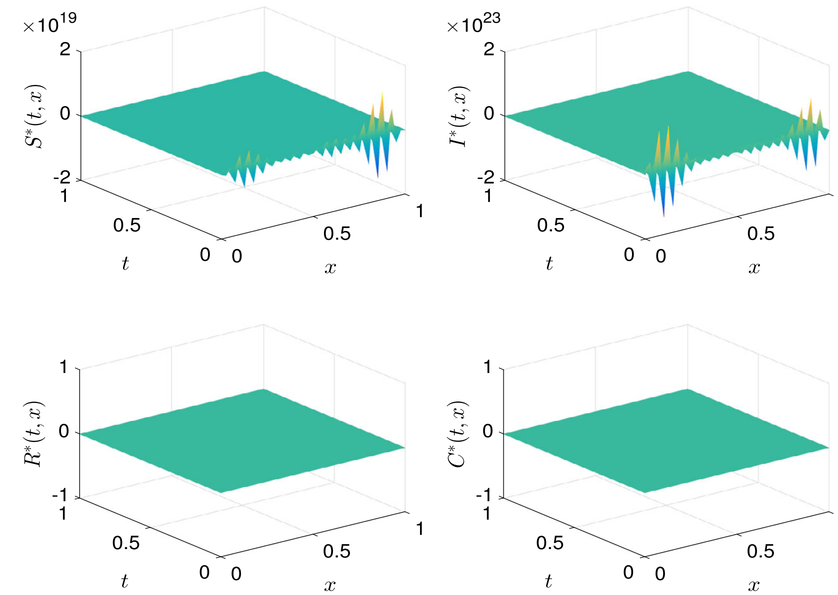

Figure 5 Time evolution of solutions $(S^*(t,x), I^*(t,x), R^*(t,x), C^*(t,x))$

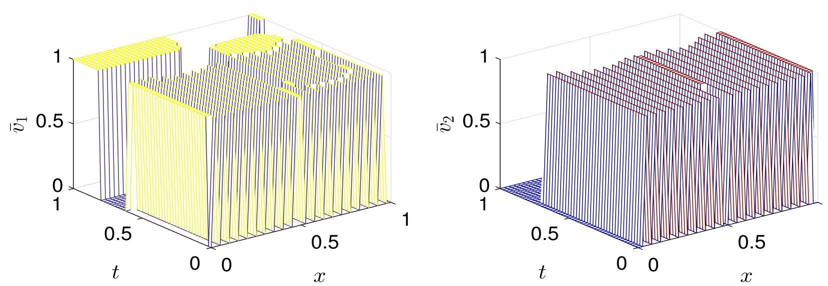

Figure 6 The optimal controls on $\Omega = [0,1] \times [0,1]$

Table 3 The values of parameters in control system (1.3)

Parameter	Value	Source
d_1	$0.002 + 0.005\sin(2\pi x)$	Assumed
d_2	$0.0125 + 0.005\sin(2\pi x)$	Assumed
d_3	$0.009 + 0.002\sin(2\pi x)$	Assumed
d_4	$0.006 + 0.003\sin(2\pi x)$	Assumed
A	$\frac{10000}{70\times52\times7} + 2\sin(2\pi x)$	Assumed
μ	$\frac{1}{70\times52\times7} + 0.05\sin(2\pi x)$	Assumed
β_1	$0.8 \times 10^4(1.1 + 0.5\cos(2\pi x))$	[33]
β_2	$0.5 \times 10^4(1.1 + 0.5\cos(2\pi x))$	Assumed
α	$0.25 + 0.25\sin(2\pi x)$	Assumed
γ	$\frac{1}{1.5\times52\times7} + 1.2\sin(2\pi x)$	Assumed
δ	$0.625 + 0.45\sin(2\pi x)$	Assumed
σ	$0.078 + 0.1\sin(2\pi x)$	Assumed

7 Conclusions

In this paper, we present a theoretical work on the optimal control problem for a class of reaction-diffusion SIRC epidemic models with the spatial spread of disease and cross-

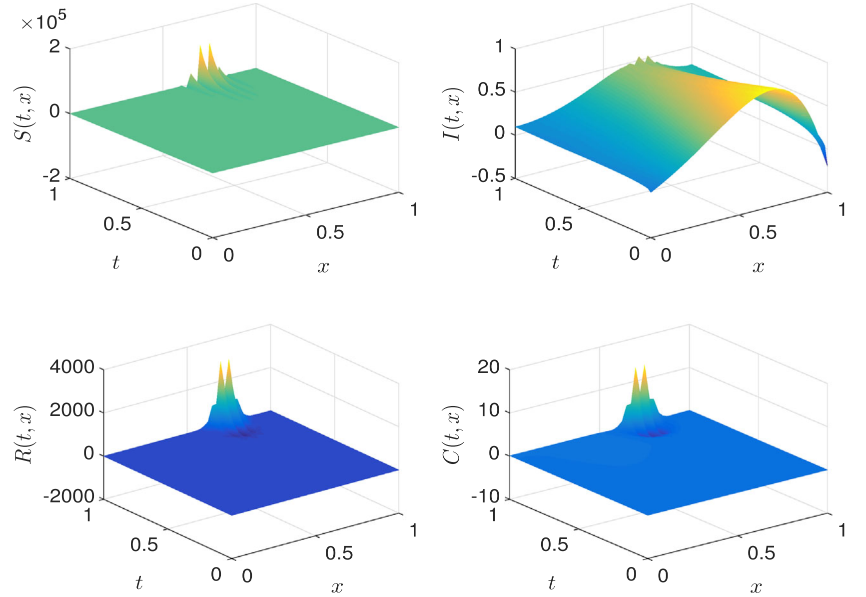

Figure 7 Time evolution of solutions $(S(t,x), I(t,x), R(t,x), C(t,x))$

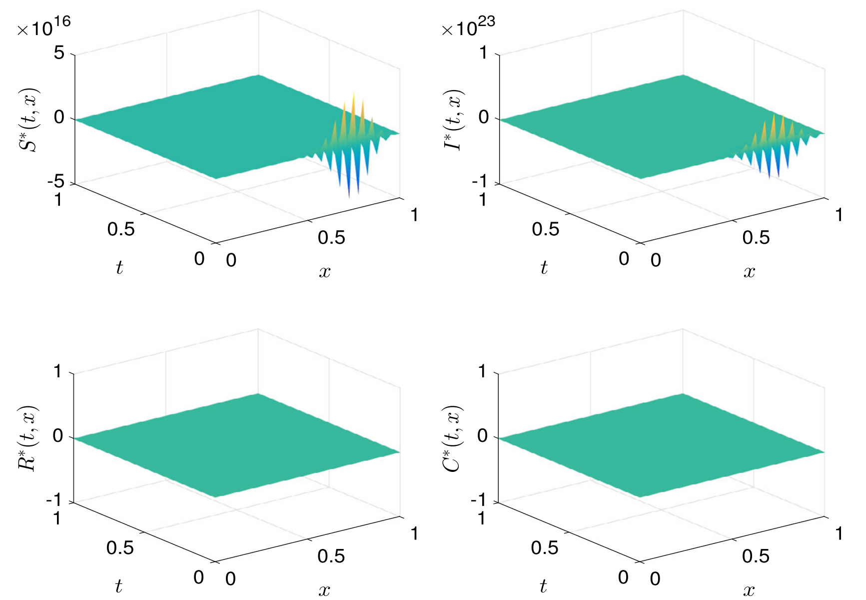

Figure 8 Time evolution of solutions $(S^*(t,x), I^*(t,x), R^*(t,x), C^*(t,x))$

immunization in order to study the effects of control strategies on susceptible and infected individuals. Due to the non-repeatability of infectious disease transmission experiments, our work has certain research significance. Firstly, based on the semigroup theories of operators and truncation function techniques, we have proved the existence and uniqueness of global positive strong solution in the control system (1.3)–(1.5). Secondly, the existence of the optimal control problem has been established using an effective method based on some properties within the weak topology. Then, the first-order necessary conditions for

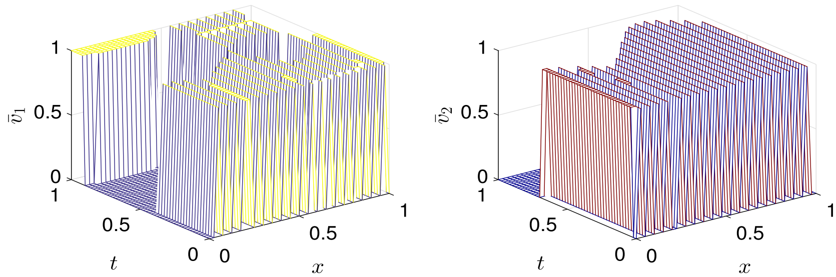

Figure 9 The optimal controls on $\Omega = [0,1] \times [0,1]$

the optimal control have been obtained. It is worth mentioning that we have found that the optimal control $\overline{v}_1$ and $\overline{v}_2$ are all of the Bang-Bang form.

Different from the existing results, the diffusion behaviors of individuals have been considered in the model and two control strategies for vaccines and treatment were considered at the same time. There is no doubt that our work is not perfect, and there are still some interesting open problem that we need to look into in the future. For example, an open question is that we consider a bilinear incidence function, which is a simple incidence function in the model (1.3), therefore, in the future, we will also consider incorporating the nonlinear incidence rate function, such as saturation incidence $\frac{\beta SI}{1+\alpha I}$, Beddington–DeAngelis incidence $\frac{\beta SI}{1+\omega S+\alpha I}$, general nonlinear incidence $\beta Sf(I)$ and $\beta f(S,I)$ and soon on, they are more realistic and achieve more exact results. In this article, we mainly give the theoretical results for optimal control of the reaction-diffusion SIRC infectious disease model, so another open problem is to combine the theoretical results of the optimal control with the specific disease to give certain suggestions for prevention and control, which is what we should further study.

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Author contributions
Pan Zhou: Mathematical analysis, numerical simulation and writing. Jianpeng Wang and Yanling Zheng: Mathematical analysis. Kai Wang: Modelling, data collection and numerical simulation. Zhidong Teng: Modelling, review and editing. All authors read and approved the final manuscript.

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Data availability
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current research.

Declarations

Competing interests
The authors declare no competing interests.

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