Advances in Continuous and Discrete Models

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Ergodicity of a stationary distribution for a stochastic cholera model with a general functional response and higher-order perturbation

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Abstract

A general stochastic compartment model for cholera with higher-order perturbation is proposed, which incorporates direct and indirect transmission by contaminated water. Nonlinear incidence, multiple stages of infection, multiple states of pathogen, and second-order white-noises perturbation are introduced into the model, which includes and extends the existing cholera model. The existence and ergodicity of the stationary distribution for the cholera system are obtained by constructing a suitable Lyapunov function, which determines a sharp critical value R_0^s corresponding to the basic productive number R_0 of the ordinary differential equation. The results show that, if $R_0^s > 1$, the system has a unique and ergodic stationary distribution, which implies the persistence of the diseases. Our general results are applied to a cholera system with a Holling type-II functional response.

Keywords: Stochastic cholera model; Nonlinear incidence; Stationary distribution and ergodictiy; Higher-order perturbation

1 Introduction

Cholera is an acute diarrheal infection, which is caused by the ingestion of food or water contaminated with Vibrio cholerae. For public health, cholera remains a global threat, reflecting injustice and social underdevelopment. Researchers estimate that cholera causes approximately 1.3 to 4 million cases and 21,000 to 143,000 deaths worldwide each year [1]. Cholera is spread in various ways, including water transmission, food transmission, life contact transmission, vector insect transmission, see [2–4]. A multipronged approach is the key to preventing cholera and reducing deaths [5–9].

Establishing mathematical modeling of infectious diseases can better help understand the pathology of transmission and analyze the factors that affect disease transmission. In this process, it can promote the improvement of public health and find the best control strategy. Many experts have proposed epidemiological models of the transmission mechanism of cholera, and the influence between people and the environment is fully consid-

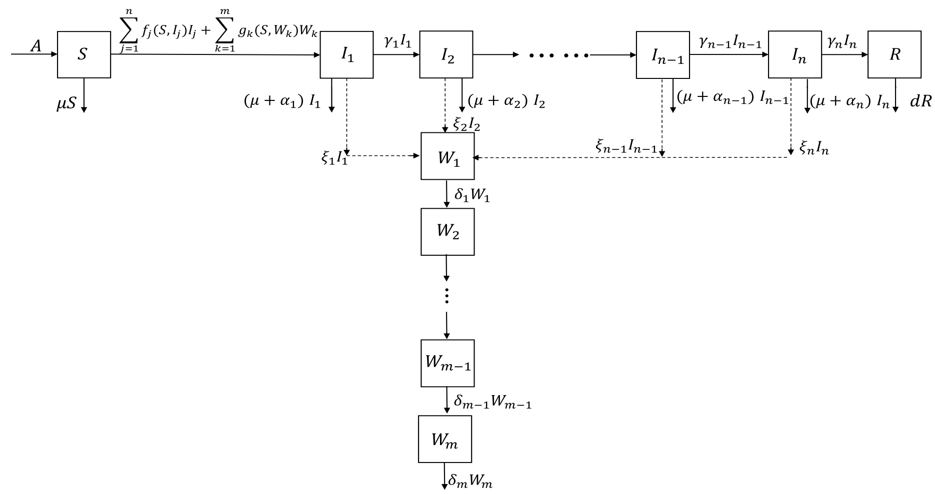

Figure 1 The transfer diagram for model (1.1)

ered, such as [10–15]. In particular, considering the nonlinear morbidity, multistage infection and multiple states of pathogen factors, Shuai et al. [10] divided the total number of people into $n + m + 2$ compartments: susceptible population S, n infected compartments $I_1, I_2, \dots, I_n$ as n latent stages, m contaminated water W_1, W_2, $\dots$, W_m according to the pathogen concentration, and the removed individuals R. In the stages $I_i, i = 1, 2, \dots, n$, the infectivity of individuals is assumed to be zero. Pathogens shed from infectious individuals in each infection stage enter W_1, then progress to W_2 and so on, see Fig. 1 for the flow diagram of this model. The incidence function is assumed to be of the form $\sum_{j=1}^{n} f_j(S, I_j)I_j + \sum_{k=1}^{m} g_k(S, W_k)W_k$, where f_j and g_k represent direct transmission and indirect transmission, respectively. Based on the above assumptions, Shuai et al. [10] established a general model with two transmission routes:

$$
\begin{aligned}
&\dot{S}(t) = A - \sum_{j=1}^{n} f_j(S, I_j)I_j - \sum_{k=1}^{m} g_k(S, W_k)W_k - \mu S, \\
&\dot{I}_1(t) = \sum_{j=1}^{n} f_j(S, I_j)I_j + \sum_{k=1}^{m} g_k(S, W_k)W_k - (\mu + \gamma_1 + \alpha_1)I_1, \\
&\dot{I}_i(t) = \gamma_{i-1}I_{i-1} - (\mu + \gamma_i + \alpha_i)I_i, \quad i = 2, 3, \dots, n, \\
&\dot{W}_1(t) = \sum_{j=1}^{n} \xi_j I_j - \delta_1 W_1, \\
&\dot{W}_k(t) = \delta_{k-1} W_{k-1} - \delta_k W_k, \quad k = 2, 3, \dots, m
\end{aligned}
\tag{1.1}
$$

and

$$
\dot{R}(t) = \gamma_n I_n - dR. \tag{1.2}
$$

Table 1 lists the definitions of the above parameters.

Since R has no effect on the dynamic behavior of other individuals, Eq. (1.2) can be eliminated. Next, we do not consider $R(t)$ throughout the article. From [10, 16], system (1.1)

Table 1 The meanings of the parameters

Symbol	Description
A	the constant recruitment
$f_j I_j, j = 1, \dots, n$	direct transmission
$g_k W_k, k = 1, \dots, m$	indirect transmission
μ	the natural mortality rate
$\gamma_i, i = 1, \dots, n-1$	the transition rates of infectious individuals from stage I_i to I_{i+1}
γ_n	the recovery rate of I_n
$\alpha_i, i = 1, \dots, n$	the mortality rates because of the disease in the ith infection stage
$\delta_k, k = 1, \dots, m-1$	the transition rates of pathogen from W_k to W_{k+1}
δ_m	the removal rate of W_m
$\xi_i, i = 1, \dots, n$	person–water pathogen shedding rates

has a unique disease-free equilibrium $P_0 = (S_0, 0, \dots, 0)$, which is globally asymptotically stable in the feasible region Γ when $R_0 \leq 1$, where $S_0 = \frac{A}{\mu}$,

$$\Gamma = \Bigg\{ (S, I_1, \dots, I_n, W_1, \dots, W_m) \in \mathbb{R}_+^{n+m+1} | \\ S + I_1 + \cdots + I_n \leq \frac{A}{\mu}, W_k \leq \frac{H}{\delta_k},\ k = 1, 2, \dots, m \Bigg\},$$

in which $H = \frac{A}{\mu} \sum_{j=1}^{n} \xi_j$. Also, if $R_0 > 1$, $P_0 = (S_0, 0, \dots, 0)$ is unstable, and there exists an endemic equilibrium $P^* = (S^*, I_1^*, \dots, W_m^*)$, which is globally asymptotically stable in Γ, where

$$\begin{aligned} R_0 :=& \frac{f_1(S_0, 0)}{\mu_1} + \frac{\gamma_1 f_2(S_0, 0)}{\mu_1 \mu_2} + \cdots + \frac{\gamma_1 \gamma_2 \cdots \gamma_{n-1} f_n(S_0, 0)}{\mu_1 \mu_2 \cdots \mu_n} \\ &+ \left(\sum_{i=1}^{m} \frac{g_i(S_0, 0)}{\delta_i} \right) \left(\frac{\xi_1}{\mu_1} + \frac{\gamma_1 \xi_2}{\mu_1 \mu_2} + \cdots + \frac{\gamma_1 \gamma_2 \cdots \gamma_{n-1} \xi_n}{\mu_1 \mu_2 \cdots \mu_n} \right) \end{aligned} \tag{1.3}$$

and $S^*, I_1^*, \dots, I_n^*, W_1^*, \dots, W_m^* > 0$ satisfy the following equalities:

$$\begin{aligned} &A = \sum_{j=1}^{n} f_j(S^*, I_j^*) I_j^* + \sum_{k=1}^{m} g_k(S^*, W_k^*) W_k^* + \mu S^*, \\ &\mu_1 I_1^* = \sum_{j=1}^{n} f_j(S^*, I_j^*) I_j^* + \sum_{k=1}^{m} g_k(S^*, W_k^*) W_k^*, \\ &\mu_i I_i^* = \gamma_{i-1} I_{i-1}^*, \quad i = 2, 3, \dots, n, \\ &\delta_1 W_1^* = \sum_{j=1}^{n} \xi_j I_j^*, \\ &\delta_k W_k^* = \delta_{k-1} W_{k-1}^*, \quad k = 2, 3, \dots, m, \end{aligned}$$

where $\mu_i = \mu + \gamma_i + \alpha_i$, $i = 1, 2, \dots, n$.

Moreover, since the epidemic system will be inevitably affected by environmental white noises in the process of transmission, some authors introduced the white noises into the population systems to reveal richer and more complex dynamics, (for example, see [17–23]). However, we note that these stochastic models are disturbed by linear white noises. As is well known, only a little work is shown for the second-order white-noises perturbation, (see [24] and [25]), which is more in line with the actual spread and development of infectious diseases. In particular, Song et al. [21] investigated a special case of system (1.1) ($f_j(S,I_j) = \beta_j S$, $g_k(S,W_k) = \lambda_k S$) disturbed by linear white noises and the critical value R_0^s is obtained. In addition, some authors studied the existence of a stationary distribution for epidemic models with the Ornstein–Uhlenbeck process, see [26, 27].

In this article, we extend the work of Song et al. [21] and Shuai et al. [10, 16] and consider the general case of $f_j(S,I_j)$ and $g_k(S,W_k)$ for system (1.1). In addition, we adopt a different approach to introduce random perturbations into it by replacing the parameters $-\mu$, $-\mu_i$ and $-\sigma_i$ with

$$
\begin{aligned}
&-\mu \to -\mu + (\sigma_{11} + \sigma_{12}S)\dot{B}_1(t), -\mu_i \to -\mu_i + (\sigma_{i+1,1} + \sigma_{i+1,2}I_i)\dot{B}_{i+1}(t),\ i = 1,2,\dots,n,\\
&-\sigma_i \to -\sigma_i + (\sigma_{n+k+1,1} + \sigma_{n+k+1,2}W_k)\dot{B}_{n+k+1}(t),\ k = 1,2,\dots m,
\end{aligned}
$$

where $B = (B_1(t), B_2(t), \dots, B_{n+m+1}(t))(t \geq 0)$ is a real-valued Brownian motion, $\sigma_{ij}^2 (i = 1,2,\dots,n+m+1;\ j = 1,2)$ denote the intensities of the white noises. Here, we introduce the second-order white-noise disturbance since the random perturbation may be dependent on the state variables S, I_i, W_k, which better reflects the reality in biology. In view of the above, we consider the following system:

$$
\begin{aligned}
dS(t) &= (A - \sum_{j=1}^{n} f_j(S,I_j)I_j - \sum_{k=1}^{m} g_k(S,W_k)W_k - \mu S)dt + (\sigma_{11} + \sigma_{12}S)SdB_1(t),\\
dI_1(t) &= (\sum_{j=1}^{n} f_j(S,I_j)I_j + \sum_{k=1}^{m} g_k(S,W_k)W_k - \mu_1 I_1)dt + (\sigma_{21} + \sigma_{22}I_1)I_1 dB_2(t),\\
dI_i(t) &= (\gamma_{i-1}I_{i-1} - \mu_i I_i)dt + (\sigma_{i+1,1} + \sigma_{i+1,2}I_i)I_i dB_{i+1}(t), \quad 2 \leq i \leq n,\\
dW_1(t) &= (\sum_{j=1}^{n} \xi_j I_j - \delta_1 W_1)dt + (\sigma_{n+2,1} + \sigma_{n+2,2}W_1)W_1 dB_{n+2}(t),\\
dW_k(t) &= (\delta_{k-1}W_{k-1} - \delta_k W_k)dt + (\sigma_{n+k+1,1} + \sigma_{n+k+1,2}W_k)W_k dB_{n+k+1}(t),\ 2 \leq k \leq m.
\end{aligned}
\tag{1.4}
$$

For biological reality and technical reasons, f_j, $j = 1,2,\dots,n$ and g_k, $k = 1,2,\dots,m$ satisfy the following assumptions throughout the article:

(H_1) there exists a $K > 0$ such that $\dfrac{\partial f_j(S,x)}{\partial x} \geq -Kf_j(S,x)$, $\dfrac{\partial g_k(S,x)}{\partial x} \geq -Kg_k(S,x)$;

(H_2) there exists a $M > 0$ such that $f_j(S,x) \leq MS$, $g_k(S,x) \leq MS$;

(H_3) there exists $p_j \in \mathbb{R}, j = 1,2,\dots,n$; $q_k \in \mathbb{R}, k = 1,2,\dots,m$ such that

$$
S^3 \frac{d^2}{dS^2}\left(\frac{1}{f_j(S,0)}\right) \leq p_j,\ S^3 \frac{d^2}{dS^2}\left(\frac{1}{g_k(S,0)}\right) \leq q_k.
$$

In addition, our results can be applied to a vast range of common functional response terms. For example:

(1) Bilinear incidence rate: $f_j(I_j)SI_j = \beta_j SI_j$, $j = 1,2,\dots,n$; $g_k(W_k)SW_k = \lambda_k SW_k$, $k = 1,2,\dots,m$, which was studied by Song et al. [21] with $\sigma_{i,2} = 0$, $i = 1,2,\dots,n+m+1$;

(2) Holling type-II saturation incidence rate:

$$f_j(S,I_j)I_j = \frac{\beta_j SI_j}{1+aI_j},\ j = 1,2,\dots,n;\ \ g_k(S,W_k)W_k = \frac{\lambda_k SW_k}{1+aW_k},\ k = 1,2,\dots,m;$$

$$f_j(S,I_j)I_j = \frac{\beta_j SI_j}{1+aS},\ j = 1,2,\dots,n;\ \ g_k(S,W_k)W_k = \frac{\lambda_k SW_k}{1+aS},\ k = 1,2,\dots,m;$$

(3) Holling type-IV incidence rate:

$$f_j(S,I_j)I_j = \frac{\beta_j SI_j}{1+aS^2},\ j = 1,2,\dots,n;\ \ g_k(S,W_k)W_k = \frac{\lambda_k SW_k}{1+aS^2},\ k = 1,2,\dots,m;$$

$$f_j(S,I_j)I_j = \frac{\beta_j SI_j}{1+aI_j^2},\ j = 1,2,\dots,n;\ \ g_k(S,W_k)W_k = \frac{\lambda_k SW_k}{1+aW_k^2},\ k = 1,2,\dots,m,$$

where β_j, λ_k are positive constants, and a represents the half-saturation constant. It is easy to check that f_j, g_k in (1)–(3) satisfy the conditions (H1)–(H3).

Based on the theory of Khasminskii [28], we mainly investigate the existence and ergodicity of the stationary distribution for the high-dimensional system (1.4) by constructing a suitable Lyapunov function, which determines the exact critical value R_0^s corresponding to the ordinary differential system (1.1). Our major innovation is to construct a suitable Lyapunov function for the high-dimensional complex system with a general functional response and high-order perturbation, which provides a new technique and a clear view.

The rest of this article is organized as follows. The existence and ergodicity of the stationary distribution for a stochastic system (1.4) are proved in Sect. 2; then, in Sect. 3, our results are applied to a stochastic cholera model with Holling type-II functional response terms, and we make a comparison with the existing results and numerical simulations are given to illustrate our results. We make a conclusion with further discussion in Sect. 4.

In this paper, let $B_i(t)$, $i = 1,2,\dots,n+m+1$ be defined on the complete probability space. Define

$$\mathbb{R}_+^{n+m+1} = \{(x_1,x_2,\dots,x_{n+m+1}) : x_i > 0, 1 \le i \le m+n+1\}$$

and let $C^2(\mathbb{R}_+^{n+m+1}; R_+)$ be the set of all nonnegative functions $V(x)$ on $\mathbb{R}_+^{n+m+1}$, which are continuously twice differentiable in x.

2 Existence and ergodicity of a stationary distribution of system (1.4)

The main research content of this section is the existence and ergodicity of the stationary distribution of system (1.4) based on the theory of Khasminskii [28], which implies the diseases will be prevalent. In the beginning, the existence and uniqueness of the global positive solution of system (1.4) will be given, which provides the basis for the following research.

Theorem 2.1 *Assume the conditions* $(H1)$–$(H3)$ *hold. For any given initial value* $(S(0), I_1(0), \dots, I_n(0), W_1(0), \dots, W_m(0)) \in \mathbb{R}_+^{n+m+1}$, *there is a unique global positive solution* $(S(t), I_1(t), \dots, I_n(t), W_1(t), \dots, W_m(t))$ *of system* (1.4) *for all* $t \geq 0$, *which remains on* $\mathbb{R}_+^{n+m+1}$ *with probability one.*

Proof The theorem is easily proved according to the classical Khasminskii–Lyapunov functional method [28], and the standard method is similar to the proof of Theorem 2.1 of [21]. Hence, we omit it here. □

Some lemmas are given before presenting the main content of this section.

Lemma 2.1 *For any* $x \geq 0$, *the following two inequalities are established*:
(a) $x^3 \geq \left(x - \frac{1}{2}\right)\left(x^2 + 1\right)$; (b) $x^4 \geq \left(\frac{3}{4}x^2 - \frac{1}{4}\right)\left(x^2 + 1\right)$.

Proof (i) Noting

$$2x^3 - (2x - 1)\left(x^2 + 1\right) = 2x^3 - 2x^3 - 2x + x^2 + 1 = (x - 1)^2 \geq 0,$$

(a) is verified.

(ii) Also,

$$4x^4 - \left(3x^2 - 1\right)\left(x^2 + 1\right) = 4x^4 - 3x^4 - 3x^2 + x^2 + 1 = \left(x^2 - 1\right)^2 \geq 0,$$

thus (b) is confirmed. □

Next, we give our main result.

Define

$$\begin{aligned} R_0^s := &\left(\frac{f_1(S_0, 0)(1 - d_1 \frac{A}{\mu} \tilde{h}_1)}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} + \sum_{i=2}^{n} \frac{f_i(S_0, 0)(1 - d_i \frac{A}{\mu} \tilde{h}_1)}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} \prod_{j=2}^{i} \frac{\gamma_{j-1}}{\mu_j + \frac{\sigma_{j+1,1}^2}{2}} \right) \\ &+ \left(\frac{g_1(S_0, 0)(1 - n_1 \frac{A}{\mu} \tilde{h}_1)}{\delta_1 + \frac{\sigma_{n+2,1}^2}{2}} + \sum_{k=2}^{m} \frac{g_k(S_0, 0)(1 - n_k \frac{A}{\mu} \tilde{h}_1)}{\delta_1 + \frac{\sigma_{n+2,1}^2}{2}} \prod_{j=2}^{k} \frac{\delta_{j-1}}{\delta_j + \frac{\sigma_{n+j+1,1}^2}{2}} \right) \cdot \\ &\left(\frac{\xi_1}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} + \sum_{i=2}^{n} \frac{\xi_i}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} \prod_{j=2}^{i} \frac{\gamma_{j-1}}{\mu_j + \frac{\sigma_{j+1,1}^2}{2}} \right), \end{aligned} \tag{2.1}$$

where $\tilde{h}_1 = 2\sqrt[3]{A^2\sigma_{12}^2} + 2\sqrt{A\sigma_{11}\sigma_{12}} + \frac{\sigma_{11}^2}{2}$, and

$$d_i > \max\left\{0, \frac{f_i(S_0, 0)p_i}{2\mu S_0^2}\right\},\ i = 1, 2, \dots, n,\ n_k > \max\left\{0, \frac{g_k(S_0, 0)q_k}{2\mu S_0^2}\right\},\ k = 1, 2, \dots, m. \tag{2.2}$$

Theorem 2.2 *Assume the conditions* $(H1)$–$(H3)$ *hold and* $R_0^s > 1$, *where* R_0^s *is defined by* (2.1), *then the system* (1.4) *has a unique stationary distribution* $\pi(\cdot)$, *which is ergodic.*

Proof We can verify Theorem 2.2 by validating conditions (A1) and (A2) in Lemma A.1 in the Appendix. Based on Lemma A.1, we need to find a nonnegative function $V \in C^2(\mathbb{R}_+^{n+m+1}; R_+)$ and a compact set $U \subset \mathbb{R}_+^{n+m+1}$ such that

$$LV(S, I_1, \ldots, I_n, W_1, \ldots, W_m) \le -1 \text{ on } (S, I_1, \ldots, I_n, W_1, \ldots, W_m) \in \mathbb{R}_+^{m+n+1}/U.$$

The construction of $V(S, I_1, \ldots, I_n, W_1, \ldots, W_m)$ is very complex, so we proceed in three main steps: (i) construct a stochastic Lyapunov function; (ii) construct a compact set; (iii) give the existence and ergodicity of the solution of system (1.4). Some transformations are shown as follows:

$$\tilde{I}_1 = \frac{I_1}{I_1^c}, \ldots, \tilde{I}_n = \frac{I_n}{I_n^c}, \tilde{W}_1 = \frac{W_1}{W_1^c}, \ldots, \tilde{W}_m = \frac{W_1}{W_m^c}, \tag{2.3}$$

where $I_1^c, I_2^c, \ldots, W_m^c$ satisfy the following equalities:

$$\begin{aligned}
&I_1^c = 1, \quad I_i^c = \frac{\gamma_{i-1} I_{i-1}^c}{\mu_i + \frac{\sigma_{i+1,1}^2}{2} + \frac{\sigma_{i+1,2}^2 p^2}{6}} := \frac{\gamma_{i-1} I_{i-1}^c}{h_{i+1}}, \quad i = 2, 3, \ldots, n, \\
&W_1^c = \frac{\sum_{i=1}^{n} \xi_i I_i^c}{\delta_1 + \frac{\sigma_{n+2,1}^2}{2} + \frac{\sigma_{n+2,2}^2 p^2}{6}} := \frac{\sum_{i=1}^{n} \xi_i I_i^c}{h_{n+2}}, \\
&W_k^c = \frac{\delta_{k-1} W_{k-1}^c}{\delta_k + \frac{\sigma_{n+k+1,1}^2}{2} + \frac{\sigma_{n+k+1,2}^2 p^2}{6}} := \frac{\delta_{k-1} W_{k-1}^c}{h_{n+k+1}}, \quad k = 2, 3, \ldots m,
\end{aligned} \tag{2.4}$$

where $0 < p < 1$.

Step 1. (Constructing a stochastic Lyapunov function)

Define a function $\hat{V}_1 \in C^2(R_+^{n+m+1}, R)$ by

$$\begin{aligned}
\hat{V}_1 =& T_2 + \sum_{j=2}^{n} C_j(-I_j^c \ln I_j + u_{j+1}) + C_{n+1}(-W_1^c \ln W_1 + u_{n+2}) \\
&+ \sum_{k=2}^{m} C_{n+k}(-W_k^c \ln W_k + u_{n+k+1}),
\end{aligned} \tag{2.5}$$

in which,

$$\begin{aligned}
T_2 =& -lnI_1 + u_2 + \frac{1}{I_1^c}\left(\sum_{j=1}^{n} f_j(S_0, 0) I_j^c e_j + \sum_{k=1}^{m} g_k(S_0, 0) W_k^c m_k\right)(S + I_1) \\
&+ \frac{1}{I_1^c}\left(\sum_{j=1}^{n} f_j(S_0, 0) I_j^c d_j + \sum_{k=1}^{m} g_k(S_0, 0) W_k^c n_k\right)\left(S - S_0 - S_0 \ln \frac{S}{S_0} + S_0 u_1\right), \\
u_1 =& \sum_{i=1}^{2} \frac{\upsilon_i (S + \omega_i)^p}{p}, \quad u_2 = \rho_0 S + \frac{\upsilon_3 (I_1 + \omega_3)^p}{p},
\end{aligned} \tag{2.6}$$

$$u_{j+1} = I_j^c \frac{\upsilon_4(I_j+p)^p}{p},\ j=2,3,\dots,n,\ \ u_{n+2} = W_1^c \frac{\upsilon_4(W_1+p)^p}{p},$$
$$u_{n+k+1} = W_k^c \frac{\upsilon_4(W_k+p)^p}{p},\ k=2,3,\dots,m,$$

where $e_j,\ d_j,\ C_j,\ j=1,2,\dots,n;\ m_k,\ n_k,\ C_{n+k},\ k=1,2,\dots,m$, and $\upsilon_i,\omega_i,\ i=1,2,\ \rho_0,\upsilon_3,\omega_3$ will be determined later.

According to the Itô formula, we obtain that

$$\begin{aligned} Lu_2 =& \rho_0\left(A-\sum_{j=1}^n f_j(S,I_j)I_j-\sum_{k=1}^m g_k(S,W_k)W_k-\mu S\right)\\ &+\upsilon_3(I_1+\omega_3)^{p-1}\left(\sum_{j=1}^n f_j(S,I_j)I_j+\sum_{k=1}^m g_k(S,W_k)W_k-\mu_1 I_1\right)\\ &-\frac{(1-p)\upsilon_3}{2}(I_1+\omega_3)^{p-2}\left(\sigma_{21}I_1+\sigma_{22}I_1^2\right)^2\\ \le& \rho_0 A-\left(\rho_0-\upsilon_3\omega_3^{p-1}\right)\left(\sum_{j=1}^n f_j(S,I_j)I_j+\sum_{k=1}^m g_k(S,W_k)W_k\right)-\frac{(1-p)\upsilon_3\omega_3^{p-2}\sigma_{22}^2I_1^4}{2\left(1+(\frac{I_1}{\omega_3})^{2-p}\right)}. \end{aligned}$$

By Lemma 2.1(b),

$$\frac{\omega_3^{p-2}I_1^4}{1+\left(\frac{I_1}{\omega_3}\right)^{2-p}}=\frac{\omega_3^{p+2}\left(\frac{I_1}{\omega_3}\right)^4}{1+\left(\frac{I_1}{\omega_3}\right)^{2-p}}\ge\frac{\omega_3^{p+2}\left(\frac{I_1}{\omega_3}\right)^4}{2\left(1+\left(\frac{I_1}{\omega_3}\right)^2\right)}\ge\frac{3\omega_3^pI_1^2}{8}-\frac{\omega_3^{p+2}}{8}.$$

Then,

$$\begin{aligned} Lu_2 \le& \rho_0 A-\left(\rho_0-\upsilon_3\omega_3^{p-1}\right)\left(\sum_{j=1}^n f_j(S,I_j)I_j+\sum_{k=1}^m g_k(S,W_k)W_k\right)\\ &+\frac{(1-p)\upsilon_3\omega_3^{p+2}\sigma_{22}^2}{16}-\frac{3(1-p)\upsilon_3\omega_3^p\sigma_{22}^2I_1^2}{16}. \end{aligned}$$

Choose $\rho_0=\upsilon_3\omega_3^{p-1}$, $\upsilon_3=\frac{8}{3(1-p)\omega_3^p}$, $\omega_3=2\sqrt[3]{\frac{A}{(1-p)\sigma_{22}^2}}$, then

$$Lu_2\le 2\sqrt[3]{\frac{A^2\sigma_{22}^2}{(1-p)^2}}-\frac{\sigma_{22}^2I_1^2}{2}. \tag{2.7}$$

Also,

$$L(-lnI_1)\le-\frac{1}{I_1}\left(\sum_{j=1}^n f_j(S,I_j)I_j+\sum_{k=1}^m g_k(S,W_k)W_k\right)+\mu_1+\frac{1}{2}(\sigma_{21}+\sigma_{22}I_1)^2. \tag{2.8}$$

Combining (2.3) and $lnx \le x-1$ for $x>0$,

$$
\begin{aligned}
-\frac{1}{I_1}\sum_{j=1}^{n} f_j(S,I_j)I_j &= -\frac{1}{I_1^c}\left(\sum_{j=1}^{n} f_j(S_0,0)I_j^c \frac{f_j(S,I_j)}{f_j(S,0)}\frac{f_j(S,0)}{f_j(S_0,0)}\frac{\widetilde{I}_j}{\widetilde{I}_1}\right)\\
&= -\frac{1}{I_1^c}\left[\sum_{j=1}^{n} f_j(S_0,0)I_j^c + \sum_{j=1}^{n} f_j(S_0,0)I_j^c\left(\frac{f_j(S,I_j)}{f_j(S,0)}\frac{f_j(S,0)}{f_j(S_0,0)}\frac{\widetilde{I}_j}{\widetilde{I}_1}-1\right)\right]\\
&\le -\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c - \frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c\left(\ln\frac{f_j(S,I_j)}{f_j(S,0)}+\ln\frac{f_j(S,0)}{f_j(S_0,0)}+\ln\frac{\widetilde{I}_j}{\widetilde{I}_1}\right).
\end{aligned}
$$

According to the differential mean-value theorem, and (H_1), there exists $\xi \in (0,I_j)$ such that

$$
\ln\frac{f_j(S,I_j)}{f_j(S,0)} = \frac{\frac{\partial}{\partial I_j}f_j(S,\xi)}{f_j(S,\xi)}I_j \ge -KI_j.
$$

Thus,

$$
\begin{aligned}
-\frac{1}{I_1}\sum_{j=1}^{n} f_j(S,I_j)I_j \le &-\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c\\
&+\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c\left(KI_j+\frac{f_j(S_0,0)}{f_j(S,0)}-1-\ln\frac{\widetilde{I}_j}{\widetilde{I}_1}\right).
\end{aligned}\tag{2.9}
$$

Similarly,

$$
\begin{aligned}
-\frac{1}{I_1}\sum_{k=1}^{m} g_k(S,W_k)W_k \le &-\frac{1}{I_1^c}\sum_{k=1}^{m} g_k(S_0,0)W_k^c\\
&+\frac{1}{I_1^c}\sum_{k=1}^{m} g_k(S_0,0)W_k^c\left(KW_k+\frac{g_k(S_0,0)}{g_k(S,0)}-1-\ln\frac{\widetilde{W}_k}{\widetilde{I}_1}\right).
\end{aligned}\tag{2.10}
$$

By (2.7)–(2.10),

$$
\begin{aligned}
L(-\ln I_1+u_2) \le &-\frac{1}{I_1^c}\left(\sum_{j=1}^{n} f_j(S_0,0)I_j^c+\sum_{k=1}^{m} g_k(S_0,0)W_k^c\right)+h_2+\sigma_{21}\sigma_{22}I_1\\
&+\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c KI_j+\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c\left(\frac{f_j(S_0,0)}{f_j(S,0)}-1\right)\\
&+\frac{1}{I_1^c}\sum_{k=1}^{m} g_k(S_0,0)W_k^c KW_k+\frac{1}{I_1^c}\sum_{k=1}^{m} g_k(S_0,0)W_k^c\left(\frac{g_k(S_0,0)}{g_k(S,0)}-1\right)\\
&-\frac{1}{I_1^c}\sum_{j=1}^{n} f_j(S_0,0)I_j^c\ln\frac{\widetilde{I}_j}{\widetilde{I}_1}-\frac{1}{I_1^c}\sum_{k=1}^{m} g_k(S_0,0)W_k^c\ln\frac{\widetilde{W}_k}{\widetilde{I}_1},
\end{aligned}\tag{2.11}
$$

where

$$h_2 = \mu_1 + 2\sqrt[3]{\frac{A^2\sigma_{22}^2}{(1-p)^2}} + \frac{\sigma_{21}^2}{2}. \tag{2.12}$$

Also,

$$L(S+I_1) = A - \mu S - \mu_1 I_1 = \mu(S_0 - S) - \mu_1 I_1, \tag{2.13}$$

where $S_0 = \frac{A}{\mu}$.

Combining Lemma 2.1(a) and (b), we have

$$\begin{aligned}
Lu_1 &= \sum_{i=1}^{2} \upsilon_i (S+\omega_i)^{p-1}\left[A - \sum_{j=1}^{n} f_j(S,I_j)I_j - \sum_{k=1}^{m} g_k(S,W_k)W_k - \mu S\right] \\
&\quad - \sum_{i=1}^{2} \frac{(1-p)\upsilon_i}{2(S+\omega_i)^{2-p}}\left(\sigma_{11}S + \sigma_{12}S^2\right)^2 \\
&\le \sum_{i=1}^{2} \frac{A\upsilon_i}{\omega_i^{1-p}} - \sum_{i=1}^{2} \frac{(1-p)\upsilon_i\omega_i^{p-2}}{2\left(1+\frac{S}{\omega_i}\right)^{2-p}}\left(\sigma_{11}S + \sigma_{12}S^2\right)^2 \\
&\le \sum_{i=1}^{2} \frac{A\upsilon_i}{\omega_i^{1-p}} - \frac{(1-p)\upsilon_1\omega_1^{p-2}\sigma_{12}^2S^4}{2\left(1+\frac{S}{\omega_1}\right)^2} - \frac{(1-p)\upsilon_2\omega_2^{p-2}\sigma_{11}\sigma_{12}S^3}{\left(1+\frac{S}{\omega_2}\right)^2} \\
&\le \sum_{i=1}^{2} \frac{A\upsilon_i}{\omega_i^{1-p}} - \frac{(1-p)\upsilon_1\omega_1^{p+2}\sigma_{12}^2\left(\frac{S}{\omega_1}\right)^4}{4\left[1+\left(\frac{S}{\omega_1}\right)^2\right]} - \frac{(1-p)\upsilon_2\omega_2^{p+1}\sigma_{11}\sigma_{12}\left(\frac{S}{\omega_2}\right)^3}{2\left[1+\left(\frac{S}{\omega_2}\right)^2\right]} \\
&\le \sum_{i=1}^{2} \frac{A\upsilon_i}{\omega_i^{1-p}} - \frac{1}{16}(1-p)\upsilon_1\omega_1^{p+2}\sigma_{12}^2\left[3\left(\frac{S}{\omega_1}\right)^2 - 1\right] \\
&\quad - \frac{1}{2}(1-p)\upsilon_2\omega_2^{p+1}\sigma_{11}\sigma_{12}\left(\frac{S}{\omega_2} - \frac{1}{2}\right) \\
&= \left[\frac{A\upsilon_1}{\omega_1^{1-p}} + \frac{(1-p)\upsilon_1\omega_1^{p+2}\sigma_{12}^2}{16}\right] + \left[\frac{A\upsilon_2}{\omega_2^{1-p}} + \frac{(1-p)\upsilon_2\omega_2^{p+1}\sigma_{11}\sigma_{12}}{4}\right] \\
&\quad - \frac{3}{16}(1-p)\upsilon_1\omega_1^{p}\sigma_{12}^2S^2 - \frac{1}{2}(1-p)\upsilon_2\omega_2^{p}\sigma_{11}\sigma_{12}S.
\end{aligned}$$

Let $\upsilon_1\omega_1^p = \frac{8}{3(1-p)}$, $\upsilon_2\omega_2^p = \frac{2}{1-p}$, then

$$Lu_1 \le \left[\frac{8A}{3(1-p)\omega_1} + \frac{\omega_1^2\sigma_{12}^2}{6}\right] + \left[\frac{2A}{(1-p)\omega_2} + \frac{\omega_2\sigma_{11}\sigma_{12}}{2}\right] - \frac{\sigma_{12}^2}{2}S^2 - \sigma_{11}\sigma_{12}S.$$

Let

$$\omega_1 = 2\sqrt[3]{\frac{A}{(1-p)\sigma_{12}^2}}, \quad \omega_2 = 2\sqrt{\frac{A}{(1-p)\sigma_{11}\sigma_{12}}}.$$

Then,

$$Lu_1 \leq 2\sqrt[3]{\frac{A^2\sigma_{12}^2}{(1-p)^2}} + 2\sqrt{\frac{A\sigma_{11}\sigma_{12}}{1-p}} - \frac{\sigma_{12}^2 S^2}{2} - \sigma_{11}\sigma_{12}S. \tag{2.14}$$

One has

$$\begin{aligned}
&L\left(S - S_0 - S_0 \ln\frac{S}{S_0}\right) \\
&= \left(1 - \frac{S_0}{S}\right)\left(\mu(S_0 - S) - \sum_{j=1}^{n} f_j(S, I_j)I_j - \sum_{k=1}^{m} g_k(S, W_k)W_k\right) + \frac{S_0}{2}(\sigma_{11} + \sigma_{12}S)^2 \\
&\leq -\frac{\mu(S-S_0)^2}{S} + S_0\sum_{j=1}^{n}\frac{f_j(S,I_j)}{S}I_j + S_0\sum_{k=1}^{m}\frac{g_k(S,W_k)}{S}W_k + \frac{S_0}{2}(\sigma_{11} + \sigma_{12}S)^2 \\
&\leq -\frac{\mu(S-S_0)^2}{S} + S_0 M\left(\sum_{j=1}^{n} I_j + \sum_{k=1}^{m} W_k\right) + \frac{S_0}{2}(\sigma_{11} + \sigma_{12}S)^2,
\end{aligned}$$

by the assumption (H_2).

Then,

$$L\left(S - S_0 - S_0\ln\frac{S}{S_0} + S_0 u_1\right) \leq -\frac{\mu(S-S_0)^2}{S} + S_0 M\left(\sum_{j=1}^{n} I_j + \sum_{k=1}^{m} W_k\right) + S_0\overline{h}_1, \tag{2.15}$$

where $\overline{h}_1 = 2\sqrt[3]{\frac{A^2\sigma_{12}^2}{(1-p)^2}} + 2\sqrt{\frac{A\sigma_{11}\sigma_{12}}{1-p}} + \frac{\sigma_{11}^2}{2}$. By (2.11), (2.13), (2.15), and (2.6), we have

$$\begin{aligned}
LT_2 \leq &-\frac{1}{I_1^c}\left(\sum_{j=1}^{n} f_j(S_0,0)I_j^c + \sum_{k=1}^{m} g_k(S_0,0)W_k^c\right) + h_2 + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0\overline{h}_1 \\
&+ \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0\overline{h}_1 + \sigma_{21}\sigma_{22}I_1 + \sum_{j=1}^{n} f_j(S_0,0)I_j^c K I_j - \sum_{j=1}^{n} f_j(S_0,0)I_j^c \ln\frac{\widetilde{I}_j}{\widetilde{I}_1} \\
&+ \sum_{k=1}^{m} g_k(S_0,0)W_k^c K W_k - \sum_{k=1}^{m} g_k(S_0,0)W_k^c \ln\frac{\widetilde{W}_k}{\widetilde{I}_1} \\
&+ \sum_{j=1}^{n} f_j(S_0,0)I_j^c\left(\frac{f_j(S_0,0)}{f_j(S,0)} - 1 + e_j\mu(S_0 - S) - d_j\frac{\mu(S-S_0)^2}{S}\right) \\
&+ \sum_{k=1}^{m} g_k(S_0,0)W_k^c\left(\frac{g_k(S_0,0)}{g_k(S,0)} - 1 + m_k\mu(S_0 - S) - n_k\frac{\mu(S-S_0)^2}{S}\right) \\
&- \sum_{j=1}^{n} f_j(S_0,0)I_j^c e_j\mu_1 I_1 - \sum_{k=1}^{m} g_k(S_0,0)W_k^c m_k\mu_1 I_1 \\
&+ \left(\sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k\right) S_0 M\left(\sum_{j=1}^{n} I_j + \sum_{k=1}^{m} W_k\right).
\end{aligned} \tag{2.16}$$

Let

$$M_j(S) = \frac{f_j(S_0,0)}{f_j(S,0)} - 1 + e_j\mu(S_0 - S) - d_j\frac{\mu(S-S_0)^2}{S},\ j = 1,2,\dots,n,$$

hence, we can easily obtain that

$$M_j(S_0) = 0,\ j = 1,2,\dots,n.$$

Also,

$$M_j'(S) = f_j(S_0,0)\frac{d}{dS}\left(\frac{1}{f_j(S,0)}\right) - e_j\mu - d_j\mu\left(1 - \frac{S_0^2}{S^2}\right),\ j = 1,2,\dots,n.$$

Let

$$M_j'(S)|_{S=S_0} = 0,\ j = 1,2,\dots,n,$$

then

$$e_j = \frac{f_j(S_0,0)\frac{d}{dS}\left(\frac{1}{f_j(S,0)}\right)|_{S=S_0}}{\mu},\ j = 1,2,\dots,n.$$

Furthermore,

$$\begin{aligned} M_j''(S) =& f_j(S_0,0)\frac{d^2}{dS^2}\left(\frac{1}{f_j(S,0)}\right) - \frac{2d_j\mu S_0^2}{S^3} \\ =& \frac{1}{S^3}\left[f_j(S_0,0)S^3\frac{d^2}{dS^2}\left(\frac{1}{f_j(S,0)}\right) - 2d_j\mu S_0^2\right] \\ \le& \frac{1}{S^3}\left(f_j(S_0,0)p_j - 2d_j\mu S_0^2\right), \end{aligned}$$

where the last inequality is based on the assumption (H_3).

By (2.2), we have $M_j''(S) < 0$ and $M_j'(S_0) = 0$, which derives

$$M_j(S) \le M_j(S_0) = 0,\ j = 1,2,\dots,n. \tag{2.17}$$

Similarly, let

$$m_k = \frac{g_k(S_0,0)\frac{d}{dS}\left(\frac{1}{g_k(S,0)}\right)|_{S=S_0}}{\mu},\ n_k > \max\left\{0, \frac{g_k(S_0,0)q_k}{2\mu(S_0)^2}\right\},\ k = 1,2,\dots,m, \tag{2.18}$$

where q_k is defined in the assumption (H_3). Therefore,

$$\begin{aligned} N_k(S) =& \frac{g_k(S_0,0)}{g_k(S,0)} - 1 + m_k\mu(S_0 - S) - n_k\frac{\mu(S-S_0)^2}{S} \\ \le& N_k(S_0) = 0,\ k = 1,2,\dots,m. \end{aligned} \tag{2.19}$$

Hence, combining (2.16), (2.17), and (2.19), we have

$$\begin{aligned} LT_2 \leq & -(R_0^s(p)-1)h_2 + \sigma_{21}\sigma_{22}I_1 + \sum_{j=1}^{n} f_j(S_0,0)I_j^c|e_j|\mu_1 I_1 + \sum_{k=1}^{m} g_k(S_0,0)W_k^c|m_k|\mu_1 I_1 \\ & + \sum_{j=1}^{n} f_j(S_0,0)I_j^c K I_j + \sum_{k=1}^{m} g_k(S_0,0)W_k^c K W_k - \sum_{k=1}^{m} g_k(S_0,0)W_k^c \left(\ln \widetilde{W}_k - \ln \widetilde{I}_1\right) \\ & + \left(\sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k\right) S_0 M \left(\sum_{j=1}^{n} I_j + \sum_{k=1}^{m} W_k\right) \\ & - \sum_{j=1}^{n} f_j(S_0,0)I_j^c \left(\ln \widetilde{I}_j - \ln \widetilde{I}_1\right), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} & R_0^s(p) \\ & := \frac{1}{h_2}\left(f_1(S_0,0)(1-d_1\frac{A}{\mu}\bar{h}_1) + \sum_{i=2}^{n} f_i(S_0,0)(1-d_i\frac{A}{\mu}\bar{h}_1)\prod_{j=2}^{i}\frac{\gamma_{j-1}}{\mu_j + \frac{\sigma_{j+1,1}^2}{2} + \frac{\sigma_{j+1,2}^2 p^2}{6}}\right) \\ & + \left(\frac{g_1(S_0,0)(1-n_1\frac{A}{\mu}\bar{h}_1)}{\delta_1 + \frac{\sigma_{n+2,1}^2}{2} + \frac{\sigma_{n+2,2}^2 p^2}{6}} + \sum_{k=2}^{m}\frac{g_k(S_0,0)(1-n_k\frac{A}{\mu}\bar{h}_1)}{\delta_1 + \frac{\sigma_{n+2,1}^2}{2} + \frac{\sigma_{n+2,2}^2 p^2}{6}}\prod_{j=2}^{k}\frac{\delta_{j-1}}{\delta_j + \frac{\sigma_{n+j+1,1}^2}{2} + \frac{\sigma_{n+j+1,2}^2 p^2}{6}}\right) \\ & \cdot \frac{1}{h_2}\left(\xi_1 + \sum_{i=2}^{n}\xi_i\prod_{j=2}^{i}\frac{\gamma_{j-1}}{\mu_j + \frac{\sigma_{j+1,1}^2}{2} + \frac{\sigma_{j+1,2}^2 p^2}{6}}\right), \end{aligned} \tag{2.21}$$

where

$$\bar{h}_1 = 2\sqrt[3]{\frac{A^2\sigma_{12}^2}{(1-p)^2}} + 2\sqrt{\frac{A\sigma_{11}\sigma_{12}}{1-p}} + \frac{\sigma_{11}^2}{2},\ h_2 = \mu_1 + 2\sqrt[3]{\frac{A^2\sigma_{22}^2}{(1-p)^2}} + \frac{\sigma_{21}^2}{2}. \tag{2.22}$$

By Itô's formula and Lemma 2.1,

$$\begin{aligned} Lu_{i+1} = & I_i^c \upsilon_4 (I_i+p)^{p-1}(\gamma_{i-1}I_{i-1} - \mu_i I_i) - I_i^c \frac{(1-p)\upsilon_4}{2}(I_i+p)^{p-2}\left(\sigma_{i+1,1}I_i + \sigma_{i+1,2}I_i^2\right)^2 \\ \leq & I_i^c \frac{\upsilon_4\gamma_{i-1}I_{i-1}}{p^{1-p}} - I_i^c \frac{(1-p)\upsilon_4 p^{p-2}\sigma_{i+1,2}^2 I_i^4}{2\left(1+\frac{I_i}{p}\right)^{2-p}} \\ \leq & I_i^c \frac{\upsilon_4\gamma_{i-1}I_{i-1}}{p^{1-p}} - I_i^c \frac{(1-p)\upsilon_4 p^{p+2}\sigma_{i+1,2}^2 \left(\frac{I_i}{p}\right)^4}{4\left[1+\left(\frac{I_i}{p}\right)^2\right]} \\ \leq & I_i^c \frac{\upsilon_4\gamma_{i-1}I_{i-1}}{p^{1-p}} + I_i^c \frac{(1-p)\upsilon_4 p^{p+2}\sigma_{i+1,2}^2}{16} - I_i^c \frac{3(1-p)\upsilon_4 p^p \sigma_{i+1,2}^2 I_i^2}{16}. \end{aligned}$$

Choose $\upsilon_4 = \frac{8}{3(1-p)p^p}$, such that

$$Lu_{i+1} \leq I_i^c \frac{8\gamma_{i-1} I_{i-1}}{3p(1-p)} + I_i^c \frac{\sigma_{i+1,2}^2 p^2}{6} - I_i^c \frac{\sigma_{i+1,2}^2 I_i^2}{2}. \tag{2.23}$$

Let

$$T_{i+1} = -I_i^c \ln I_i + u_{i+1},\ i = 2, 3, \dots, n. \tag{2.24}$$

Applying Itô's formula and by (2.4) and (2.23), one has

$$\begin{aligned} LT_{i+1} \leq & -\frac{I_i^c}{I_i}(\gamma_{i-1} I_{i-1} - \mu_i I_i) + \frac{\sigma_{i+1,1}^2}{2} I_i^c + \sigma_{i+1,1}\sigma_{i+1,2} I_i^c I_i + I_i^c \frac{8\gamma_{i-1} I_{i-1}}{3p(1-p)} + I_i^c \frac{\sigma_{i+1,2}^2 p^2}{6} \\ = & -\gamma_{i-1}\frac{I_i^c I_{i-1}}{I_i} + \frac{8\gamma_{i-1} I_i^c}{3p(1-p)} I_{i-1} + \left(\mu_i + \frac{\sigma_{i+1,1}^2}{2} + \frac{\sigma_{i+1,2}^2 p^2}{6}\right) I_i^c + \sigma_{i+1,1}\sigma_{i+1,2} I_i^c I_i \\ = & -\gamma_{i-1}\frac{I_{i-1}^c \widetilde{I}_{i-1}}{\widetilde{I}_i} + h_{i+1} I_i^c + \frac{8\gamma_{i-1} I_i^c}{3p(1-p)} I_{i-1} + \sigma_{i+1,1}\sigma_{i+1,2} I_i^c I_i \\ = & \gamma_{i-1} I_{i-1}^c \left(1 - \frac{\widetilde{I}_{i-1}}{\widetilde{I}_i}\right) + \frac{8\gamma_{i-1} I_i^c}{3p(1-p)} I_{i-1} + \sigma_{i+1,1}\sigma_{i+1,2} I_i^c I_i \\ \leq & -\gamma_{i-1} I_{i-1}^c \left(\ln \widetilde{I}_{i-1} - \ln \widetilde{I}_i\right) + \frac{8\gamma_{i-1} I_i^c}{3p(1-p)} I_{i-1} + \sigma_{i+1,1}\sigma_{i+1,2} I_i^c I_i, \end{aligned} \tag{2.25}$$

according to the inequality $x - 1 - lnx \geq 0$ for any $x > 0$.

Likewise, we derive

$$\begin{aligned} Lu_{n+2} \leq & W_1^c \frac{8\sum_{j=1}^{n} \xi_j I_j}{3p(1-p)} + W_1^c \frac{\sigma_{n+2,2}^2 p^2}{6} - W_1^c \frac{\sigma_{n+2,2}^2 W_1^2}{2}, \\ L(-W_1^c \ln W_1 + u_{n+2}) \leq & -\sum_{i=1}^{n} \xi_i I_i^c \left(\ln \widetilde{I}_i - \ln \widetilde{W}_1\right) + W_1^c \frac{8\sum_{i=1}^{n} \xi_i I_i}{3p(1-p)} + \sigma_{n+2,1}\sigma_{n+2,2} W_1^c W_1, \\ Lu_{n+j+1} \leq & W_j^c \frac{8\delta_{j-1} W_{j-1}}{3p(1-p)} + W_j^c \frac{\sigma_{n+j+1,2}^2 p^2}{6} - W_j^c \frac{\sigma_{n+j+1,2}^2 W_j^2}{2}, \\ L(-W_j^c \ln W_j + u_{n+j+1}) \leq & -\delta_{j-1} W_{j-1}^c \left(\ln \widetilde{W}_{j-1} - \ln \widetilde{W}_j\right) + W_j^c \frac{8\delta_{j-1} W_{j-1}}{3p(1-p)} \\ & + \sigma_{n+j+1,1}\sigma_{n+j+1,2} W_j^c W_j. \end{aligned} \tag{2.26}$$

Combining (2.20), (2.25), and (2.26), we obtain

$$\begin{aligned} & L\hat{V}_1(S, I_1, \dots, I_n, W_1, \dots, W_m) \\ \leq & -(R_0^s(p) - 1)h_2 + \left(\sigma_{21}\sigma_{22} + \sum_{j=1}^{n} f_j(S_0, 0) I_j^c |e_j| \mu_1 + \sum_{k=1}^{m} g_k(S_0, 0) W_k^c |m_k| \mu_1 \right. \\ & + f_1(S_0, 0)K + \sum_{j=1}^{n} f_j(S_0, 0) I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0, 0) W_k^c n_k S_0 M \end{aligned}$$

$$
\begin{aligned}
&+C_{n+1} W_1^c \frac{8\xi_1}{3p(1-p)} + C_2 \frac{8\gamma_1 I_2^c}{3p(1-p)}\Bigg) I_1 \\
&+ \sum_{j=2}^{n-1} \Bigg(f_j(S_0,0) I_j^c K + \sum_{j=1}^{n} f_j(S_0,0) I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0) W_k^c n_k S_0 M \\
&+C_{j+1} \frac{8\gamma_j I_{j+1}^c}{3p(1-p)} + C_{n+2} \frac{8\xi_j W_1^c}{3p(1-p)} + C_j \sigma_{j+1,1}\sigma_{j+1,2} I_j^c \Bigg) I_j \\
&+ \Bigg(f_n(S_0,0) K I_n^c + \sum_{j=1}^{n} f_j(S_0,0) I_j^c d_j S_0 M \\
&+ \sum_{k=1}^{m} g_k(S_0,0) W_k^c n_k S_0 M + C_{n+1} \frac{8\xi_n W_1^c}{3p(1-p)} + C_n \sigma_{n+1,1}\sigma_{n+1,2} I_n^c \Bigg) I_n \\
&+ \sum_{k=1}^{m-1} \Bigg(g_k(S_0,0) W_k^c K + \sum_{j=1}^{n} f_j(S_0,0) I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0) W_k^c n_k S_0 M \\
&+C_{n+k}\sigma_{n+k+1,1}\sigma_{n+k+1,2} W_k^c + C_{n+k+1} \frac{8\delta_k W_{k+1}^c}{3p(1-p)} \Bigg) W_k \\
&+ \Bigg(g_m(S_0,0) W_m^c K + \sum_{j=1}^{n} f_j(S_0,0) I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0) W_k^c n_k S_0 M \\
&+ C_{n+m}\sigma_{n+m+1,1}\sigma_{n+m+1,2} W_m^c \Bigg) W_m \\
&+ \Bigg(\sum_{j=1}^{n} f_j(S_0,0) I_j^c + \sum_{k=1}^{m} g_k(S_0,0) W_k^c - f_1(S_0,0) I_1^c - C_2\gamma_1 I_1^c - C_{n+1}\xi_1 \Bigg) \ln \tilde{I}_1 \\
&+ \sum_{j=2}^{n-1} \left(C_j \gamma_{j-1} I_{j-1}^c - f_j(S_0,0) I_j^c - C_{j+1}\gamma_j I_j^c - C_{n+1}\xi_j I_j^c \right) \ln \tilde{I}_j \\
&+ \left(C_n \gamma_{n-1} I_{n-1}^c - f_n(S_0,0) I_n^c - C_{n+1}\xi_n I_n^c \right) \ln \tilde{I}_n \\
&+ \Bigg(C_{n+1} \sum_{j=1}^{n} \xi_j I_j^c - g_1(S_0,0) W_1^c - C_{n+2}\delta_1 W_1^c \Bigg) \ln \tilde{W}_1 \\
&+ \sum_{k=2}^{m-1} \left(C_{n+k}\delta_{k-1} W_{k-1}^c - C_{n+k+1}\delta_k W_k^c - g_k(S_0,0) W_k^c \right) \ln \tilde{W}_k \\
&+ \left(C_{n+m}\delta_{m-1} W_{m-1}^c - g_m(S_0,0) W_m^c \right) \ln \tilde{W}_m.
\end{aligned}
$$

Choose

$$
\begin{cases}
C_{n+m} = \frac{g_m(S_0,0) W_m^c}{\delta_{m-1} W_{m-1}^c}, \quad C_{n+k} = \frac{\sum_{i=k}^{m} g_i(S_0,0) W_i^c}{\delta_{k-1} W_{k-1}^c}, \quad k = m-1, m-2, \ldots, 2, \\
C_{n+1} = \frac{\sum_{k=1}^{m} g_k(S_0,0) W_k^c}{\sum_{j=1}^{n} \xi_j I_j^c}, \; C_n = \frac{f_n(S_0,0) I_n^c + C_{n+1}\xi_n I_n^c}{\gamma_{n-1} I_{n-1}^c}, \\
C_\alpha = \frac{\sum_{j=\alpha}^{n} f_j(S_0,0) I_j^c + C_{n+1} \sum_{j=\alpha}^{n} \xi_j I_j^c}{\gamma_{\alpha-1} I_{\alpha-1}^c}, \quad \omega = n-1, n-2, \ldots, 3, 2,
\end{cases}
\tag{2.27}
$$

such that

$$\begin{cases} \sum_{j=1}^{n} f_j(S_0,0)I_j^c + \sum_{k=1}^{m} g_k(S_0,0)W_k^c - f_1(S_0,0)I_1^c - C_2\gamma_1 I_1^c - C_{n+1}\xi_1 = 0, \\ C_j\gamma_{j-1}I_{j-1}^c - f_j(S_0,0)I_j^c - C_{j+1}\gamma_j I_j^c - C_{n+1}\xi_j I_j^c = 0, \\ C_n\gamma_{n-1}I_{n-1}^c - f_n(S_0,0)I_n^c - C_{n+1}\xi_n I_n^c = 0, \\ C_{n+1}\sum_{j=1}^{n}\xi_j I_j^c - g_1(S^0,0)W_1^c - C_{n+2}\delta_1 W_1^c = 0, \\ C_{n+k}\delta_{k-1}W_{k-1}^c - C_{n+k+1}\delta_k W_k^c - g_k(S_0,0)W_k^c = 0, \\ C_{n+m}\delta_{m-1}W_{m-1}^c - g_m(S_0,0)W_m^c = 0. \end{cases}$$

Define the Lyapunov function

$$V_1 = \hat{V}_1 + \sum_{j=2}^{n}\psi_j I_j + \sum_{k=1}^{m}\chi_k W_k,$$

where $\psi_j, \chi_k, j = 2,3,\dots,n, k = 1,2,\dots,m$ are determined later. Therefore,

$$\begin{aligned} LV_1 &= L\hat{V}_1 + L\left(\sum_{j=2}^{n}\psi_j I_j + \sum_{k=1}^{m}\chi_k W_k\right) \\ &\le -(R_0^s(p)-1)h_2 + \Bigg(\sigma_{21}\sigma_{22} + \sum_{j=1}^{n} f_j(S_0,0)I_j^c|e_j|\mu_1 + \sum_{k=1}^{m} g_k(S_0,0)W_k^c|m_k|\mu_1 \\ &\quad + f_1(S_0,0)I_1^c K + \psi_2\gamma_1 + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0 M \\ &\quad + C_{n+1}W_1^c\frac{8\xi_1}{3p(1-p)} + C_2\frac{8\gamma_1 I_2^c}{3p(1-p)} + \chi_1\xi_1\Bigg)I_1 \\ &\quad + \sum_{j=2}^{n-1}\Bigg(f_j(S_0,0)I_j^c K + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0 M + C_{j+1}\frac{8\gamma_j I_{j+1}^c}{3p(1-p)} \\ &\quad + C_{n+2}\frac{8\xi_j W_1^c}{3p(1-p)} + C_j\sigma_{j+1,1}\sigma_{j+1,2}I_j^c + \psi_{j+1}\gamma_j + \chi_1\xi_j - \psi_j\mu_j\Bigg)I_j \\ &\quad + \Bigg(f_n(S_0,0)I_n^c K + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0 M + C_{n+1}\frac{8\xi_n W_1^c}{3p(1-p)} \\ &\quad + C_n\sigma_{n+1,1}\sigma_{n+1,2}I_n^c + \chi_1\xi_n - \mu_n\psi_n\Bigg)I_n \\ &\quad + \sum_{k=1}^{m-1}\Bigg(g_k(S_0,0)W_k^c K + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0 M \\ &\quad + C_{n+k}\sigma_{n+k+1,1}\sigma_{n+k+1,2}W_k^c + C_{n+k+1}\frac{8\delta_k W_{k+1}^c}{3p(1-p)} + \chi_{k+1}\delta_k - \chi_k\delta_k\Bigg)W_k \\ &\quad + \Bigg(g_m(S_0,0)W_m^c K + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_j S_0 M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_k S_0 M \end{aligned}$$

$$+ C_{n+m}\sigma_{n+m+1,1}\sigma_{n+m+1,2}W_m^c - \chi_m\delta_m\Bigg)W_m. \tag{2.28}$$

Take

$$\begin{cases} \chi_m = & \frac{1}{\delta_m}\Bigg(g_m(S_0,0)W_m^cK + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_jS_0M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_kS_0M \\ & +C_{n+m}\sigma_{n+m+1,1}\sigma_{n+m+1,2}W_m^c\Bigg), \\ \chi_k = & \chi_{k+1} + \frac{1}{\delta_k}\Bigg(g_k(S_0,0)W_k^cK + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_jS_0M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_kS_0M \\ & +C_{n+k}\sigma_{n+k+1,1}\sigma_{n+k+1,2}W_k^c + C_{n+k+1}\frac{8\delta_k W_{k+1}^c}{3p(1-p)}\Bigg),\ k = m-1, m-2, \dots, 1, \\ \psi_n = & \frac{1}{\mu_n}\Bigg(f_n(S_0,0)I_n^cK + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_jS_0M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_kS_0M + C_{n+1}\frac{8\xi_n W_1^c}{3p(1-p)} \\ & +C_n\sigma_{n+1,1}\sigma_{n+1,2}I_n^c + \chi_1\xi_n\Bigg), \\ \psi_j = & \psi_{j+1}\frac{\gamma_j}{\mu_j} + \frac{\chi_1\xi_j}{\mu_j} + \frac{1}{\mu_j}\Bigg(f_j(S_0,0)I_j^cK + \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_jS_0M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_kS_0M \\ & +C_{j+1}\frac{8\gamma_j I_{j+1}^c}{3p(1-p)} + C_{n+2}\frac{8\xi_j W_1^c}{3p(1-p)} + C_j\sigma_{j+1,1}\sigma_{j+1,2}I_j^c\Bigg),\ j = n-1, n-2, \dots, 2, \end{cases}$$

then by (2.28), we have that

$$LV_1 \le -(R_0^s(p) - 1)h_2 + JI_1, \tag{2.29}$$

where

$$\begin{aligned} J = \Bigg(&\sigma_{21}\sigma_{22} + \sum_{j=1}^{n} f_j(S_0,0)I_j^c|e_j|\mu_1 + \sum_{k=1}^{m} g_k(S_0,0)W_k^c|m_k|\mu_1 + \psi_2\gamma_1 \\ &+ \sum_{j=1}^{n} f_j(S_0,0)I_j^c d_jS_0M + \sum_{k=1}^{m} g_k(S_0,0)W_k^c n_kS_0M + C_{n+1}W_1^c\frac{8\xi_1}{3p(1-p)} \\ &+C_2\frac{8\gamma_1 I_2^c}{3p(1-p)} + f_1(S_0,0)K + \chi_1\xi_1\Bigg). \end{aligned} \tag{2.30}$$

Applying Itô's formula, (2.14), and the assumption (H_2), we have

$$\begin{aligned} L(-\ln S + u_1) &= -\frac{A}{S} + + \sum_{j=1}^{n}\frac{f_j(S,I_j)I_j}{S} + \sum_{k=1}^{m}\frac{g_k(S,W_k)W_k}{S} + \mu + \bar{h}_1 \\ &\le -\frac{A}{S} + \mu + \bar{h}_1 + M\left(\sum_{j=1}^{n} I_j + \sum_{k=1}^{m} W_k\right), \end{aligned} \tag{2.31}$$

where u_1, $\bar{h}_1$ are defined by (2.6) and (2.22), respectively.

By (2.23)–(2.25), we have that

$$
\begin{aligned}
&L\left(\sum_{j=2}^{n}\left(-\ln I_j+\frac{1}{I_j^c}u_{j+1}\right)\right)\\
&=-\sum_{j=2}^{n}\gamma_{j-1}\frac{I_{j-1}}{I_j}+\sum_{j=2}^{n}\mu_j+\frac{1}{2}\sum_{j=2}^{n}\left(\sigma_{j+1,1}+\sigma_{j+1,2}I_j\right)^2+\sum_{j=2}^{n}\frac{1}{I_j^c}Lu_{j+1}\\
&\le-\sum_{j=2}^{n}\gamma_{j-1}\frac{I_{j-1}}{I_j}+\sum_{j=2}^{n}\frac{8\gamma_{j-1}I_{j-1}}{3p(1-p)}+\sum_{j=2}^{n}\sigma_{j+1,1}\sigma_{j+1,2}I_j+\sum_{j=2}^{n}h_{j+1},
\end{aligned}
\tag{2.32}
$$

where $h_{j+1}=\mu_j+\frac{\sigma_{j+1,1}^2}{2}+\frac{\sigma_{j+1,2}^2p^2}{6}$ and u_{j+1}, $j=2,3,\dots,n$ are defined by (2.6).

Furthermore, let

$$
v_1=(-lnW_1+\frac{1}{W_1^c}u_{n+2})+\sum_{k=2}^{m}\left(-\ln W_k+\frac{1}{W_k^c}u_{n+k+1}\right),
$$

where u_{n+2}, u_{n+k+1}, $k=2,3,\dots,m$ are defined by (2.6). Applying Itô's formula and (2.26), one has

$$
\begin{aligned}
Lv_1&=-\sum_{j=1}^{n}\frac{\xi_jI_j}{W_1}+\delta_1+\frac{1}{W_1^c}Lu_{n+2}-\sum_{k=2}^{m}\frac{\delta_{k-1}W_{k-1}}{W_k}+\sum_{k=2}^{m}\delta_k+\sum_{k=2}^{m}\frac{1}{W_k^c}Lu_{n+k+1}\\
&\quad+\frac{1}{2}\left(\sigma_{n+2,1}+\sigma_{n+2,2}W_1\right)^2+\frac{1}{2}\sum_{k=2}^{m}\left(\sigma_{n+k+1,1}+\sigma_{n+k+1,2}W_k\right)^2\\
&\le-\sum_{j=1}^{n}\frac{\xi_jI_j}{W_1}+\frac{8\sum_{j=1}^{n}\xi_jI_j}{3p(1-p)}-\sum_{k=2}^{m}\frac{\delta_{k-1}W_{k-1}}{W_k}+\frac{8\sum_{k=2}^{m}\delta_{k-1}W_{k-1}}{3p(1-p)}\\
&\quad+\sum_{k=1}^{m}\sigma_{n+k+1,1}\sigma_{n+k+1,2}W_k+\sum_{k=1}^{m}h_{n+k+1},
\end{aligned}
\tag{2.33}
$$

where $h_{n+k+1}=\delta_k+\frac{1}{2}\sigma_{n+k+1,1}^2+\frac{1}{6}\sigma_{n+k+1,2}^2p^2$.

Let

$$
v_2=\frac{(\sigma_{11}+\sigma_{12}S)^p}{p}+\sum_{j=1}^{n}\frac{(\sigma_{j+1,1}+\sigma_{j+1,2}I_j)^p}{p}+\sum_{k=1}^{m}\frac{(\sigma_{n+k+1,1}+\sigma_{n+k+1,2}W_k)^p}{p}.
$$

Then,

$$
\begin{aligned}
Lv_2=&(\sigma_{11}+\sigma_{12}S)^{p-1}\sigma_{12}\left(A-\sum_{j=1}^{n}f_j(S,I_j)I_j-\sum_{k=1}^{m}g_k(S,W_k)W_k-\mu S\right)\\
&-\frac{1-p}{2}(\sigma_{11}+\sigma_{12}S)^p\sigma_{12}^2S^2-\frac{1-p}{2}(\sigma_{21}+\sigma_{22}I_1)^p\sigma_{22}^2I_1^2\\
&+(\sigma_{21}+\sigma_{22}I_1)^{p-1}\sigma_{22}\left(\sum_{j=1}^{n}f_j(S,I_j)I_j+\sum_{k=1}^{m}g_k(S,W_k)W_k-\mu_1I_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=2}^{n}\left(\sigma_{j+1,1}+\sigma_{j+1,2}I_j\right)^{p-1}\sigma_{j+1,2}\left(\gamma_{j-1}I_{j-1}-\mu_j I_j\right)\\
&-\frac{1-p}{2}\sum_{j=2}^{n}(\sigma_{j+1,1}+\sigma_{j+1,2}I_j)^p\sigma_{j+1,2}^2 I_j^2-\frac{1-p}{2}(\sigma_{n+2,1}+\sigma_{n+2,2}W_1)^p\sigma_{n+2,2}^2W_1^2\\
&+\left(\sigma_{n+2,1}+\sigma_{n+2,2}W_1\right)^{p-1}\sigma_{n+2,2}\left(\sum_{j=1}^{n}\xi_j I_j-\delta_1 W_1\right)\\
&+\sum_{k=2}^{m}\left(\sigma_{n+k+1,1}+\sigma_{n+k+1,2}W_k\right)^{p-1}\sigma_{n+k+1,2}\left(\delta_{k-1}W_{k-1}-\delta_k W_k\right)\\
&-\frac{1-p}{2}\sum_{k=2}^{m}(\sigma_{n+k+1,1}+\sigma_{n+k+1,2}W_k)^p\sigma_{n+k+1,2}^2W_k^2\\
\le&\sigma_{12}\sigma_{11}^{p-1}A-\frac{1-p}{2}\left(\sigma_{12}^{p+2}S^{p+2}+\sum_{j=1}^{n}\sigma_{j+1,2}^{p+2}I_j^{p+2}+\sum_{k=1}^{m}\sigma_{n+k+1,2}^{p+2}W_k^{p+2}\right)\\
&+MS\sigma_{22}\sigma_{21}^{p-1}\left(\sum_{j=1}^{n}I_j+\sum_{k=1}^{m}W_k\right)+\sum_{j=2}^{n}\sigma_{j+1,2}\sigma_{j+1,1}^{p-1}\gamma_{j-1}I_{j-1}\\
&+\sigma_{n+2,2}\sigma_{n+2,1}^{p-1}\sum_{j=1}^{n}\xi_j I_j+\sum_{k=2}^{m}\sigma_{n+k+1,2}\sigma_{n+k+1,1}^{p-1}\delta_{k-1}W_{k-1}.
\end{aligned}
\tag{2.34}
$$

Denote

$$
V_2=(-\ln S+u_1)+\sum_{j=2}^{n}\left(-\ln I_j+\frac{1}{I_j^c}u_{j+1}\right)+v_1+v_2.
$$

Combining (2.31)–(2.34),

$$
\begin{aligned}
L(V_2)\le&-\frac{A}{S}-\sum_{j=2}^{n}\gamma_{j-1}\frac{I_{j-1}}{I_j}-\sum_{j=1}^{n}\frac{\xi_j I_j}{W_1}-\sum_{k=2}^{m}\delta_{k-1}\frac{W_{k-1}}{W_k}+C\\
&-\frac{1-p}{4}\left(\sigma_{12}^{p+2}S^{p+2}+\sum_{j=1}^{n}\sigma_{j+1,2}^{p+2}I_j^{p+2}+\sum_{k=1}^{m}\sigma_{n+k+1,2}^{p+2}W_k^{p+2}\right),
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{aligned}
C=&\sup_{(S,I_1,\dots,I_n,W_1,\dots,W_m)\in\mathbb{R}_+^{n+m+1}}\{\sigma_{12}\sigma_{11}^{p-1}A+M(S\sigma_{22}\sigma_{21}^{p-1}+1)\left(\sum_{j=1}^{n}I_j+\sum_{k=1}^{m}W_k\right)\\
&-\frac{1-p}{4}\left(\sigma_{12}^{p+2}S^{p+2}+\sum_{j=1}^{n}\sigma_{j+1,2}^{p+2}I_j^{p+2}+\sum_{k=1}^{m}\sigma_{n+k+1,2}^{p+2}W_k^{p+2}\right)+\sum_{j=2}^{n}\sigma_{j+1,2}\sigma_{j+1,1}^{p-1}\gamma_{j-1}I_{j-1}\\
&+\sigma_{n+2,2}\sigma_{n+2,1}^{p-1}\sum_{j=1}^{n}\xi_j I_j+\sum_{k=2}^{m}\sigma_{n+k+1,2}\sigma_{n+k+1,1}^{p-1}\delta_{k-1}W_{k-1}+\mu+\bar{h}_1+\sum_{j=2}^{n+m+1}h_{j+1}
\end{aligned}
$$

$$+\sum_{j=2}^{n}\frac{8\gamma_{j-1}}{3p(1-p)}I_{j-1}+\sum_{j=1}^{n}\frac{8\xi_j}{3p(1-p)}I_j+\sum_{k=2}^{m}\frac{8\delta_{k-1}}{3p(1-p)}W_{k-1}$$
$$+\sum_{j=2}^{n}\sigma_{j+1,1}\sigma_{j+1,2}I_j+\sum_{k=1}^{m}\sigma_{n+k+1,1}\sigma_{n+k+1,2}W_k\}. \tag{2.36}$$

Define a function $\bar{V}\in C^2(\mathbb{R}_+^{n+m+1};R)$,

$$\bar{V}(S,I_1,\dots,I_n,W_1,\dots,W_m):=HV_1+V_2,$$

where H must meet a certain condition, hence, we define it later.

Step 2. (Constructing a compact set)

$\bar{V}$ tends to ∞ when $(S,I_1,\dots,I_n,W_1,\dots,W_m)$ approaches the boundary of R_+^{n+m+1} because of the continuity and the monotonicity of the function $\bar{V}$. Therefore, it must have a minimum point $(\hat{S},\hat{I}_1,\dots,\hat{W}_m)$ in the interior of R_+^{n+m+1}. Thus, define a C^2-function V: $\mathbb{R}_+^{n+m+1}\to\bar{\mathbb{R}}_+$ as follows:

$$V(S,I_1,\dots,I_n,W_1,\dots,W_m)=\bar{V}(S,I_1,\dots,I_n,W_1,\dots,W_m)-\bar{V}(\hat{S},\hat{I}_1,\dots,\hat{W}_m).$$

Combining (2.29) and (2.35), we obtain

$$\begin{aligned}LV\le&-H\lambda+HJI_1-\frac{1-p}{8}\sigma_{22}^{p+2}I_1^{p+2}\\&-\frac{A}{S}-\sum_{j=2}^{n}\gamma_{j-1}\frac{I_{j-1}}{I_j}-\sum_{j=1}^{n}\frac{\xi_jI_j}{W_1}-\sum_{k=2}^{m}\delta_{k-1}\frac{W_{k-1}}{W_k}+C\\&-\frac{1-p}{8}\left(\sigma_{12}^{p+2}S^{p+2}+\sum_{j=1}^{n}\sigma_{j+1,2}^{p+2}I_j^{p+2}+\sum_{k=1}^{m}\sigma_{n+k+1,2}^{p+2}W_k^{p+2}\right),\end{aligned}$$

where $\lambda\triangleq(R_0^s(p)-1)h_2$ and C and J can be found in (2.36) and (2.30), respectively. H is large enough satisfying $-H\lambda+C<-2$.

Then, we have

$$LV\le\begin{cases}-H\lambda+C+D-\frac{A}{S}-\frac{1-p}{8}\sigma_{12}^{p+2}S^{p+2}\to-\infty,\ \text{as } S\to0^+ \text{ or } S\to+\infty,\\-H\lambda+C+HJI_1<-1,\ \text{as } I_1\to0^+,\\-H\lambda+C+D-\frac{1-p}{8}\sum_{j=1}^{n}\sigma_{j+1,2}^{p+2}I_j^{p+2}\to-\infty,\ \text{as } I_j\to+\infty,\ j=1,2,\dots,n,\\-H\lambda+C+D-\frac{1-p}{8}\sum_{k=1}^{m}\sigma_{n+k+1,2}^{p+2}W_k^{p+2}\to-\infty,\ \text{as } W_k\to+\infty,\ k=1,2,\dots,m,\\-H\lambda+C+D-\sum_{j=2}^{n}\gamma_{j-1}\frac{I_{j-1}}{I_j}\to-\infty,\ \text{as } I_{j-1}\nrightarrow0^+,I_j\to0^+,\ j=2,3,\dots,n,\\-H\lambda+C+D-\frac{\xi_1I_1}{W_1}\to-\infty,\ \text{as } I_1\nrightarrow0^+,W_1\to0^+,\\-H\lambda+C+D-\sum_{k=2}^{m}\delta_{k-1}\frac{W_{k-1}}{W_k}\to-\infty,\ \text{as } W_{k-1}\nrightarrow0^+,W_k\to0^+,\ k=2,3,\dots,m,\end{cases}$$

where

$$D=\sup_{I_1\in(0,+\infty)}\left\{HJI_1-\frac{1-p}{8}\sigma_{22}^{p+2}I_1^{p+1}\right\}.$$

Thus, there exists a compact set $U \subset R_+^{n+m+1}$ such that if $R_0^s(p) > 1$,

$$LV(S, I_1, \dots, W_m) \le -1, \; (S, I_1, \dots, W_m) \in \mathbb{R}_+^{n+m+1} \setminus U.$$

Noting that $R_0^s(p)$ defined in (2.21) is continuous and nonincreasing in p, $\lim_{p \to 0^+} R_0^s(p) = R_0^s(0) \triangleq R_0^s$ makes senses. Therefore, if $R_0^s > 1$, then there exists a $p \in (0,1)$ such that $R_0^s(p) > 1$. Thus, if $R_0^s > 1$, the condition (A2) in Lemma A.1 in the Appendix holds.

Step 3. (Proving the ergodicity)

Next, we verify the condition (A1) in Lemma A.1 in the Appendix. The diffusion matrix of system (1.4) is as follows:

$$\widetilde{A} = diag\Big((\sigma_{11} + \sigma_{12}S)^2 S^2, \; (\sigma_{21} + \sigma_{22}I_1)^2 I_1^2, \; \dots, \; (\sigma_{n+m+1,1} + \sigma_{n+m+1,2}W_m)^2 W_m^2\Big).$$

It is obvious that for any compact subset of $\mathbb{R}_+^{n+m+1}$, the matrix A is positive-definite. Therefore, the condition (A1) in Lemma A.1 is satisfied.

Therefore, the conditions (A1) and (A2) in Lemma A.1 in the Appendix are proved. That is to say, there is a unique and ergodic stationary distribution $\pi(\cdot)$ for system (1.4). □

Remark 2.1 If $\sigma_{i,j} = 0$, $i = 1, 2, \dots, n + m + 1$, $j = 1, 2$, then R_0^s is consistent with the basic reproduction number R_0 of the ODE system (1.1) defined in (1.3). That is, our work includes and extends the work of Song et al. [21], and partially extends the result of Shuai et al. [10].

Remark 2.2 If there exist white-noise intensities $\sigma_{i,j}$, $i = 1, 2, \dots, n + m + 1$, $j = 1, 2$ such that $\sigma_{i,j}^2 > 0$, then $R_0^s < R_0$, which implies that the white noises are beneficial to the control of the diseases.

3 Numerical simulations

Consider the same parameters in this section as Example 2 of Song et al. [21] in system (1.4) as follows:

$$\begin{aligned} &A = 0.5, \; \mu = 0.25, \; \mu_1 = 0.85, \; \mu_2 = 0.65, \; \xi_1 = 0.15, \; \xi_2 = 0.15, \; \gamma_1 = 0.55, \; \delta_1 = 0.3, \\ &\delta_2 = 0.2, \; \delta_3 = 0.15, \; \beta_1 = \beta_2 = \lambda_1 = \lambda_2 = 0.15, \; \lambda_3 = 0.12. \end{aligned} \tag{3.1}$$

Next, as an example of (1.4), we consider the stochastic cholera model wtih a Holling type-II functional response disturbed by higher-order perturbation:

$$
\begin{aligned}
dS(t) &= \left(A - \sum_{j=1}^{2} \frac{\beta_j I_j S}{1 + aI_j} - \sum_{k=1}^{3} \frac{\lambda_k W_k S}{1 + aW_k} - \mu S\right) dt + (\sigma_{11} + \sigma_{12} S) S dB_1(t), \\
dI_1(t) &= \left(\sum_{j=1}^{2} \frac{\beta_j I_j S}{1 + aI_j} + \sum_{k=1}^{3} \frac{\lambda_k W_k S}{1 + aW_k} - \mu_1 I_1\right) dt + (\sigma_{21} + \sigma_{22} I_1) I_1 dB_2(t), \\
dI_2(t) &= (\gamma_1 I_1 - \mu_2 I_2) dt + (\sigma_{31} + \sigma_{32} I_2) I_2 dB_3(t), \\
dW_1(t) &= (\sum_{j=1}^{2} \xi_j I_j - \delta_1 W_1) dt + (\sigma_{41} + \sigma_{42} W_1) W_1 dB_4(t), \\
dW_k(t) &= (\delta_{k-1} W_{k-1} - \delta_k W_k) dt + (\sigma_{3+k,1} + \sigma_{3+k,2} W_k) W_k dB_{3+k}(t),\ k = 2, 3.
\end{aligned}
\tag{3.2}
$$

Obviously,

$$
f_j(S,x) = \frac{\beta_j S}{1 + ax},\ j = 1, 2;\ \ g_k(S,x) = \frac{\lambda_k S}{1 + ax},\ k = 1, 2, 3.
$$

Next, we verify that the conditions (H1)–(H3) hold.

(i) $\frac{-\partial f_j(S,x)}{\partial x} = -\frac{\beta_j a S}{(1+ax)^2} = -f_j(S,x)\frac{a}{1+ax} \geq -af_j(S,x)$.
Similarly, $\frac{-\partial g_k(S,x)}{\partial x} \geq -ag_k(S,x)$. We choose $K = a$ such that (H1) holds.
(ii) We choose $M = \max_{j=1,2;\ k=1,2,3} \{\beta_j,\ \lambda_k\}$ satisfying

$$
f_j(S,x) \leq \beta_j S \leq MS,\ g_k(S,x) \leq \lambda_k S \leq MS.
$$

(iii) It is easy to compute that

$$
S^3 \frac{d^2}{dS^2}\left(\frac{1}{f_j(S,0)}\right) = \frac{2}{\beta_j} = p_j,\ S^3 \frac{d^2}{dS^2}\left(\frac{1}{g_k(S,0)}\right) = \frac{2}{\lambda_k} = q_k,\ j = 1, 2;\ k = 1, 2, 3.
$$

Thus, (H1)–(H3) hold. By (1.3), we obtain that

$$
\begin{aligned}
R_0 &= \frac{A}{\mu}\left(\frac{\beta_1}{\mu_1} + \frac{\gamma_1 \beta_2}{\mu_1 \mu_2}\right) + \frac{A}{\mu}\left(\frac{\lambda_1}{\delta_1} + \frac{\lambda_2}{\delta_2} + \frac{\lambda_3}{\delta_3}\right)\left(\frac{\xi_1}{\mu_1} + \frac{\gamma_1 \xi_2}{\mu_1 \mu_2}\right) \\
&= 1.9873 > 1,
\end{aligned}
$$

which is the same as the R_0 in Example 2 of Song et al. [21] and implies that the positive equilibrium of the corresponding ODE system of (3.2) is uniformly persistent, illustrated in the red dotted lines of Fig. 2. Next, we further investigate the effects of higher-order disturbance of white noises.

Case 1. Choosing $d_j,\ n_k, j = 1, 2; k = 1, 2, 3$ and the white noises σ_{ij} as follows:

$$
d_j = 2.1 > \max\left\{0, \frac{f_j(S_0, 0) p_j}{2\mu S_0^2}\right\} = \frac{1}{A} = 2,\ j = 1, 2,
$$

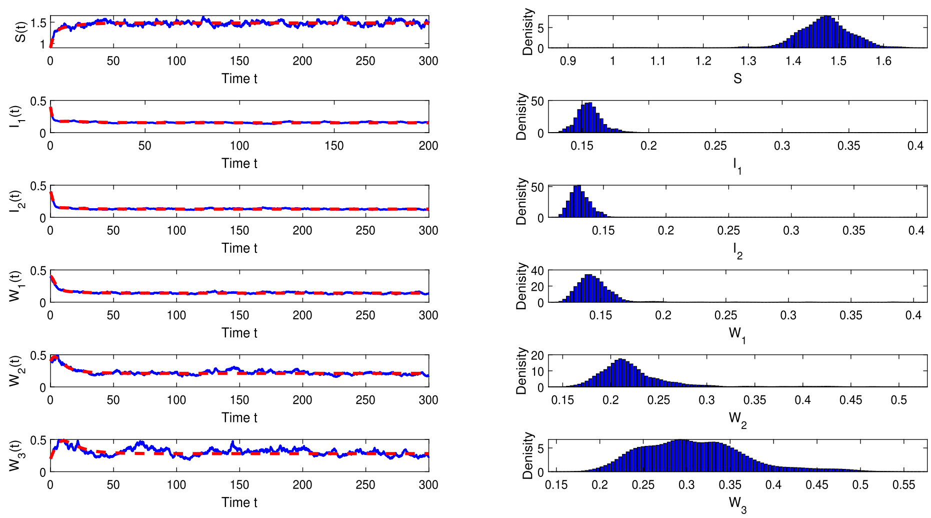

Figure 2 Left: The solutions of the corresponding deterministic and stochastic system (3.2). Initial value: $(S(0), I_1(0), I_2(0), W_1(0), W_2(0), W_3(0)) = (0.9, 0.4, 0.4, 0.4, 0.4, 0.2)$. Right: Distribution densities of $S(t)$, $I_1(t)$, $I_2(t)$, $W_1(t)$, $W_2(t)$, $W_3(t)$

$$n_k = 2.1 > \max\left\{0, \frac{g_k(S_0, 0)q_k}{2\mu S_0^2}\right\} = \frac{1}{A} = 2,\ k = 1, 2, 3$$

and $\sigma_{11} = \sigma_{12} = 0.01$, $\sigma_{21} = \sigma_{22} = 0.02$, $\sigma_{31} = \sigma_{32} = 0.03$, $\sigma_{41} = \sigma_{42} = 0.04$, $\sigma_{51} = \sigma_{52} = 0.05$, $\sigma_{61} = \sigma_{62} = 0.06$, we have that from (2.1),

$$\begin{aligned} R_0^s = & \left(\frac{\beta_1 S_0 \left(1 - d_1 S_0 \widetilde{h}_1\right)}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} + \frac{\beta_2 S_0 \left(1 - d_2 S_0 \widetilde{h}_1\right)}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} \cdot \frac{\gamma_1}{\mu_2 + \frac{\sigma_{31}^2}{2}} \right) \\ & + \left(\frac{\lambda_1 S_0 \left(1 - n_1 S_0 \widetilde{h}_1\right)}{\delta_1 + \frac{\sigma_{41}^2}{2}} + \sum_{k=2}^{3} \frac{\lambda_k S_0 \left(1 - n_k S_0 \widetilde{h}_1\right)}{\delta_1 + \frac{\sigma_{41}^2}{2}} \prod_{j=2}^{k} \frac{\delta_{j-1}}{\delta_j + \frac{\sigma_{3+j,1}^2}{2}} \right) \cdot \\ & \left(\frac{\xi_1}{\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}} + \frac{\gamma_1 \xi_2}{\left(\mu_1 + 2\sqrt[3]{A^2\sigma_{22}^2} + \frac{\sigma_{21}^2}{2}\right)\left(\mu_2 + \frac{\sigma_{31}^2}{2}\right)} \right) \\ = & 1.2342 > 1, \end{aligned}$$

which means that there exists a unique ergodic stationary distribution of system (3.2) according to Theorem 2.2, illustrated in the blue lines and the right graphs of Fig. 2. Moreover, $R_0^s = 1.2342 < R_0^* = 1.9699$, where R_0^* is the critical value of Song et al. [21], which implies that the higher-order disturbances $\sigma_{i,2}$, $i = 1, 2, \ldots, n + m + 1$ may decrease the critical value and speed up the extinction of the diseases.

Case 2. Fixing the parameters as in (3.1) and $\sigma_{i,2}$, $i = 1, 2, \ldots, 6$ as in Case 1, we increase the white-noise intensities σ_{i1}, $i = 1, 2, \ldots, 6$ as follows:

$$\sigma_{11} = 0.61,\ \sigma_{21} = 0.62,\ \sigma_{31} = 0.63,\ \sigma_{41} = 0.64,\ \sigma_{51} = 0.65,\ \sigma_{61} = 0.66.$$

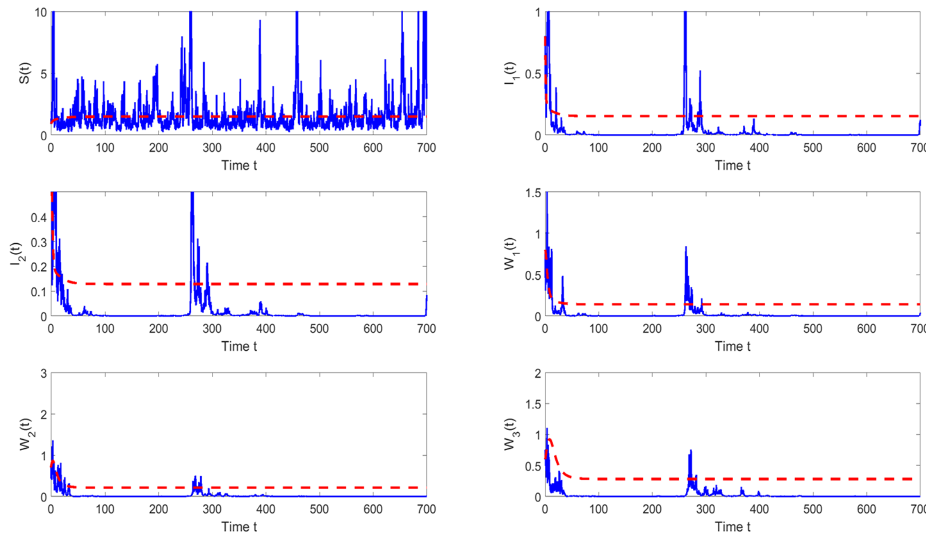

Figure 3 The solutions of the corresponding deterministic and stochastic system (3.2), which implies that the stronger white noises may result in the extinction of the diseases. Initial value: $(S(0), I_1(0), I_2(0), W_1(0), W_2(0), W_3(0)) = (0.9, 0.8, 0.7, 0.8, 0.7, 0.6)$

We have that $R_0^s = 0.2895 < 1$ from (2.1), which shows that the diseases $I_i(t)$, $W_i(t)$ tend to extinction from Fig. 3. For the general response function, we do not give the proof of extinction of the diseases in this article.

4 Conclusion and discussion

The spread and control of the infectious disease cholera disturbed by higher-order environment noises have always been hot topics discussed by many scholars and experts. In this article, we propose a stochastic cholera model, which is the summarization and generalization of Song et al. [21] and the corresponding ODE system. The distinguishing features are the general nonlinear incidence rates $f_j(S, I_j)I_j$, $g_k(S, W_k)W_k$, $j = 1, 2, \dots, n$; $k = 1, 2, \dots, m$ and the higher-order perturbations are only dependent on S (or I_j (W_k)) and include many existing results about cholera epidemics.

For the general stochastic system (1.4), we obtain the existence and ergodicity of the stationary distribution by using the theory of Khasminskii [28]. The most essential and difficult step is to construct a Lyapunov function satisfying Lemma A.1 in the Appendix. Due to the complexity and generality of system (1.4), we divide the construction of the Lyapunov function into three steps to determine the critical value R_0^s corresponding to the basic reproductive number R_0 of the ODE system (1.1). Finally, we apply our results to the stochastic system with a Holling type-II functional response and show that if $R_0^s > 1$, the diseases will prevail and the stronger white noises will result in the extinction of the diseases.

In this article, we show that the higher-order white noises can decrease the critical value and result in the extinction of the diseases, which provides us new insights into controlling the spread of the diseases. However, there are still many interesting problems worth studying in the near future. For example, when $f_j(I_j) = \frac{\beta_j I_j^2}{1+aI_j^2}$, $g_k(I_j) = \frac{\lambda_k I_k^2}{1+aI_k^2}$, that is, a Holling type-III functional response, our condition (H1) does not hold. It is challenging to deal with the existence of a stationary distribution for this case. We will leave these investigations to future work.

Appendix

Let $X(t)$ be a homogeneous Markov process in $\mathbb{R}_+^{n+m+1}$, which can be described by the following equation:

$$dX(t) = b(X(t))dt + \sum_{r=1}^{n+m+1} \sigma_r(X(t))dB_r(t). \tag{A.1}$$

The diffusion matrix is defined as follows:

$$A(x) = (a_{ij}(x))_{1\leq i,j\leq n+m+1},\ a_{ij}(x) = \sum_{r=1}^{n+m+1} \sigma_r^i(X)\sigma_r^j(X).$$

Applying the differential operator L to a function $V \in C^2(\mathbb{R}_+^{n+m+1}, R_+)$, Eq. (A.1) can be defined by

$$LV(X) = \sum_{i=1}^{n+m+1} b_i(X)\frac{\partial V}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{n+m+1} a_{ij}(X)\frac{\partial^2 V}{\partial X_i X_j}.$$

Lemma A.1 [28] *Assume there exists a bounded domain $D \subset \mathbb{R}_+^{n+m+1}$ with regular boundary Γ, and the following conditions hold*:

(A1) *there is a positive constant $\tilde{M}$ satisfying* $\sum_{i,j=1}^{n+m+1} a_{ij}(x)\varsigma_i\varsigma_j \geq \tilde{M}|\varsigma|^2$, *for* $x \in D$, $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_{n+m+1}) \in \mathbb{R}_+^{n+m+1}$;

(A2) *there exists a function* $V \in C^2(R_+^{n+m+1}; R_+)$ *and a positive constant* $\hat{C}$ *such that* $LV(x) \leq -\hat{C}$ *for any* $\mathbb{R}_+^{n+m+1}\backslash\bar{D}$.

Then, system (1.4) *is a unique and ergodic Markov process with a stationary distribution* $\pi(\cdot)$, *and letting* $f(\cdot)$ *be a function integrable with respect to the measure* π, *then*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{R_+^{n+m+1}} f(x)\pi(dx)\right\} = 1.$$

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 12371160), the Shandong Provincial Natural Science Foundation (Nos. ZR2023MA021 and ZR2021MA053), and the Fundamental Research Funds for the Central Universities of China (No. 22CX03013A).

Author contributions

Wenjie Zuo: Methodology, Investigation, Supervision, Writing, Reviewing, Editing. Beibei Liao: Writing, Revison, Simulation, Reviewing, Editing. Haile Wang: Investigation, Writing the original draft, Reviewing, Editing. Na Zhao: Investigation, Simulation. Daqing Jiang: Methodology, Investigation. All authors read and approved the final manuscript.

Data availability

No data was used for the research described in the article.

Declarations

Competing interests

The authors declare no competing interests.

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Received: 27 March 2024 Accepted: 15 July 2024 Published online: 16 December 2024

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