Approximation algorithms for the weighted edge-dominating set problem

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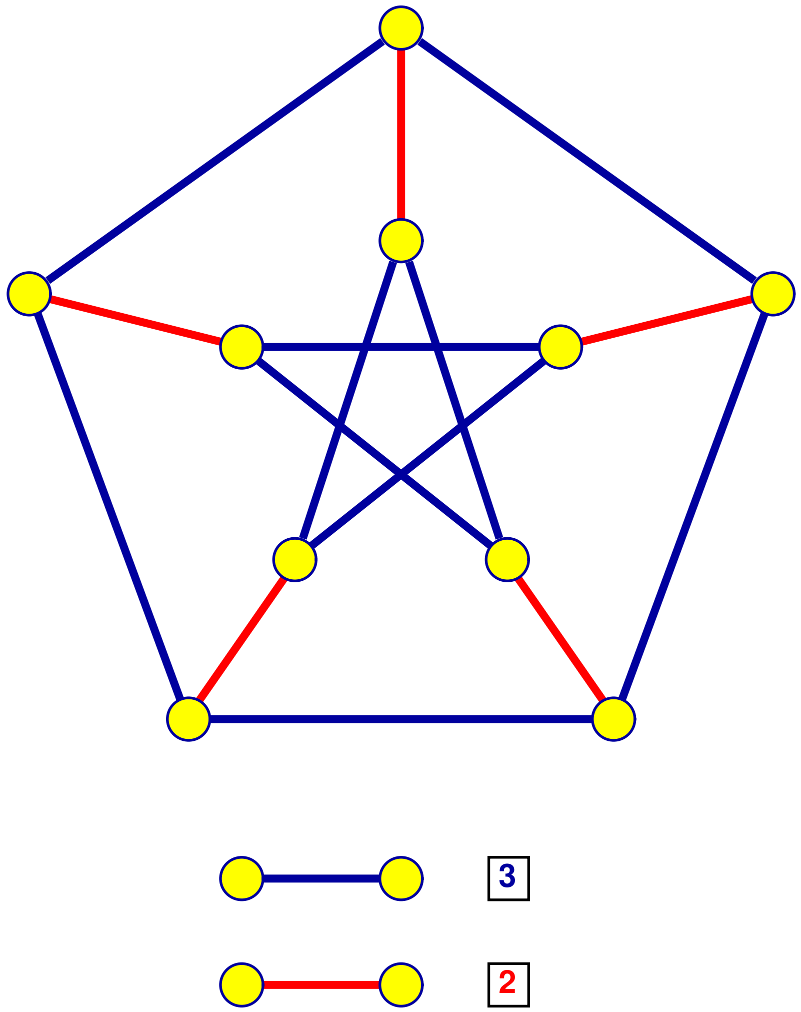
Coming attractions

1. Introduction
   - Weighted EDS
2. Hardness
3. Previous Work
4. Polyhedra
5. Algorithm
6. Gaps

The problem with definitions

Given a finite, simple, and undirected graph $G(V, E)$ and cost function $c : E \to \mathbb{Q}_+$, we seek to find a minimum cost set of edges which dominates E.



The red edges indicate an eds of cost 10.

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1. Introduction
2. Hardness
   - *NP*-completeness
   - Approximation hardness
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It sure ain't easy

Yannakakis and Gavril showed in '80 that the corresponding unweighted decision problem is *NP*-complete for even:

- Planar graphs
- Bipartite graphs of max degree ≤ 3

Settling for less

Here is a simple approximation-preserving reduction from weighted Vertex Cover to weighted EDS:

Thus a constant approximation ratio better than 2 for weighted EDS is worth a PhD.

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1. Introduction
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3. Previous Work
   - 2-Approx. for unweighted EDS
   - $(4 - o(|V|))$-Approx. for weighted EDS
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Reducto ad maximal matching

Note that every maximal matching is an eds.

[Harary, '69] Every minimal eds can be transformed into a maximal matching with at most as many edges.

Finally we have that $\frac{|A|}{|B|} \leq 2$ for any maximal matchings A and B.

So in fact any maximal matching provides a 2-Approximation.

Reducto ad Vertex Cover

[Folklore] We can transform an instance of weighted EDS into one of weighted VC by setting $w_u = \min_{e \in \delta(u)} c_e$:

If C^* is an optimal vc and D^* an optimal eds,

$$\sum_{u \in C^*} w_u \leq \sum_{u \in V(D^*)} w_u \leq 2 \sum_{e \in D^*} c_e .$$

We lose another factor of $2 - \frac{\log \log |V|}{2 \log |V|}$ since in general we can only approximate C^*.

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1. Introduction
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4. Polyhedra
   - Integer program (EDS(G))
   - Fractional relaxation (FEDS(G))
   - Edge Cover relaxation (FEC(G))
   - Full linear desc. of Edge Cover (EC(G))
5. Algorithm
6. Gaps

Formulate Away

We use $x(\delta(S))$ as shorthand for $\sum_{e\in\delta(S)} x_e$, omitting braces for singletons.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{e\in E} c_e x_e & \\
\text{subject to:} & & \\
(\mathrm{EDS}(G)) & x(\delta(u)) + x(\delta(v)) - x_{uv} \geq 1 & \{u,v\} \in E \\
 & x_e \in \{0,1\} & e \in E.
\end{array}
$$

The red constraint ensures that edge $\{u,v\}$ is dominated:

Relax

Taking a respite from the integrality conditions yields:

$$
\begin{array}{rll}
\min & \displaystyle\sum_{e \in E} c_e x_e & \\
\text{subject to:} & & \\
(\mathrm{FEDS}(G)) & x(\delta(u)) + x(\delta(v)) - x_{uv} \geq 1 & \{u, v\} \in E \\
& x_e \geq 0 & e \in E,
\end{array}
$$

which we can solve in polynomial time. However, $x \in$(FEDS(G)) can have rational elements with arbitrarily large denominators:

Help please

We will call to the aid of the following linear relaxation of the Edge Cover problem.

$$
\begin{array}{rll}
\min & \displaystyle\sum_{e \in E} c_e x_e & \\
\text{subject to:} & & \\
(\mathrm{FEC}(G)) & x(\delta(u)) \geq 1 & u \in V \\
 & x_e \geq 0 & e \in E,
\end{array}
$$

whose extreme points are $\frac{1}{2}$-integral:

Even better

In fact we can one-up the previous slide.

$$\begin{array}{rll} \min & \sum_{e\in E} c_e x_e & \\ \text{subject to:} & & \\ & x(\delta(u)) \geq 1 & u \in V \\ & x_e \in \{0,1\} & e \in E \end{array}$$

[Edmonds & Johnson, '70] is equivalent to

$$\begin{array}{rll} \min & \sum_{e\in E} c_e x_e & \\ \text{subject to:} & & \\ (\mathrm{EC}(G)) & x(\delta(S)) + x(E(S)) \geq \left\lceil \frac{|S|}{2} \right\rceil & S \subseteq V \\ & x_e \geq 0 & e \in E, \end{array}$$

which, despite the large number of constraints, we can solve efficiently.

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5. Algorithm
   - Reduction to edge cover
   - Take 1: 4-Approximation
   - Take 2: $2\frac{2}{3}$-Approximation
   - Take 3: $2\frac{1}{10}$-Approximation
6. Gaps

Reducto ad edge cover

Recall that any feasible solution $x \in (\text{FEDS}(G))$ satisfies

$$x(\delta(u)) + x(\delta(v)) \geq 1 \quad \{u, v\} \in E,$$

hence either

$$x(\delta(u)) \geq \frac{1}{2}, \text{ or } x(\delta(v)) \geq \frac{1}{2}.$$

We let $V_+ = \{v \in V \mid x(\delta(v)) \geq \frac{1}{2}\}$:

Note that V_+ is a vertex cover of G.

A bona fide instance

Since V_+ is a vc, any set of edges covering the vertices in V_+ is an eds. We have a cost-preserving transformation from an instance (G, V_+) to one of EC, $\bar{G}$:

What is the quality of our reduction from EDS to EC?

A Genuine home-cooked square rounding

Suppose we have V_+ derived from $x \in$(FEDS(G)) as before. We let $y = 2x$ and $\bar{y}$ be the extension of y to $\bar{G}$. Since

$$V_+ = \{v \in V \mid y(\delta(v)) \geq 1\},$$

$\bar{y} \in$(FEC($\bar{G}$)):

Take 1: 4

So the cost of a minimum weight fractional ec, $\bar{y}^*$, of $\bar{G}$ is at most twice that of a minimum weight (fractional) eds, x^*, of G:

$$\sum_{e \in \bar{E}} \bar{c}_e \bar{y}^*_e \leq 2 \sum_{e \in E} \bar{c}_e \bar{x}^*_e = 2 \sum_{e \in E} c_e x^*_e.$$

But we can assume that $\bar{y}^*$ is $\frac{1}{2}$-integral, so we immediately have a 4-approximation:

Take 2: $2\frac{2}{3}$

Yes, $\bar{y}^*$ is $\frac{1}{2}$-integral, but do we really have to multiply by 2 for an intergral solution?

So we all we really need to do is to decompose $\bar{y}^*$ into a convex combination of integral soultions, for which a factor $\frac{4}{3}$ sufficed above.

It turns out that $\frac{4}{3}$ does indeed suffice. Looking at (FEC($\bar{G}$)), we have

$$\bar{y}(\delta(u)) \geq 1 \quad u \in \bar{V}.$$

Summing these over some $S \subseteq V$ we have

$$\bar{y}(\delta(S)) + 2\bar{y}(E(S)) = \sum_{u \in S} \bar{y}(\delta(u)) \geq |S|,$$

which yields

$$\bar{y}(\delta(S)) + \bar{y}(E(S)) \geq \frac{|S|}{2}.$$

From the description of $(\mathrm{EC}(\bar{G}))$, we need the rhs of the above inequality to be $\left\lceil \frac{|S|}{2} \right\rceil$ instead of $\frac{|S|}{2}$.

We need only check that $\max_{S \subseteq \bar{V}} \left\{ \frac{\left\lceil \frac{|S|}{2} \right\rceil}{\frac{|S|}{2}} \right\} = \frac{4}{3}$, when $|S| = 3$.

Take 3: $2\frac{1}{10}$

Can we elminate the intermediate step of rounding into $(\text{FEC}(\bar{G}))$? We actually have

$$\bar{y}(\delta(u)) + \bar{y}(\delta(v)) \geq \begin{cases} 2 + \bar{y}_{uv} & \text{if } uv \in \bar{E}, \\ 2 & \text{otherwise.} \end{cases}$$

Attempting an analogous approach, we set $s = |S|$ and sum the appropriate inequality above for each pair $\{u, v\}$ in $S \times S$, where $u \neq v$ to get

$$\begin{array}{ll} (s-1)\bar{y}(\delta(S)) + 2(s-1)\bar{y}(\bar{E}(S)) & = \\ (s-1)\sum_{u \in S} \bar{y}(\delta(u)) & = \\ \sum_{\{\{u,v\} \in S \times S \mid u \neq v\}} \bar{y}(\delta(u)) + \bar{y}(\delta(v)) & \geq \\ s(s-1) + \bar{y}(\bar{E}(S)). & \end{array}$$

Isolating the desired left hand side yields

$$\begin{aligned}\bar{y}(\delta(S)) + \bar{y}(\bar{E}(S)) &\geq \frac{s(s-1) + (s-2)\bar{y}(\delta(S))}{2s-3} \\ &\geq \frac{s(s-1)}{2s-3}, \text{ for } s \geq 3.\end{aligned}$$

Using standard optimization techniques,

$$\max_{s \geq 3, \text{ odd}} \left\{ \frac{\left\lceil \frac{|S|}{2} \right\rceil}{\frac{s(s-1)}{2s-3}} \right\} = \frac{21}{20},$$

which is achieved when $s = 5$.

At the end of the rainbow

Our algorithm is simply:

1. Compute an optimal solution x^* for (FEDS(G)).
2. Determine V_+.
3. Output a minimum weight edge cover of V_+.

Note that although we may transform (G, V_+) to $\bar{G}$ for step 3 above, we are free to use any specialized algorithm we wish.

From our previous arguments we have that $\frac{21}{20}\bar{y}$, which is the extension of $y = 2x$ to $\bar{G}$, is a convex combination of integral edge covers of $\bar{G}$.

An optimal integral edge cover of $\bar{G}$ therefore costs at most

$$\frac{21}{20}(2x^*) \leq \frac{21}{10}(\text{cost of an optimal integral eds})$$

and corresponds to an integral eds of G of the same cost.

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    - Algorithm

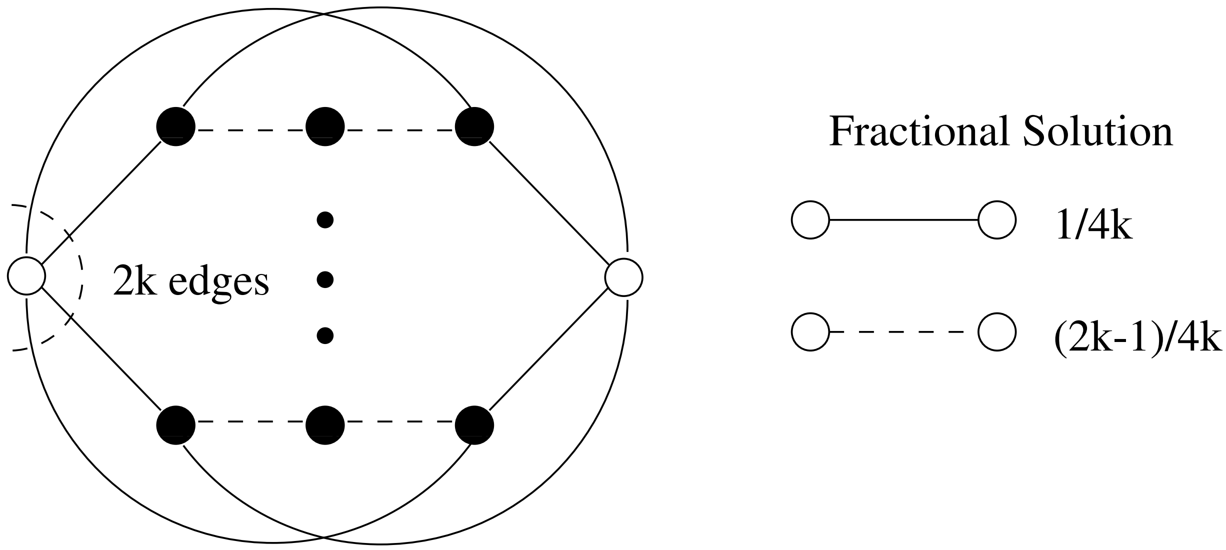
Closing the gap

Is there hope we might improve the approximation guarantee? The integrality gap of our EDS polyhedra is defined to be

$$\max_{G=(V,E),\, c\in\mathbb{Q}_+^{|E|}} \left\{ \frac{\min_{x\in(\mathrm{EDS}(G))} c\cdot x}{\min_{x\in(\mathrm{FEDS}(G))} c\cdot x} \right\}$$

Our algorithm shows this is at most $\frac{21}{10}$, and in fact this is tight:

We cannot hope to prove a better ratio exclusively using (FEDS(G)) as a lower bound. We do know that the algorithm is at least asymptotically a 2-approximation:



A fractional extreme point of cost $k + \frac{1}{2}$. The algorithm chooses the darkened vertices as V_+, yielding a solution of cost $2k$, while the optimal integral solution costs $k + 1$.