Mehryar Mohri
Foundations of Machine Learning
Courant Institute of Mathematical Sciences
Homework assignment 1
Due: Feb 15th, 2008
Credit: Ashish Rastogi, Afshin Rostamizadeh
Ameet Talwalkar, and Eugene Weinstein.

1. **Probability Review**:

   (a) Imagine you are given one fair die, and you need to decide which task is harder: (i) guessing the value of one die toss or (ii) tossing the die twice and getting the same value twice. Given that the die is fair (every side has weight 1/6), does event (i) have a greater chance of success or event (ii), or do they have the same probability of success? Make sure to give justification.

   (b) We will now generalize this result to n-sided dice with any (possibly non-uniform) distribution. First prove the following useful fact, for any $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\sum_i \alpha_i = 1$, the following holds,
$$0 \leq \sum_{i=1}^{n} (\alpha_i - 1/n)^2 = \sum_{i=1}^{n} \alpha_i^2 - 1/n$$

   (c) Let X_1 be the value of the first toss, and X_2 be the value of the second toss. Show that $\Pr(X_1 = X_2) \geq 1/n$ (hint: use part b). For what distribution is the inequality tight?

2. **Concentration Bounds:**

   (a) Given a sample of m bounded points $X = (x_1, x_2, \ldots, x_m), \forall i, |x_i| \leq M$, define the function
$$f(X) = \frac{1}{m} \sum_i x_i.$$
   Can you give a bound on the probability $\Pr[|f(X) - \mathrm{E}[f(X)] \geq \epsilon]$?

   (b) Let X and X' be two sets of size m that differ in exactly one point. That is, $|X \cap X'| = m - 1$. We say a function h is *stable* if for all such X, X', $|h(X) - h(X')| \leq g(m)$ for some decreasing function g. How quickly does g need to decrease as a function of

m in order for McDiarmid's inequality to provide a bound on the event $\Pr[|h(X) - \mathrm{E}[h(X)]| \geq \epsilon]$ that converges to zero as $m \to \infty$?

(c) Is the function f from part (a) stable (still assuming the bound $|x_i| \leq M, \forall i$)? Will McDiarmid's inequality provide provide a convergent bound? If so give the bound. Now define the function $f'(X) = \max(X)$, is f' stable? Can you give a bound with McDiarmid's inequality?

3. **PAC Learning:** Here we will consider an alternative PAC learning scenario, called the two-oracle model. Imagine you are given the ability to explicitly ask for a positive or negative sample, which are drawn from different distributions D_+ and D_- respectively. A concept is efficiently PAC-learnable if there exists an algorithm L that can generate a hypothesis h, such that $\Pr_{x \sim D_+}[h(x) = 0] \leq \epsilon$ and $\Pr_{x \sim D_-}[h(x) = 1] \leq \epsilon$ with confidence $(1 - \delta)$, after sampling $m = \mathrm{poly}(1/\epsilon, 1/\delta)$ points.

   (a) Show that if a problem is efficiently PAC-learnable in the classic sense, it is also always efficiently PAC-learnable in the two-oracle model.

   (b) (Bonus) Show that the reverse direction is also true.