Repeated Communication and Ramsey Graphs

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Preliminaries

Graph Theoretic Notation

Given a graph $G = (V, E)$, we recall the following notation.

- $\alpha(G)$ - G's independence number.
- $\omega(G)$ - G's clique number.
- $\chi(G)$ - G's chromatic number.

Graph Products

Definition (AND Product)

Given graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, *their AND product,* $G_1 \wedge G_2$, *is a graph with vertex set* $V_1 \times V_2$ *and edges given by* $(u_1, u_2) \sim (v_1, v_2) \iff \forall i \in [2],\ u_i = v_i$ *or* $u_i \sim v_i$ *in* E_i.

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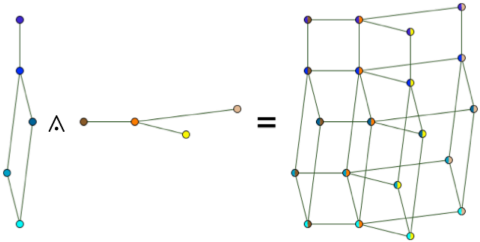

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- Note: $\overline{G_1} \wedge \cdots \wedge \overline{G_n} = \overline{G_1 \vee \cdots \vee G_n}$.
- We abbreviate $G^{\wedge n} \triangleq (\overbrace{G \wedge \cdots \wedge G}^{n})$ and $G^{\vee n} \triangleq (\overbrace{G \vee \cdots \vee G}^{n})$.

Ramsey Numbers

Definition (Ramsey Numbers)

The Ramsey Number $r_n(l_1, l_2, \ldots, l_n)$ is the maximum r such that there exists a coloring with n colors of the edges of the complete graph on r vertices, K_r, such that every obtained i-monochromatic clique has size at most l_i.

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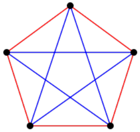

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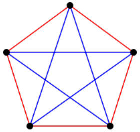

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- $r_2(2) = 5$.
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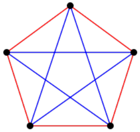
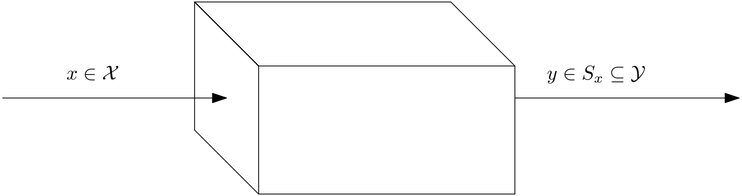
- Generally, $\sqrt{2}^l \leq r_2(l) \leq 4^l$.



Channel Coding

Definition (Channel Coding [Shannon '56])

A channel $\mathcal{C}$ is given by an input set $\mathcal{X}$ and output set $\mathcal{Y}$, as well as fan-out sets S_x for every $x \in \mathcal{X}$. When a sender sends an input $x \in \mathcal{X}$ the receiver receives some $y \in S_x \subseteq \mathcal{Y}$.



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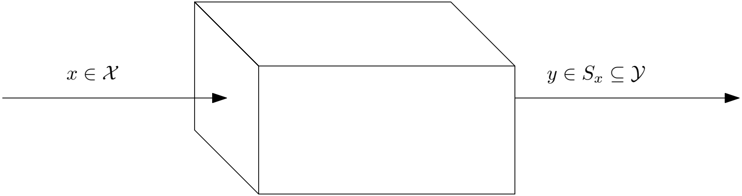

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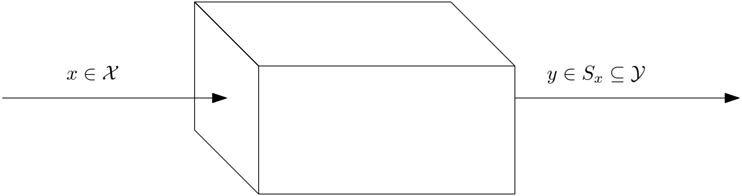

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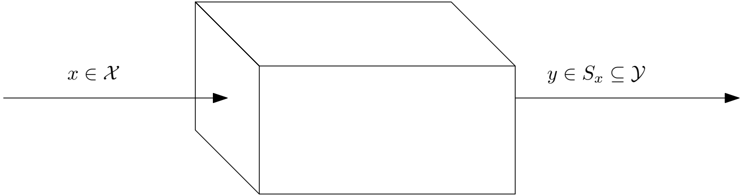

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- The pentagon channel: $\mathcal{X} = \mathcal{Y} = \{0, 1, 2, 3, 4\}$ and $S_x = \{x, x + 1 \mod 5\}$ for all $x \in \mathcal{X}$.

Definition (Single-Use Capacity)

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Theorem

For every channel $\mathcal{C}$ with characteristic graph $\mathcal{G}$,

$$\gamma^{(1)} = \log \omega(\mathcal{G}).$$

Definition (Multiple-use Capacity)

The n-use capacity of a channel $\mathcal{C}$, denoted by $\gamma^{(n)}$, is the maximum number of bits the sender can transmit to the receiver with no error in n uses of the channel $\mathcal{C}$.

Theorem

For every $\gamma^{(1)}$ there exists a channel $\mathcal{C}$ with $\gamma^{(2)} \geq 2^{\gamma^{(1)}-1}$.
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Corollary

There exists a channel $\mathcal{C}$ with exponentially larger per-use capacity than its single-use capacity.

Characterising $\gamma^{(n)}$ vs. $\gamma^{(1)}$ - first step

The n-use use of the channel $\mathcal{C}$ is equivalent to one use of a channel $\mathcal{C}^n$ with input and output sets $\mathcal{X}^n$ and $\mathcal{Y}^n$. For every n-tuple $\bar{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ we have $S_{\bar{x}} = S_{x_1} \times \cdots \times S_{x_n}$. The characteristic graph of $\mathcal{C}^n$ is therefore $\mathcal{G}^{\vee n}$.

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For every channel $\mathcal{C}$ with characteristic graph $\mathcal{G}$,

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Proof of Capacity Increase

Definition

Let $\rho_n(l_1, \ldots, l_n) \triangleq \max\{\omega(\mathcal{G}_1 \wedge \cdots \wedge \mathcal{G}_n) \mid \omega(\mathcal{G}_i) \leq l_i,\ \forall i \in [n]\}$.

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Recall that $\sqrt{2}^l \le r_2(l) \le 4^l$.
$\therefore$ for every $\gamma^{(1)}$ there exists a channel $\mathcal{C}$ with $\gamma^{(2)} \ge 2^{\gamma^{(1)}-1}$,
and every such channel $\mathcal{C}$ satisfies $\gamma^{(2)} \le 2^{\gamma^{(1)}+1}$.

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Proof - Part I.

$\underline{\rho_n(l) \geq r_n(l)}$. Let $r = r_n(l)$. Fix an n-coloring of the edges of K_r such that all i-monochromatic cliques have size $\leq l$. Let G_i be the graph induced by the i-colored edges of K_r. Clearly $\omega(G_i) \leq l$ and $\{(1, \ldots, 1), \ldots, (r, \ldots, r)\}$ is a clique in $G_1 \veebar \cdots \veebar G_n$.

□

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Proof - Part II.

$\underline{r_n(l) \geq \rho_n(l)}$. Let $\rho = \rho_n(l)$. Let $G_1, \ldots, G_n$ and $G = G_1 \veebar \cdots \veebar G_n$ be graphs with $\omega(G_i) \leq l$ such that $\omega(G) = \rho$. Fix a clique of size ρ in G, $S = \{(x_1^1, \ldots, x_n^1), \ldots, (x_1^\rho, \ldots, x_n^\rho)\}$. We n-color K_ρ by assigning each edge (u, v) a color i such that $(x_i^u, x_i^v) \in E[G_i]$. Every monochromatic set under this n-coloring has size $\leq l$. $\square$

An Open Question: Per-Use Capacity

Definition (Per-Use Capacity)

The n-use per-use capacity of a channel is $C^{(n)} = \gamma^{(n)}/n$.
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Open Question: Can $C^{(n)} = \gamma^{(n)}/n$ grow with n for fixed $\gamma^{(1)} = c$? If true, this would resolve an open question of Erdős, of whether $r_n(c')$ grows faster than any exponential in c'.

Dual-Source Coding

Definition (Dual-Source)

A dual-source consists of a finite set $\mathcal{X}$, a set $\mathcal{Y}$ and a support set $\mathcal{S} \subseteq \mathcal{X} \times \mathcal{Y}$. The fan-out set $S_x = \{y : (x, y) \in \mathcal{S}\}$.

Definition (Dual-Source Instance)

In each dual-source instance sender $P_{\mathcal{X}}$ and receiver $P_{\mathcal{Y}}$ are given (before hand) symbols x and y, respectively, where $(x, y) \in \mathcal{S}$.

We are interested in the minimum number of bits needed (irrespective of the starting instance), that $P_{\mathcal{X}}$ needs to send (noiselessly) for $P_{\mathcal{Y}}$ to learn x.

Definition (Single instance cost)

The single-instance (n-instance) cost of a dual-source $\mathcal{S}$, denoted by $\sigma^{(1)}$ ($\sigma^{(n)}$), is the minimum number of bits the sender needs to transmit irrespective of starting instance (n instances) of $\mathcal{S}$.

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For every $\epsilon, t > 0$, there is a graph $\mathcal{G}$, such that $\forall n$, $\mathcal{X}(\mathcal{G}) \geq \epsilon t$ but $\mathcal{X}(\mathcal{G}^{\wedge n}) \leq O(nt(2+\epsilon)^{n+1})$.

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Corollary

For every $\epsilon > 0$, and arbitrarily large $\sigma^{(1)}$, there are dual sources such that, such that $R^{(1)} \geq \sigma^{(1)}$ and $R^{(\infty)} \leq 1 + \epsilon$.

Preliminaries of Proof

Definition

The vertices of the Kneser graph $K(u, t)$ *correspond to all the* $\binom{u}{t}$ *subsets of size* t *of* $\{1, \ldots u\}$. *Two vertices are connected iff they are disjoint.*

Coloring Kneser graphs

Observation

Let $\{1, \ldots u\}$ be a set of colors. Assigning to each vertex v any element $x \in v$ as it's color, generates a valid coloring of $K(u,t)$.

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proof

Theorem

$\mathcal{X}(K(u,t)) = u - 2t + 2$ *[Lovasz]*

While, $\mathcal{X}(K^{\wedge n}(u,t)) \leq n(\frac{u}{t})^n \ln \binom{u}{t} = m$

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Prob. of a color being representative for a vertex v : $(t/u)^n$.

□

proof

Theorem

$\mathcal{X}(K(u,t)) = u - 2t + 2$ *[Lovasz]*

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Proof.

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The claim follows by union bound over all $\binom{u}{t}^n$ vertices.

□

Open Question

Definition (Normalized per-use capacity)

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Open Question: Is it true that for every $\epsilon > 0$ there is a dual-source such that $\tilde{R}^{(1)} \geq 1 - \epsilon$ but $\tilde{R}^{(\infty)} \leq \epsilon$?

A positive answer would have interesting applications to communication complexity.

Thank You.