List Decoding Reed-Muller Codes over $\mathbb{F}_2$

Sahil Singla (ssingla@cmu.edu)
Manzil Zaheer (manzil@cmu.edu)

Original Paper by Gopalan, Klivans and Zuckerman

December 3, 2014

Algebraic Code

Algebraic Coding Theory

Linear Block Codes

- Partition message into blocks and encode as polynomials
- 1 codeword $\leftrightarrow$ 1 message
  - Reed-Solomon codes: Univariate polynomials
  - Reed-Muller codes: Multivariate polynomials
- List decoding

Convolutional Codes

- Message treated as series and encoded into series
- 1 codeword is weighted sum input messages
  - Turbo codes
- Viterbi algorithm
- Historically used commonly as easier to implement

Both posses same error correcting power!

Background

Reed-Muller Codes

Given a field size q, a number m of variables, and a total degree bound r, the $\mathrm{RM}_q[m, r]$ code is the linear code over $\mathbb{F}_q$ defined by the encoding map:

$$f(X_1, \dots, X_m) \to \langle f(\alpha) \rangle_{\alpha \in \mathbb{F}_q^m}$$

applies to the domain of all polynomials in $\mathbb{F}_q[X_1, \dots, X_m]$ of total degree $\deg(f) \le r$.

For the binary case, i.e. $q = 2$

- Block length $n = 2^m$
- Dimension $k = \sum_{i=0}^{r} \binom{m}{i}$
- Distance $d = 2^{m-r}, \delta = d/n = 2^{-r}$

For $r = 1$ boils down to Hadamard code.

Decoding RM Codes

- Unique Decoding:
  - Majority Logic Circuit Decoder [Reed, 1954, Muller, 1954]
  - Works when error rate $\eta < 2^{-r-1} - \epsilon$

Decoding RM Codes

- Unique Decoding:
  - Majority Logic Circuit Decoder [Reed, 1954, Muller, 1954]
  - Works when error rate $\eta < 2^{-r-1} - \epsilon$
- List Decoding for the case $r = 1$
  - Goldreich-Levin Method [Goldreich and Levin, 1989]
  - When error rate $\eta < \frac{1}{2} - \epsilon$
  - Outputs a list of size $\leq 2m/\epsilon^2$
  - In time poly$(m, 1/\epsilon)$

Decoding RM Codes

- Unique Decoding:
  - Majority Logic Circuit Decoder [Reed, 1954, Muller, 1954]
  - Works when error rate $\eta < 2^{-r-1} - \epsilon$
- List Decoding for the case $r = 1$
  - Goldreich-Levin Method [Goldreich and Levin, 1989]
  - When error rate $\eta < \frac{1}{2} - \epsilon$
  - Outputs a list of size $\leq 2m/\epsilon^2$
  - In time poly$(m, 1/\epsilon)$
- List Decoding for the case $r \geq 2$ – This talk!
  - Built by generalizing GL as in [Gopalan et al., 2008]
  - When error rate $\eta < 2^{-r} - \epsilon$
  - Outputs a list of size $O(\epsilon^{-8r})$
  - In time poly$_r(m, 1/\epsilon)$

Marketing of GKZ I

Beats Johnson Bound!

- Recall Johnson Bound
  - When $\eta < J(\delta) - \epsilon$, then
  - code is list decodable with list size $O(\epsilon^2)$
  - where $J(\delta) = \frac{1}{2}(1 - \sqrt{1 - 2\delta})$
- For RM codes, we have $\delta = 2^{-r}$

	Johnson Bound	GKZ List Decoding
List Size	$O(\epsilon^2)$	$O(\epsilon^2)$
Time	–	$\text{poly}_r(m, 1/\epsilon)$
Max Error	$J(2^{-r}) - \epsilon$	$2^{-r} - \epsilon$
Example ($r = 2$)	0.146	0.25

Marketing of GKZ II

Can we do better?

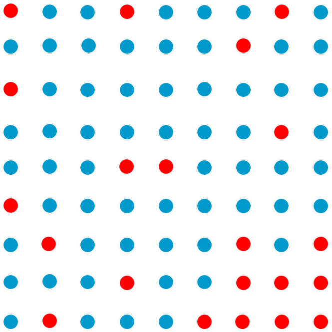
- No! as exponentially many codewords at distance of 2^{-r}
- An example:
  - Let $\mathbf{V}_1, \ldots, \mathbf{V}_t \subset \mathbb{F}_2^m$ such that $\forall i : \dim(\mathbf{V}_i) = m - r$.
  - Each $\mathbf{V}_i$ has a parity check matrix $[H^{(i)}]_{r \times m}$
  - Consider the polynomials

$$P_i(x) = \prod_{j=1}^{r} (1 + \langle H_j^{(i)}, x \rangle) = \begin{cases} 1 & \text{if } x \in \mathbf{V}_i \\ 0 & \text{else} \end{cases}$$

  - All P_i's are unique
  - They are valid codewords in $\mathrm{RM}(m, r)$ code!
  - If we receive $R = 0$, then all these are at distance 2^{-r}
  - Note $t =$ Number of subspace of dimension $m - r > 2^{r(m-r)}$

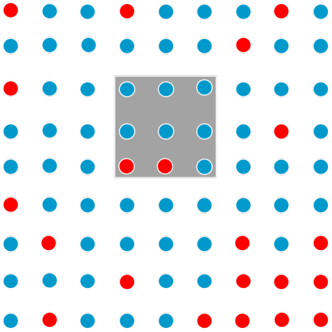
GL: Hadamard List Decoding

- Let the message be $s \in \mathbb{F}_2^m$ and define $P(x) = \langle s, x \rangle$
- Then $\mathrm{Had}(s) = \langle P(\alpha) \rangle_{\alpha \in \mathbb{F}_2^m}$
- We receive a noisy function $R : \mathbb{F}_2^m \to \mathbb{F}_2$ such that $\Delta(P, R) \leq \eta < \frac{1}{2} - \epsilon$
- Goal: Recover the message s (or equivalently P) from R



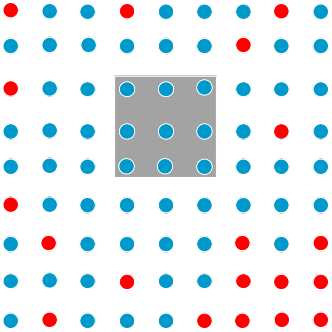
- Enumerated R
- Error $R(x) \neq P(x)$
- Correct $R(x) = P(x)$

GL: Hadamard List Decoding



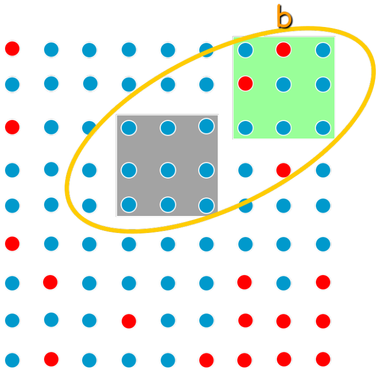
- Set $k := O(\log(m/\epsilon))$
- Begin by selecting a random subspace A of $\dim(A) = k$
- Assume $\forall x \in A : R(x) = P(x)$
- Call them "hints"

GL: Hadamard List Decoding



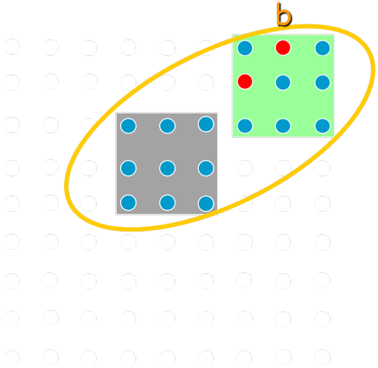
- Set $k := O(\log(m/\epsilon))$
- Begin by selecting a random subspace A of $\dim(A) = k$
- Assume $\forall x \in A : R(x) = P(x)$
- Call them "hints"

GL: Hadamard List Decoding



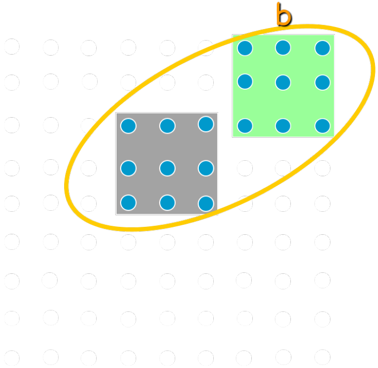
- Given the hints
- For any $b \in \mathbb{F}_2^m$
- Consider the space $b + A$
- Error in $A = 0$ (assumed)
- Error in $b + A < \eta + \epsilon$ (with constant probability)

GL: Hadamard List Decoding



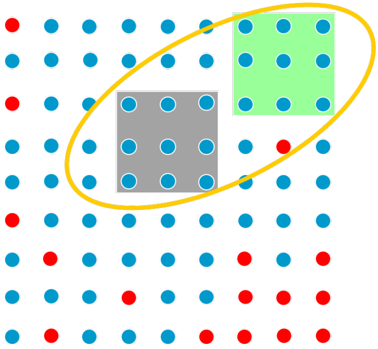
- Error in $A = 0$
- Error in $b + A < \eta + \epsilon$
- Error in combined subspace $< \frac{\eta+\epsilon}{2} < \frac{1}{4}$
- Unique Decode!

GL: Hadamard List Decoding



- Error in $A = 0$
- Error in $b + A < \eta + \epsilon$
- Error in combined subspace $< \frac{\eta+\epsilon}{2} < \frac{1}{4}$
- Unique Decode!

GL: Hadamard List Decoding

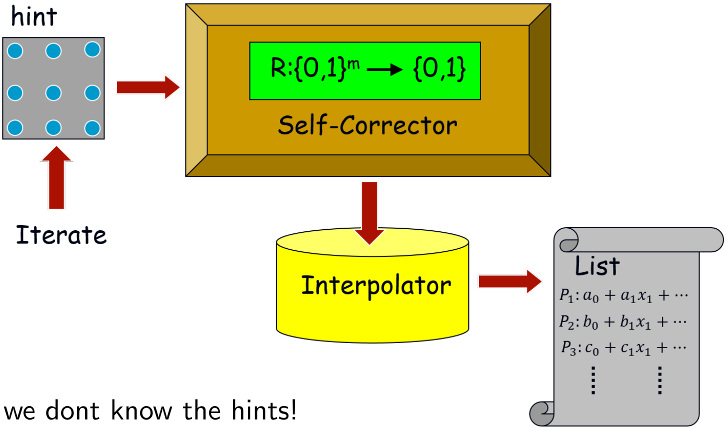


- Error in $A = 0$
- Error in $b + A < \eta + \epsilon$
- Error in combined subspace $< \frac{\eta+\epsilon}{2} < \frac{1}{4}$
- Unique Decode!

Interpolating Sets

- Q: For how many b's do we need to run this?
- A: As many times as it needs to uniquely determine the polynomial P
  - In case of Hadamard codes, P is linear in m variables
  - It suffices to run for $b = e_1, \dots, e_m$
- In general, for a degree r polynomial in m variables
  - The set sufficient to efficiently determine the polynomial uniquely is called the interpolating set
  - Any Hamming ball of radius r is an interpolating set having $O(m^r)$ points.

Summary



- But we dont know the hints!
- Iterate over all possible hints
- $\#$ hints $= 2^k =$ poly$(m, 1/\epsilon)$
- $\therefore$ still polynomial in list size and time

Problems porting to RM

- Most of the steps for GL can be directly ported for general $\mathrm{RM}[r, m]$ codes
- Brute forcing over guess doesn't work any more
  - Too many choices for $r \geq 2$
  - For being able to evaluate $Q(a+b)$, we need to make $2^{O(k^r)}$ guess

Finding restriction P_A

- Note with high probability $\Delta(P_A, R_A) \leq \eta + \epsilon$
- Thus, find list $\mathcal{L}$ of every degree r polynomial Q on k dimensions s.t. $\Delta(Q, R_A) \leq \eta + \epsilon$
- Moreover, since $k = O(\log \frac{m}{\epsilon})$, we can use a global list decoding algorithm

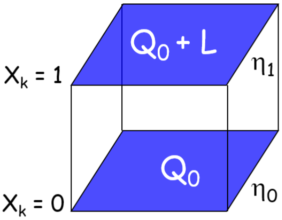
Finding restriction P_A

- Note with high probability $\Delta(P_A, R_A) \leq \eta + \epsilon$
- Thus, find list $\mathcal{L}$ of every degree r polynomial Q on k dimensions s.t. $\Delta(Q, R_A) \leq \eta + \epsilon$
- Moreover, since $k = O(\log \frac{m}{\epsilon})$, we can use a global list decoding algorithm

Challenges:

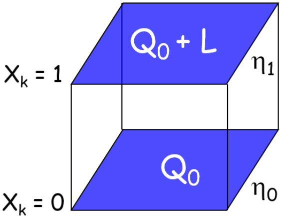
1. Design a global RM list decoding algorithm.
2. Argue $|\mathcal{L}|$ is $O(\epsilon^{-8r})$

Global RM list decoding



- $\eta = \frac{1}{2}(\eta_0 + \eta_1)$
- Assume $\eta_0 \leq \eta_1$
- Thus, $\eta_0 \leq \eta$ and $\eta_1 \leq 2\eta$

Global RM list decoding



- $\eta = \frac{1}{2}(\eta_0 + \eta_1)$
- Assume $\eta_0 \leq \eta_1$
- Thus, $\eta_0 \leq \eta$ and $\eta_1 \leq 2\eta$

- Note $Q = Q_0(X_1, \ldots, X_{k-1}) + X_k Q'(X_1, \ldots, X_{k-1})$
- Recurse over Q_0: $\eta_0 \leq \eta$ and degree at most k
- Recurse over Q': $\eta_1 \leq 2\eta$ and degree at most $k-1$

Since we don't know if $\eta_0 \leq \eta_1$, try every possible 2^k orders

Reduction of $A's$ dimension

- The original algorithm has $k \geq O(\log \frac{m}{\epsilon})$.
- Instead, $k \geq O(\log \frac{1}{\epsilon})$ suffices
- First showed using clever interpolating sets, Dvir-Shpilka [Dvir and Shpilka, 2008]
- Later showed by implementing Reed's Majority Logic Decoder locally
- Hence, $l(r, m, 2^{-r} - \epsilon) = O(l(r, k, 2^{-r}))$
- We bound $l(r, k, 2^{-r})$ by $O(\epsilon^{-8r})$

Deletion lemma

Johnson Bound

For any code $\mathcal{C}$ with distance δn and any $R \in \{0,1\}^n$

- Number of C such that $\Delta(R, C) < J(\delta) - \gamma$ is at most $O(\gamma^{-2})$
- Number of C such that $\Delta(R, C) < J(\delta)$ is at most $2n$

Deletion lemma

Johnson Bound

For any code $\mathcal{C}$ with distance δn and any $R \in \{0,1\}^n$

- Number of C such that $\Delta(R, C) < J(\delta) - \gamma$ is at most $O(\gamma^{-2})$
- Number of C such that $\Delta(R, C) < J(\delta)$ is at most $2n$

Let $A(\alpha)$ be number of codewords of weight less than α

Deletion lemma

For any linear code $\mathcal{C}$ and $\alpha \in [0,1]$ and $R \in \{0,1\}^n$

- Number of C such that $\Delta(R, C) < J(\alpha) - \gamma$ is at most $A(\alpha)O(\gamma^{-2})$
- Number of C such that $\Delta(R, C) < J(\alpha)$ is at most $2A(\alpha)n$

- Generalization of Johnson Bound for $\alpha = \delta$ and $A(\delta) = 1$

Bounding list size $|\mathcal{L}|$

Let $\alpha = 2(2^{-r} - 2^{-2r})$

Corollary of Kasami-Tokura lemma

$A(\alpha) \leq 2.2^{(4r-2)(k+1)}$

Recollect $l(r, m, 2^{-r} - \epsilon) = O(l(r, k, 2^{-r}))$

$$\begin{aligned} l(r, k, 2^{-r}) &\leq 2A(\alpha)n, \text{ by Deletion lemma} \\ &= 2A(\alpha)2^k \\ &= O(\epsilon^{-8r}), \text{ using above corollary} \end{aligned}$$

Open Problem

Conjecture

For field $\mathbb{F}_q$ and $\epsilon > 0$, $\exists c(q, \epsilon, r)$ independent of n s.t. for all m and r

$$l_q(r, m, \delta_q(r) - \epsilon) \leq c(q, \epsilon, r)$$

- GKZ also proves for small q when $q - 1$ divides r
- Proven for quadratic polynomials $r = 2$ [Gopalan, 2010]
- List decoding over $\mathbb{F}_p$ for prime p shown [Bhowmick and Lovett, 2014]

Reference I

Many of the images were adopted from David Zuckerman's presentation!

Bhowmick, A. and Lovett, S. (2014).
List decoding reed-muller codes over small fields.
arXiv preprint arXiv:1407.3433.

Dvir, Z. and Shpilka, A. (2008).
Noisy interpolating sets for low degree polynomials.
In *Computational Complexity, 2008. CCC '08. 23rd Annual IEEE Conference on*, pages 140–148.

Goldreich, O. and Levin, L. A. (1989).
A hard-core predicate for all one-way functions.
In *Proceedings of the twenty-first annual ACM symposium on Theory of computing*, pages 25–32. ACM.

Reference II

Gopalan, P. (2010).
A fourier-analytic approach to reed-muller decoding.
In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 685–694.

Gopalan, P., Klivans, A. R., and Zuckerman, D. (2008).
List-decoding reed-muller codes over small fields.
In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 265–274. ACM.

Muller, D. (1954).
Application of boolean algebra to switching circuit design and to error detection.
Electronic Computers, Transactions of the I.R.E. Professional Group on, EC-3(3):6–12.

Reference III

Reed, I. (1954).
A class of multiple-error-correcting codes and the decoding scheme.
Information Theory, Transactions of the IRE Professional Group on, 4(4):38–49.