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① More examples of convex sets

[CONSTRUCTION]

- convex combination : $\alpha_1 x_1 + \dots \alpha_n x_n$ $\sum_{i=1}^{n} \alpha_i = 1$ $\alpha_i \geq 0$ $i = \overline{1, n}$
- a set is cvx iif it contains every convex comb. of its points
- convex hull of a set C conv(C) = set of all cvx comb. of points in C
- cvx. comb. generalize to inf. sums (p. distributions)
- note: even if $X \in \mathbb{R}^n$ is infinite, each point in conv(X) can be represented as a cvx comb of at most n+1 pts in X

$\forall x \in \text{conv}(C)$ $\exists$ $x_1 \dots x_m$, $m \leq n$ s.t. $x = \sum_{i=1}^{m} \alpha_i x_i$ | $\sum \alpha_i = 1$, $\alpha_i \geq 0$

Reminder: construction via intersection, cartesian prod, minkowski sum, projection, affine transf.

perspective transformation $(x, t) \mapsto x/t$

cvx set $C \subset \mathbb{R}^n \times \mathbb{R}_{++}$ $\mapsto$ $P(C)$ also cvx

② convex cones

- cone = a set closed under positive scalar multiplication

$\forall x \in K$, $\alpha x \in K$ $\forall \alpha > 0$

cone is cvx iif it is also closed under addition

$\Leftrightarrow$ $\forall x_1, x_2 \in K$ and $\alpha_1, \alpha_2 \geq 0$ $\alpha_1 x_1 + \alpha_2 x_2 \in K$

- conic combination : a point of the form $\alpha_1 x_1 + \dots \alpha_m x_m$ (can be generalized to inf. sums & integrals) $\alpha_1 \dots \alpha_m \geq 0$
- K is a cvx cone iif it contains all conic comb. of its elements : $\forall x_1 \dots x_m$, $\alpha_1 \dots \alpha_m \geq 0$ $\sum_{i=1}^{m} x_i \alpha_i \in K$
- conic hull of a set C = set of all conic comb. in C

$\text{cone} = \{ \alpha_1 x_1 \dots \alpha_m x_m \mid x_i \in C, \alpha_i \geq 0, i = 1 \dots m \}$, $\Leftrightarrow$ smallest cvx cone that contains C

Reminder: set of positive semi-def mat. is a cone

$S_+^n = \{ X \in S^n \mid X \succeq 0 \}$. Let $\alpha_1, \alpha_2 \geq 0$, $A, B \in S_+^n$

$\forall x \in \mathbb{R}^n$ $x^T(\alpha_1 A + \alpha_2 B) x = \alpha_1 x^T A x + \alpha_2 x^T B x \geq 0$

$\rightarrow$ more polyhedral cone

Also: copositive cone CP_n

$\{ A$ st. $\forall x \in \mathbb{R}_+^n$ nonnegative $\mid x^T A x \geq 0 \}$

example of non-polyhedral cone: $\mathbb{R}_+^n$ (non negative orth)
other: Lorentz cone $\{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{++} \mid \|x\|_2 \le t\}$

Properties
- intersection of cvx cones is a cvx cone
- a cone K is cvx iff $K+K \subset K$
- $= \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{++} \mid \|x\| \le t\}$ ← cone for any norm $\|\cdot\|$ on $\mathbb{R}^n$
- spectrahedron: inverse image of $\mathbb{P}_+^m$ under the affine map $A(x) = \sum_i' x_i A_i$
  $S := \{x \in \mathbb{R}^m \mid x_1 A_1 + \cdots + x_m A_m \succeq 0\}$ convex for A_i = symmetric
- $\mathbb{P}_+^n = \operatorname{conv}(\{xx^T \mid x \in \mathbb{R}^n\})$

Connection to convex sets
- cones are easier to handle ⟹ we'll use them to subs. for cvx sets

Proposition: Every convex set $C \subset \mathbb{R}^n$ can be regarded as the cross-section of some convex cone K in $\mathbb{R}^{n+1}$.

Dual cone

dual depends on inner product. in B&V, they consider dot prod.

K cone, $K^* := \{y \mid \langle x, y\rangle \ge 0 \ \forall x \in K\}$

K^* is always convex

Note: The dual cone of a subspace is its orthogonal complement: $S \subseteq \mathbb{R}^n$, $S^* = S^\perp = \{y \mid x^T y = 0, \forall x \in S\}$

If, for norm $\langle\rangle$, $K = K^*$ ⟹ K is self-dual under norm $\langle\rangle$

Examples: $\mathbb{R}_+^n$, $\mathbb{P}_+^n$ under $\langle x, y\rangle = \operatorname{tr}(XY)$ ← Proof in BV Ex 2.24

$\boxed{\text{Exercise: Dual of } CP_n}$

Properties of dual cones
- K^* is closed and convex
- $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$ "order reversing property"
- if K has nonempty interior ⟹ K^* is pointed
- if closure(K) is pointed ⟹ K^* has nonempty int.
- K^{**} is the closure of conv(K)
  corolary: if K is conv closed ⟹ $K^{**} = K$

(b) Polyhedra + simplexes

$P = \{x \mid a_i^T x \leq b_i, \; i = 1 \dots m, \; c_i^T x = d_i, \; i = \overline{1,p}\}$

solution set of a finite number of linear eq & ineq.

$P = \{x \mid Ax \preceq b, \; Cx = d\}$

Simplexes

$k+1$ pts affinely indep $v_0 \dots v_k$

$(\Leftrightarrow v_1 - v_0 \dots v_k - v_0$ linearly indep$)$

simplex is $\operatorname{conv}(\{v_0 \dots v_k\})$

(c) Ellipsoids

$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$

$P = P^T$ is symmetric and positive definite

$\sqrt{\lambda_i}$ (λ_i = eigenvalues of P) give the length of the semi-axes

equivalent representation: $\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}$

If $P = r^2 I \Rightarrow$ Euclidean ball

If A is singular $\Rightarrow$ degenerate ellipsoid (also convex)
affine dimension $= \operatorname{rank}(A)$

Norm balls / cones. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$.

$\{x \mid \|x - x_c\| \leq r\}$ convex

$K = \{(x,t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$ convex cone

② Convex functions : examples + Fenchel duals

$$f((1-\alpha)x+\alpha y) \leq (1-\alpha)f(x)+\alpha f(y)$$

Determining convexity

- continuous + midpoint cvx
- if f is diff., it is cvx iff $f(x) \geq f(y) + \langle \Delta f(y), x-y \rangle \ \forall x, y \in \mathrm{dom} f$
- if f is twice diff, convex iff
  $\nabla^2 f(x) \succeq 0 \ \forall x \in \mathrm{dom} f$
- $f: I \to \mathbb{R}$ cvx iff f' is $\uparrow$
- $f: \mathbb{R}^n \to \mathbb{R}$ cvx iff f' is a monotone operator
  i.e. $\langle \nabla f(x) - \nabla f(y), x-y \rangle \geq 0 \ \forall x, y \in \ldots$

ⓐ Examples

- all affine functions are both convex & concave
- on $\mathbb{R}$: $\underset{\text{cvx}}{\exp}$, $\underset{\text{cvx}}{\text{powers}}$, $|x|^p_{\text{cvx}}$, $\log$ is concave
  negative entropy $x \log x$ is convex

Discussed in class:

- norms (every norm on $\mathbb{R}^n$), including on mat.
- support, Fenchel conjugates*, epigraph cvx. set $\Leftrightarrow$ f. is cvx.
- logdet = concave

Examples from B.V.

- maximum is convex on $\mathbb{R}^n$
- Quadratic-over-linear $f: \mathbb{R} \times \mathbb{R}_{++}$ cvx
  $f(x,y) = x^2/y$
- Log-sum-exp is convex on $\mathbb{R}^n$
  $f(x) = \log(e^{x_1} + \ldots + e^{x_n})$
- Geometric mean is concave on $\mathbb{R}^n_{++}$
  $f(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$

Demonstrations in 3.1.5 BV

- α-sublevel set of a function $f: \mathbb{R}^n \to \mathbb{R}$
  $C_\alpha = \{x \in \mathrm{dom} f \mid f(x) \leq \alpha$
  f cvx $\Rightarrow C_\alpha$ cvx $\forall \alpha$

ⓑ Conjugates

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$f^* : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f^*(y) := \sup_{x \in \operatorname{dom} f} \left(\langle y, x \rangle - f(x) \right)$ convex (!)

f^* Domain: $\{ y \in \mathbb{R}^n \mid y^T x - f(x) < \infty \quad \forall\, x \in \operatorname{dom} f \}$

Inverse

$f(x) = 1/x$ on $\mathbb{R}_{++}$

$$\begin{cases} y > 0 \Rightarrow yx - 1/x \text{ is unbounded above} \\ y = 0 \Rightarrow \text{supremum is } 0 \\ y < 0 \Rightarrow \text{supremum is attained at } x = (-y)^{-1/2} \end{cases}$$

$\Rightarrow f^*(y) = -2(-y)^{1/2}$ $\quad \operatorname{dom} f^* = -\mathbb{R}_+$

Sqrt

$f(x) = \sqrt{x}$ on $\mathbb{R}_{++}$

$$\begin{cases} y > 0 \Rightarrow yx - \sqrt{x} \text{ is unbounded} \\ y = 0 \Rightarrow \sup \text{ is } 0 \\ y < 0 \Rightarrow \sup \text{ is attained at } 0 \end{cases} \Rightarrow f^* = 0$$

Log-determinant

$f(X) = \log \det X^{-1}$ on S^n_{++}

$$f^*(Y) = \sup_{X \succ 0} \big(\underbrace{\operatorname{tr}(YX)}_{\text{standard inner product}} + \log \det X \big)$$

- If $Y \not\prec 0 \Rightarrow Y$ has eigenvector v with $\|v\|_2$ and eigval $\lambda \geq 0$, then

let $X = I + t v v^T$,

$\operatorname{tr}(YX) + \log\det X = \operatorname{tr} Y + t\lambda + \log\det(I + t v v^T) = \underbrace{\operatorname{tr} Y + t\lambda + \log(1+t)}_{\text{unbounded as } t \to \infty}$

- when $Y \prec 0$, we find the maximizing X: $\in S^n_{++}$

$\nabla_X \left(\operatorname{tr}(YX) + \log\det X \right) = Y + X^{-1} = 0 \Rightarrow X = -Y^{-1}$ (which is $\succ 0$)

$\Rightarrow f^*(Y) = \log\det(-Y)^{-1} - n$

(c) Properties of conjugates

- Fenchel's inequality (follows from definition)

  $f(x) + f^*(x) \geq x^T y$

- If f is convex and closed, then $f^{**} = f$
  (epi (f) is closed set)

- Legendre transform: the conjugate when f is diff.

  Assume f is cvx and diff, x^* a maximizer of $f^*(y)$
  $(y = \nabla f(x^*))$

  $\Rightarrow f^*(y) = x^{*T} \nabla f(x^*) - f(x^*)$

  we can find f^* for any y for which we can solve $y = \nabla f(z)$ in

- Scaling / composition

  $a > 0$, $b \in \mathbb{R}$, $g(x) = a f(x) + b$

  $\Rightarrow g^*(x) = a f^*(y/a) - b$

  $A \in \mathbb{R}^{n \times n}$ nonsingular, $b \in \mathbb{R}^n$, $g(x) = f(Ax + b)$

  $\Rightarrow g^*(x) = f^*(A^{-T} y) - b^T A^{-T} y$

  $\mathrm{dom}\, g^* = A^T \mathrm{dom}\, f^*$

- Sums

  $f(w, v) = f_1(w) + f_2(v)$ where f_1, f_2 cvx with FCs f_1^*, f_2^*

  $f^*(w, z) = f_1^*(w) + f_2^*(z)$

  conjugate of sum of independent functions is the sum of the conjugates.

B.V → operations that preserve convexity
(no time to cover it)

Log convexity/concavity

- $f: \mathbb{R}^n \to \mathbb{R}$ is log concave if $f(x) > 0 \quad \forall x \in \mathrm{dom} f$
  $\log f$ is concave

  log-convex " " is convex

- f is log-cvx iff $1/f$ is log-concave
- if $f = 0 \Rightarrow \log f = -\infty$
  "extended-value function $\log f$"
- equivalent def for log-concave
  $\forall\, x, y \in \mathrm{dom} f \,,\quad 0 \le \alpha \le 1$
  $$f(\alpha x + (1-\alpha) y) \ge f(x)^{\alpha} f(y)^{1-\alpha}$$
- log cvx functions are cvx (bc. e^x is cvx)
- non-negative concave function are log-concave

Examples (3.39 in BV)
- Affine functions $f(x) = a^T x + b$ log-concave where $f(x) > 0$
- Powers $f(x) = x^a$ on $\mathbb{R}_{++}$ — log cvx for $a \le 0$, log conc for $a \ge 0$
- Exponentials $f(x) = e^{ax}$ is log cvx and log concave
- CDF of Gaussian is log concave
- Gamma function is log cvx for $x \ge 1$
- Determinant $\det X$ is log concave on S^n_{++}
- $\det X / \mathrm{tr}\, X$ is log concave on S^n_{++}
- log concave { pdf of
  - multivariate normal dist.
  - uniform dist. over a conv. set.
  - Wishart dist.

Properties of log-cvx / log-concave

- f (twice diff, dom f cvx) is log-convex iff $\forall\ x \in \text{dom} f$ $\quad f(x)\nabla^2$
- f is log-concave iff $\forall\ x \in \text{dom} f$
- log-cvx, log-conc are closed under multiplication/pos. scaling
- log-cvx is preserved under sums but <u>not</u> log concavity
- ~~marginals of log-cvx functions are log-cvx~~

if $f(x,y)$ is log-cvx in x $\forall\ y \in C$ then

$$g(x) = \int_C f(x,y)\,dy \quad \text{is log-convex}$$

- If $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave then

! $$g(x) = \int f(x,y)\,dy \quad \text{is a log-conc. function of } x \text{ on } \mathbb{R}^n$$

This is non-trivial to prove

$\Rightarrow$ marginals of log-concave functions are log-concave

$\Rightarrow$ log-concavity is closed under convolution