Barrier Methods

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Convex Optimization 10-725/36-725

Optimization Methods Using Lagrange Multipliers

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    - Primal Dual Methods

Inequality Constrained Problems

Inequality constrained problem

minimize $f(x)$

subject to $x \in X, \qquad g_j(x) \leq 0,\ j = 1, \ldots, r,$

where f and g_j are continuous and X is closed.

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where f and g_j are continuous and X is closed. We assume that the set

$$S = \left\{ x \in X \mid g_j(x) < 0,\ j = 1, \ldots, r \right\}$$

is nonempty and any feasible point is in the closure of S.

S is the **interior, relative to X,** of the set defined by inequality constraints

Barrier Method

- Consider a *barrier function*, that is continuous and goes to ∞ as any one of the constraints $g_j(x)$ approaches 0 from negative values. Examples:

$$B(x) = -\sum_{j=1}^{r} \ln\left\{-g_j(x)\right\}, \qquad B(x) = -\sum_{j=1}^{r} \frac{1}{g_j(x)}.$$

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- Barrier Method:

$$x^k = \arg\min_{x \in S} \big\{ f(x) + \epsilon^k B(x) \big\}, \qquad k = 0, 1, \ldots,$$

where the parameter sequence $\{\epsilon^k\}$ satisfies $0 < \epsilon^{k+1} < \epsilon^k$ for all k and $\epsilon^k \to 0$.

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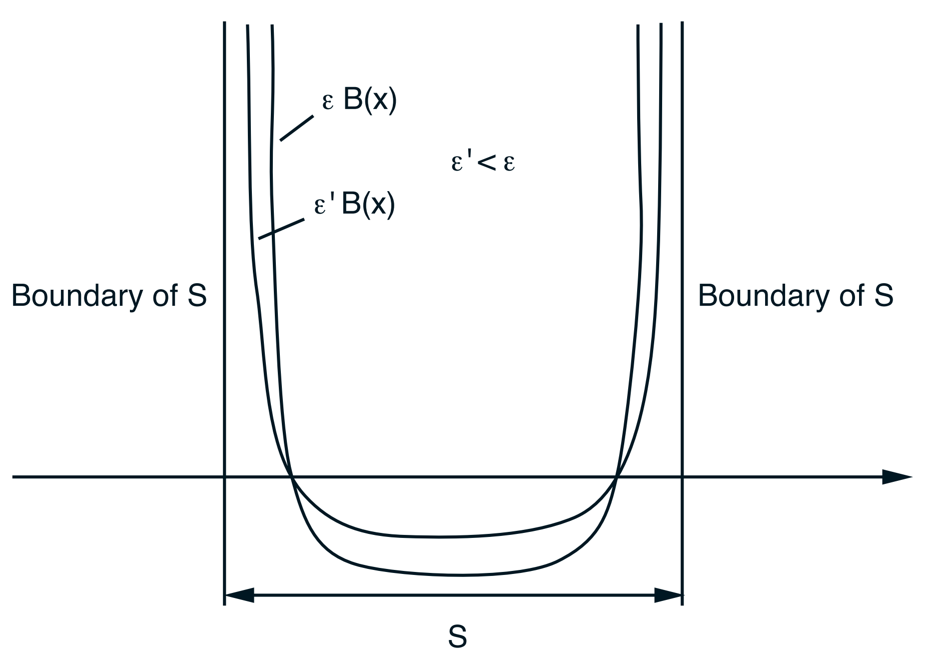

Barrier & Interior Point Methods

- Barrier function is only defined on the interior set S (i.e. interior of set of all feasible points)
- So if we start at an interior point, successive points will also be interior points
- Hence also referred to as interior point methods

Finding Interior Feasible Points

- Transformations of original optimization problem might have easily available feasible points

Example:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad a_i'x = b_i, \quad i = 1, \ldots, m, \qquad x \geq 0, \end{aligned}$$

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Equivalent, for c being a very large positive number:

$$\begin{aligned} &\text{minimize} \quad f(x) + cy \\ &\text{subject to} \quad a_i'x + \left(b_i - \sum_{j=1}^{n} a_{ij}\right) y = b_i, \quad i = 1, \ldots, m, \quad x \geq 0, \; y \geq 0, \end{aligned}$$

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$\bar{x} = \mathbf{1}, \bar{y} = 1$ is an interior feasible point.

Finding Interior Feasible Points

- Transformations of original optimization problem might have easily available feasible points

Example:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad a_j'x \le b_j, \quad j = 1, \ldots, r, \end{aligned}$$

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For any $\bar{x}$ infeasible for original problem, letting $\bar{y} = \max_{j=1}^{r}\{a_j'\bar{x} - b_j\}$, $(\bar{x}, \bar{y} + 1)$ is an interior feasible point.

Convergence

Every limit point of a sequence $\{x^k\}$ generated by a barrier method is a global minimum of the original constrained problem

Convergence

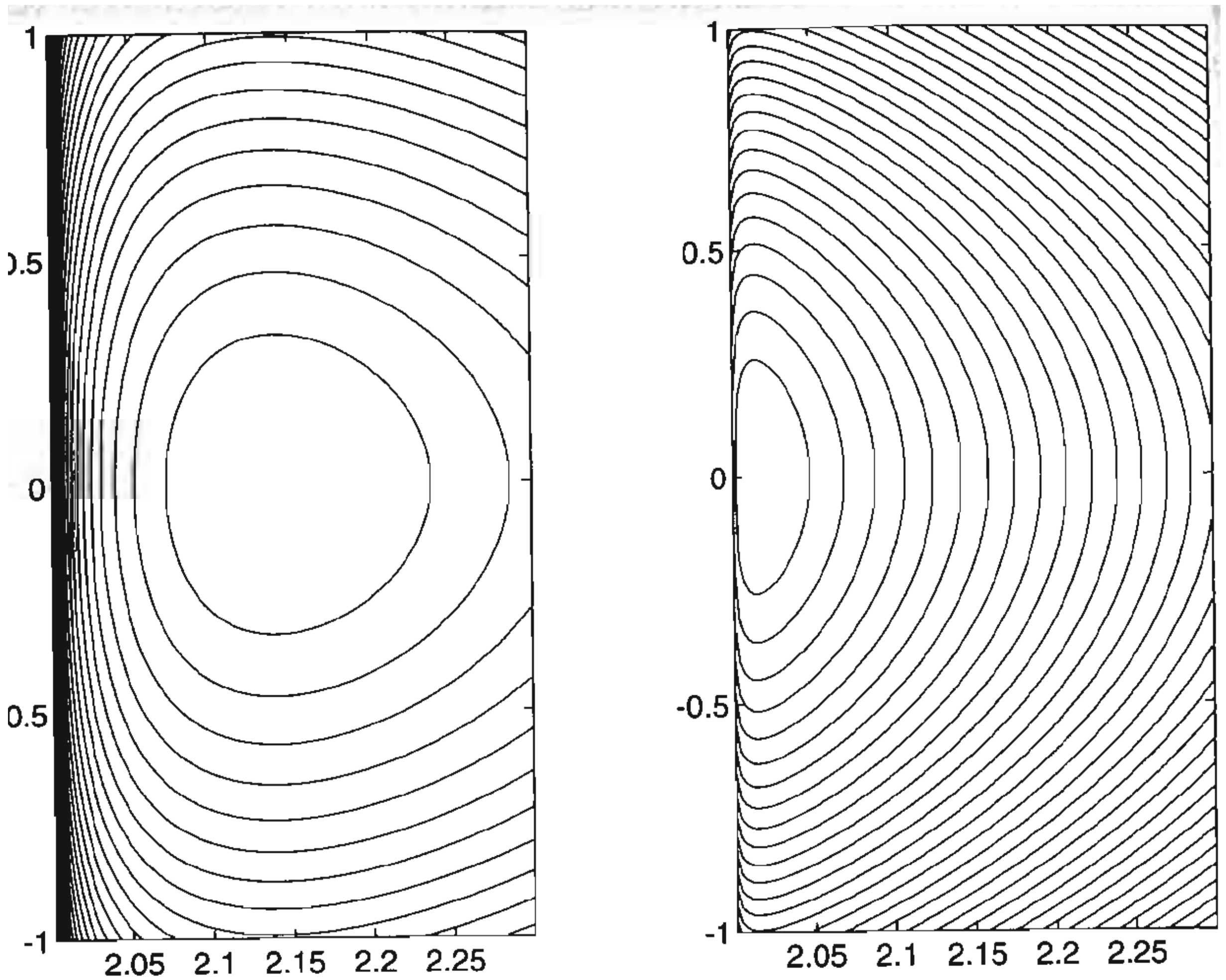

figure shows contours of f(x) + \epsilon B(x)

Left: \epsilon = 0.3
Right: \epsilon = 0.03

Optimal solution:
x^* = (2,0)

$$\text{minimize} \quad f(x) = \tfrac{1}{2}\left(x_1^2 + x_2^2\right)$$

$$\text{subject to} \quad 2 \le x_1,$$

with optimal solution $x^* = (2, 0)$. For the case of the logarithmic barrier function $B(x) = -\ln(x_1 - 2)$, we have

$$x^k = \arg\min_{x_1 > 2} \left\{ \tfrac{1}{2}\left(x_1^2 + x_2^2\right) - \epsilon^k \ln(x_1 - 2) \right\} = \left(1 + \sqrt{1 + \epsilon^k}\,, 0\right),$$

LINEAR PROGRAMS/LOGARITHMIC BARRIER

- Apply logarithmic barrier to the linear program

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad Ax = b, \qquad x \geq 0, \end{aligned} \tag{LP}$$

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The method finds for various $\epsilon > 0$,

$$x(\epsilon) = \arg\min_{x \in S} F_\epsilon(x) = \arg\min_{x \in S} \left\{ c'x - \epsilon \sum_{i=1}^{n} \ln x_i \right\},$$

where $S = \big\{x \mid Ax = b,\, x > 0\big\}$. We assume that S is nonempty and bounded.

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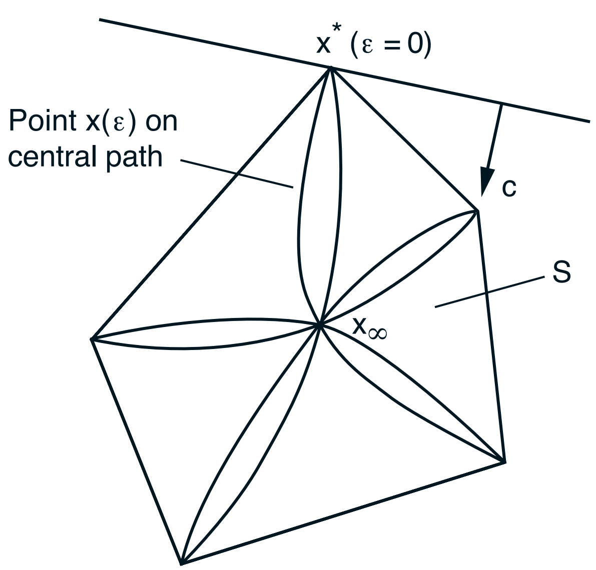
- As $\epsilon \to 0$, $x(\epsilon)$ follows the *central path*



All central paths start at the *analytic center*

$$x_\infty = \arg\min_{x \in S} \left\{ - \sum_{i=1}^{n} \ln x_i \right\},$$

and end at optimal solutions of (LP).

PATH FOLLOWING W/ NEWTON'S METHOD

- Newton's method for minimizing F_ϵ:

$$\tilde{x} = x + \alpha(\overline{x} - x),$$

where $\overline{x}$ is the pure Newton iterate

$$\overline{x} = \arg \min_{Az=b} \left\{ \nabla F_\epsilon(x)'(z - x) + \tfrac{1}{2}(z - x)'\nabla^2 F_\epsilon(x)(z - x) \right\}$$

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- By straightforward calculation

$$\overline{x} = x - Xq(x, \epsilon),$$

$$q(x, \epsilon) = \frac{Xz}{\epsilon} - e, \quad e = (1 \ldots 1)', \quad z = c - A'\lambda,$$

$$\lambda = (AX^2A')^{-1}AX\Big(Xc - \epsilon e\Big),$$

and X is the diagonal matrix with x_i, $i = 1, \ldots, n$ along the diagonal.

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- View $q(x, \epsilon)$ as the Newton increment $(x - \overline{x})$ transformed by X^{-1} that maps x into e.
- Consider $\|q(x, \epsilon)\|$ as a *proximity measure* of the current point to the point $x(\epsilon)$ on the central path.

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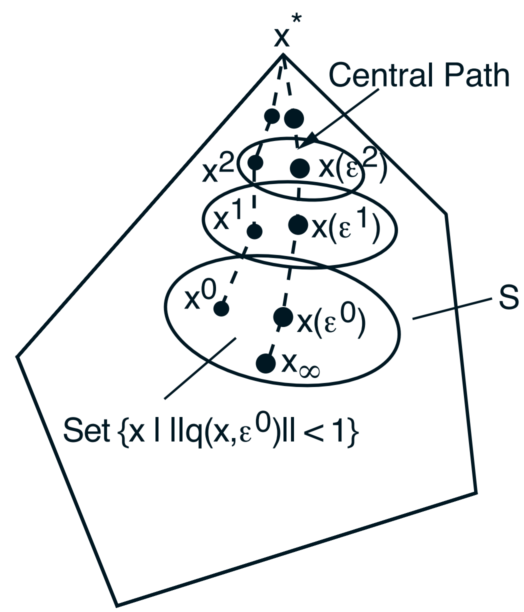
q(x,\epsilon) = 0 iff
x = x(\epsilon)

Path following

- x(\epsilon): solution of barrier problem with parameter \epsilon
- \bar{x}: one Newton iterate from x
- q(x,\epsilon): depends on one Newton iterate from x, approximates distance of x from x(\epsilon)
    - can be used to determine if we need more Newton iterations

KEY RESULTS

- It is sufficient to minimize F_ϵ approximately, up to where $\|q(x,\epsilon)\| < 1$.



If $x > 0$, $Ax = b$, and $\|q(x,\epsilon)\| < 1$, then

$$c'x - \min_{Ay=b,\ y\geq 0} c'y \leq \epsilon\left(n + \sqrt{n}\right).$$

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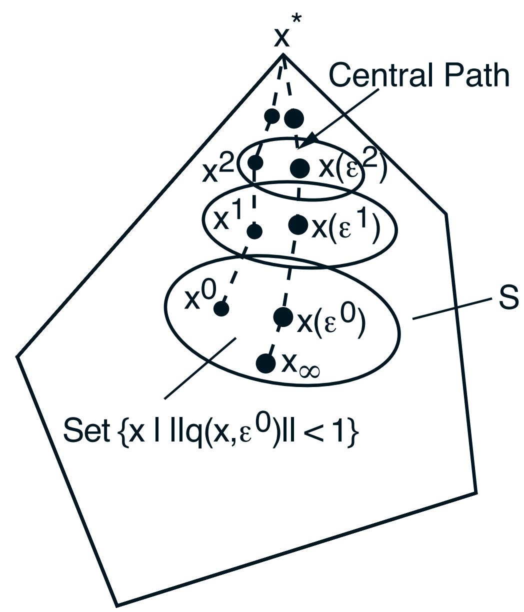

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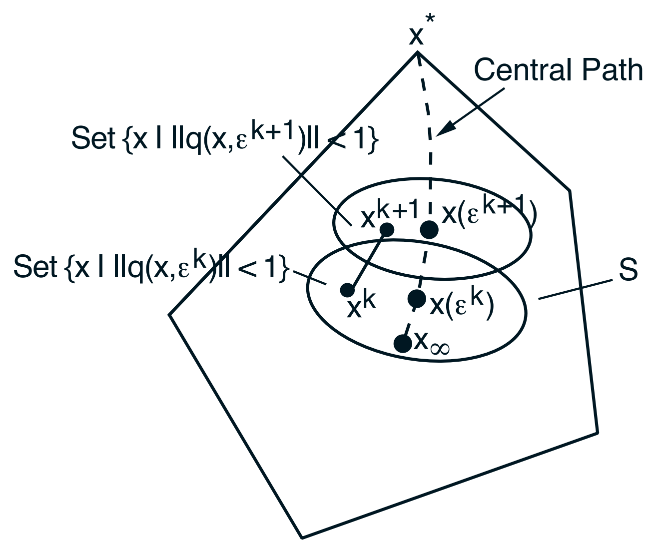
- The "termination set" $\big\{x \mid \|q(x,\epsilon)\| < 1\big\}$ is part of the region of quadratic convergence of the pure form of Newton's method. In particular, if $\|q(x,\epsilon)\| < 1$, then the pure Newton iterate $\overline{x} = x - Xq(x,\epsilon)$ is an interior point, that is, $\overline{x} \in S$. Furthermore, we have $\|q(\overline{x},\epsilon)\| < 1$ and in fact

$$\|q(\overline{x},\epsilon)\| \leq \|q(x,\epsilon)\|^2.$$

SHORT STEP METHODS



Following approximately the central path by using a single Newton step for each ϵ^k. If ϵ^k is close to ϵ^{k+1} and x^k is close to the central path, one expects that x^{k+1} obtained from x^k by a single pure Newton step will also be close to the central path.

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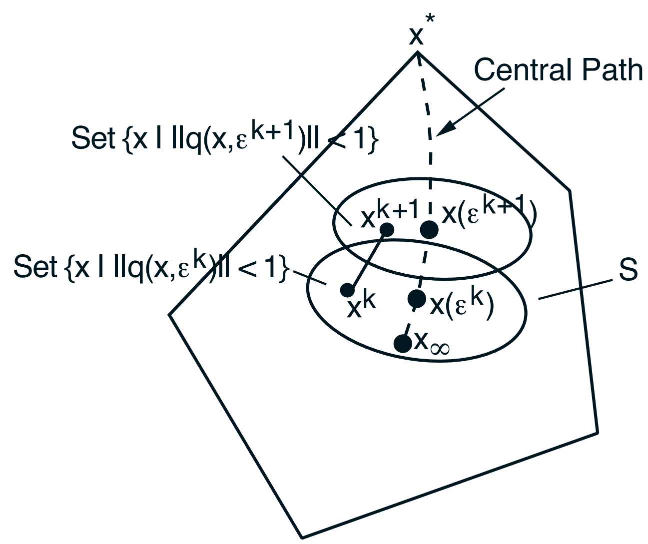

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Proposition Let $x > 0$, $Ax = b$, and suppose that for some $\gamma < 1$ we have $\|q(x,\epsilon)\| \leq \gamma$. Then if $\overline{\epsilon} = (1 - \delta n^{-1/2})\epsilon$ for some $\delta > 0$,

$$\|q(\overline{x},\overline{\epsilon})\| \leq \frac{\gamma^2 + \delta}{1 - \delta n^{-1/2}}.$$

In particular, if

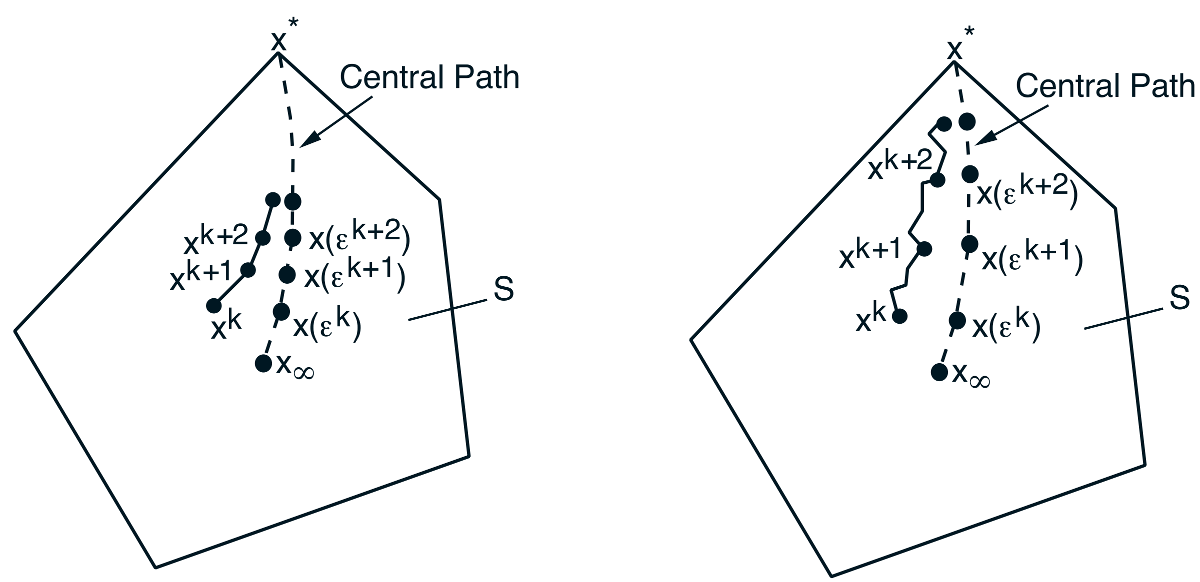
$$\delta \leq \gamma(1-\gamma)(1+\gamma)^{-1},$$

we have $\|q(\overline{x},\overline{\epsilon})\| \leq \gamma$.

- Can be used to establish nice complexity results; but ϵ must be reduced VERY slowly.

LONG STEP METHODS

- Main features:
  - Decrease ϵ faster than dictated by complexity analysis.
  - Require more than one Newton step per (approximate) minimization.
  - Use line search as in unconstrained Newton's method.
  - Require much smaller number of (approximate) minimizations.



(a) (b)

Short Step Method Long Step Method