Lecture 18 - Analysis of Boolean Functions

[[Scribe. analysisofbooleanfunctions.org]]

[[Major tool in many/most areas of TCS — complexity, learning, pseudorandomness, We'll use it in future lecs on comm cxty, quantum, hierarchies, inapprox

Also key in additive comb, rand. graph theory, metric/Banach spaces Gaussian geom., social choice...

Today's app: Property testing.]]

[[The Boolean fcn.:]] $f: \{0,1\}^n \to \{0,1\}$.

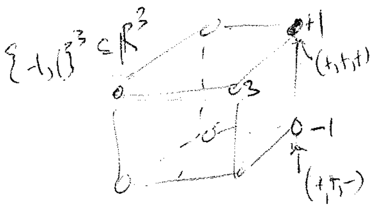
$\mathbb{F}_2$ $\{-1,+1\} \subseteq \mathbb{R}$ $\{T,F\}$. "-1" :+1 [we'll see why...]

[[be flexible abt. bit rep^n]]

[one of the hallmarks of AoBF is letting the range be $\mathbb{R}$]

[most typically:] $f: \{-1,1\}^n \to \mathbb{R}$, $f: \mathbb{F}_2^n \to \mathbb{R}$ [depends on if you're adding strings.]

Perspective 1: Writing fcns as polynomials (over $\mathbb{R}$)



Interpolate

$$f(x_1, x_2, x_3) = \left(\tfrac{1}{2}+\tfrac{x_1}{2}\right)\left(\tfrac{1}{2}+\tfrac{x_2}{2}\right)\left(\tfrac{1}{2}+\tfrac{x_3}{2}\right)\cdot(+1)$$
$$+\left(\tfrac{1}{2}+\tfrac{x_1}{2}\right)\left(\tfrac{1}{2}+\tfrac{x_2}{2}\right)\left(\tfrac{1}{2}-\tfrac{x_3}{2}\right)\cdot(-1)$$
$$+ \cdots$$

written x_1

= a "multilinear" poly. in $x_1 \ldots x_n$.

[lot of cancellation]

e.g. "Majority": $\{-1,1\}^3 \to \{-1,1\}$ $\quad \mathrm{Maj}_3(x_1,x_2,x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1x_2x_3$

$\mathrm{Parity}_3(x_1,x_2,x_3) = x_1x_2x_3$

[true if odd # of (-1) (-1)]

Working over $\mathbb{R}$

$\mathrm{Par}_3: \mathbb{F}_2^3 \to \mathbb{F}_2$ is $x_1+x_2+x_3$

[we'll see in a sec.]

[Interpolation] $\Rightarrow$ Prop: Every $f:\{-1,1\}^n \to \mathbb{R}$ is (uniquely*) expressible as a multilin. polynomial.

[param. counting!]

$$f(x) = \sum_{S\subseteq[n]} c_S \prod_{i\in S} x_i$$

[for some reason] $\boxed{\text{Fourier expansion}}$

c_S: [show dep. on f] $\hat{f}(S)$

$\prod_{i\in S} x_i$: $\mathrm{Parity}_S(x)$ $\to$ $\chi_S(x)$.

$\chi_S : \{-1,1\}^n \to \{-1,1\}$
$x \mapsto \prod_{i\in S} x_i$

: $\mathbb{F}_2^n \to \mathbb{F}_2$
$x \mapsto \sum_{i\in S} x_i$ [not in this class]

: $\mathbb{F}_2^n \to \{-1,1\}$
$x \mapsto \prod_{i\in S} (-1)^{x_i}$

e.g.: $\widehat{\mathrm{Maj}_3}(\{1,2,3\}) = -\frac{1}{2}$

$\widehat{\mathrm{Maj}_3}(\{1,2\}) = 0$

[etc.] $\widehat{\mathrm{Par}_3}(\{1,2,3\}) = 1$

Perspective 2

[so far didn't use anything about the "Hamming ~~space~~ of the geom" pts]

$f:$ (Hamming Cube) $\to \mathbb{R}$
a regular graph

"π" = unif. distrib. ☺

AOBF = spectral gh theory on

- mean: $\mathop{E}_{x\sim\mathbb{F}_2^n}[f(x)]$ [unif] , • $\mathrm{Var}[f] = E[f(x)^2] - E[f]^2$.
  $E[f(x)^2] \to 1$ if $f\to\{-1,1\}$ [our most interested case]

rec:
- $\{f:\mathbb{F}_2^n\to\mathbb{R}\}$ is a vector space [basis?]
- $\langle f,g\rangle = \mathop{E}_{x\sim\mathbb{F}_2^n}[f(x)g(x)]$

prop: For $f,g:\mathbb{F}_2^n\to\{-1,1\}$, $\langle f,g\rangle = 1 - 2\cdot\Pr_x[f(x)\neq g(x)]$

[Measures "corr" of 2 boolean fcns....]

[[Put the "spectral" in spectral gh theory.]]

Eigvectors of "L", equiv. of "K"?

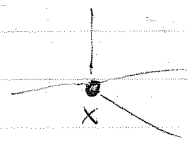
rec: $Kf(x) = \mathop{E}_{y \sim x}[f(x)]$.

Flip a rand coord of x.

Claim: the parity fcns $\chi_S : \{-1,1\}^S \to \{-1,1\}$ are all eigenvectors [[ex?]]

Pf:

[cons]



[[n nbrs]].

$\chi_S(x) = b \in \{\pm 1\}$.

$\chi_S(y) = -b$ for $|S|$ nbrs

$ b$ for $n-|S|$ nbrs

$\therefore \mathop{\text{avg}}_{y \sim x}$ is $\frac{(n-2|S|)b}{n} = \left(1-\frac{2|S|}{n}\right)\chi_S(x)$

Ks eigenval: $1-\frac{2|S|}{n}$. $\therefore L = id - K$; eigenval: $\frac{2|S|}{n}$.

$0, \frac{2}{n} \times n, \ldots\ldots, 2$

$\chi_\varnothing \equiv 1$. $\quad$ 2 bip. ✓

Claim: χ_S's are orthonormal. $\Rightarrow$ basis $\because$ there's $\underbrace{2^n}_{=\dim}$ of them

Pf: $\langle \chi_S, \chi_T \rangle = \mathop{E}_x\left[\prod_{i \in S} x_i \cdot \prod_{i \in T} x_i\right]$ $\quad$ $x_i^2 = 1$

$= \mathop{E}_x\left[\prod_{i \in S \Delta T} x_i\right]$

$= \begin{cases} 1 & \text{if } S = T. \\ \prod_{i \in S\Delta T} \underbrace{E[x_i]}_{0} & \text{if } S \neq T. \end{cases}$ $\quad \square$

$\therefore$ Fourier expansion $\equiv$ ~~eigenfunction exp~~ eigenvect expansion

$f = \sum_{S \subseteq [n]} \hat{f}(S)\, \chi_S$. $\quad$ Prop: $\hat{f}(S) = \langle f, \chi_S \rangle$

Cor: "Parseval Thm": $\langle f, g \rangle = \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S)$

$$\Rightarrow \langle f, f \rangle = \sum_S \hat{f}(S)^2$$

$$\langle f,f\rangle = \mathop{E}_x[f(x)^2] = 1 \text{ for } f: \{-1,1\}^n \to \{-1,1\}.$$

[Sum of squares of coeffs $= 1$]

Let $f: \mathbb{F}_2^n \to \{-1,1\}$

$$\mathcal{E}[f] = \frac{1}{2} \mathop{E}_{x\sim y}\left[(f(x)-f(y))^2\right]$$

$$\| $$

$$\langle f, Lf \rangle = \sum_{S\subseteq[n]} \frac{2|S|}{n} \cdot \hat{f}(S)^2$$

$$\Rightarrow \sum_{S\subseteq[n]} |S|\hat{f}(S)^2 = n \mathop{E}_{x\sim y}\left[\left(\frac{f(x)-f(y)}{2}\right)^2\right]$$

$$= \sum_{i=1}^{n} \mathop{E}_x\left[\left(\frac{f(x)-f(x+e_i)}{2}\right)^2\right]$$

$\begin{cases} 1 & \text{if } f(x) \neq f(x+e_i) \\ 0 & \text{else.} \end{cases}$

def: $\mathrm{Inf}_i[f] = \Pr[f(x) \neq f(x+e_i)]$

Influence

Total Infl.

AKA

$$\sum_i \mathop{E}_x[1[f \text{ sens @ coord } i \text{ on } x]]$$

$$= \mathop{E}_x[\#\text{ sens. coords for } f \text{ @ } x]$$

$=$ average sensitivity

(maybe skip)

Rec: $\sum_{S\subseteq[n]} \hat{f}(S)^2 = 1$

$\hat{f}(S) =$ coeff on $\prod_{i\in S} x_i$. $|S| =$ degree.

$\therefore$ "average degree" of Fourier expansion

[very common theme in AOBF: reading off a "combinatorial/probabilistic" ppty from an "analytic" one/Fourier expansion.]

Property Testing

"random spot-checking"
"sublinear-time algs"

[[Think: the input is so enormous, can't read it all. Just "query" it in a few bits.]]

Testing property $\mathcal{P}$

Input: "query access" to $f: \{-1,1\}^n \to \{-1,1\}$

[[Popular]] (Graph testing: $n = \binom{v}{2}$, f is edge-indic. of v-vtx graph)

Goal: Ⓒ ["completeness"] f has ppty $\mathcal{P} \Rightarrow$ say YES whp. (say, 2/3)

Ⓢ ["soundness"] f doesn't $\Rightarrow$ say NO w.h.p.

- Min # queries

[[Ⓢ is too strong to get any interesting sublin. algs]]

☹ $\mathcal{P}$ is "$f \equiv 1$".
Disting. $f \equiv 1$ from f which is -1 on 1 out of 2^n inputs
reqs $\Omega(2^n)$ queries

(Relaxed) Ⓢ: f ε-far from having $\mathcal{P} \Rightarrow$ say no w.h.p.

$\forall g$ with $\mathcal{P}$, $\Pr_x[f(x) \neq g(x)] > \varepsilon$.
$\iff \langle f, g \rangle < 1 - 2\varepsilon$.

rem: If $f \notin \mathcal{P}$ but $< \varepsilon$-close, alg may say YES or NO.

prop: $\mathcal{P}$ of "$f \equiv 1$" testable with $O(1/\varepsilon)$ queries

pf: Alg: Query f on 1 rand input x, Say YES if $f(x) = 1$.
f has $\mathcal{P} \Rightarrow \Pr[\text{YES}] = 1$
f ε-far from $\mathcal{P} \Rightarrow \Pr[\text{YES}] < 1 - \varepsilon$. [[hmm, not $< \frac{1}{3}$. Now what?]]

[Repeat $O(1/\varepsilon)$ times to drive 'NO' err. down.]

late '80s

thm [BLR ~~'90~~]: P = "f is a parity fcn" testable w/ $O(\frac{1}{\varepsilon})$ queries

rem: [[... partial birth of PCP theorem]] → "linearity testing"

($\mathbb{F}_2$) $f: \mathbb{F}_2^n \to \mathbb{F}_2$, $f(x) = \sum_{i=1}^{n} a_i x_i$?

rem: f linear $\iff f(x+y) = f(x) + f(y) \mod 2$.

in $\{\pm 1\} \subseteq \mathbb{R}$ notation, $\iff f(x \circ y) = f(x) f(y)$
$\uparrow$ entrywise prod.

Idea:
- Choose $x, y \sim \{-1,1\}^n$ unif.
- Set $z = x \circ y$. $(z_i = x_i y_i)$
- Query $f(x), f(y), f(z)$ (3)
- Say YES iff $f(z) = f(x)f(y)$?

Prop: If f has P, $\Pr[\text{YES}] = 1$

[[Analysis:]]. Fix $f: \{-1,1\}^n \to \{-1,1\}$ $\Pr[\text{YES}] = \mathop{\mathbb{E}}_{x,y}\left[\frac{1}{2} + \frac{1}{2} f(x) f(y) f(z)\right]$

$$= \frac{1}{2} + \frac{1}{2} \mathop{\mathbb{E}}_{x,y}\left[\left(\sum_S \hat{f}(S)\chi_S(x)\right)\left(\sum_T \hat{f}(T)\chi_T(y)\right)\left(\sum_U \hat{f}(U)\chi_U(z)\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T,U} \hat{f}(S)\hat{f}(T)\hat{f}(U) \mathbb{E}\left[\left(\prod_{i \in S} x_i\right)\left(\prod_{i \in T} y_i\right)\left(\prod_{i \in U} x_i y_i\right)\right]$$

$$\mathop{\mathbb{E}}_{\substack{x,y\\ \text{indep}}}\left[\prod_{i \in S \triangle U} x_i \prod_{i \in T \triangle U} y_i\right] = \begin{cases} 0 & \text{unless } S \triangle U, T \triangle U = \emptyset \iff S = T = U \\ & \text{in which case } 1. \end{cases}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_S \hat{f}(S)^3. \quad [[!]] \quad [[\text{Remember: } \sum \hat{f}(S)^2 = 1]].$$

Check: $= 1$ if f has P? $\Leftarrow f = \chi_{S^*} \Leftarrow \hat{f}(S^*) = 1$, $\hat{f}(S) = 0\ \forall S \neq S^*$

$$= \frac{1}{2} + \frac{1}{2} \sum_S \hat{f}(S)^2 \cdot \hat{f}(S) \leq \frac{1}{2} + \frac{1}{2} \max_S \{\hat{f}(S)\} \cdot \sum_S \hat{f}(S)^2 \; (\to 1).$$

"f ε-far from P" $\iff \langle f, \chi_S \rangle < 1 - 2\varepsilon\ \forall S \iff \max_S \{\hat{f}(S)\} < 1 - 2\varepsilon$.

($\langle f, \chi_S \rangle = \hat{f}(S)$; χ_S elts of P)

$\Rightarrow \Pr[\text{YES}] < 1 - \varepsilon$.

Repeat $O(1/\varepsilon)$ times $\to$ $3 \cdot O(1/\varepsilon)$ queries to finish. $\square$