Lecture 6 — Spectral Graph Theory I [[first of 2 or 3]]

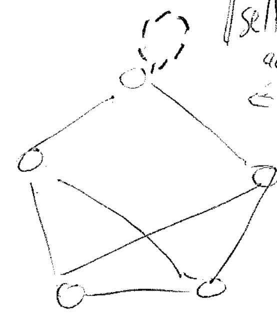
[[SCRIBE]]

[[A topic beloved by alg/cxty people alike.]]
[[expanders, sparsest cut, max cut, UGC, ...]]

[[We always work with:]] <u>Undirected</u> graph $G = (V, E)$.

[[Technicalities:]]

- finite
- multiple edges, self-loops <u>OK</u>
- no deg-0 vertices [[isolated]] [[multiple conn. comps. OK]]



[[self-loop adds $\frac{1}{2}$ for each endpt]]

Never seen this before? → Assume G <u>regular</u>. [[Don't have to, but conceptually <u>easier</u>.]]

[[Everything we do generalizes to "reversible Markov chains".
A lot of what " " " non-reversible " "/dir'd graphs.]]

[[We'll be looking at:]] Labeling V with real #s:

$$f : V \to \mathbb{R}$$

e.g.: f is temperature, voltage, coord. of an embedding into $\mathbb{R}^d$, 0/1-indic. of $S \subseteq V$, ...

$$\equiv \text{ col. vec. } \begin{array}{c c} \begin{bmatrix} \\ \\ f \\ \\ \\ \end{bmatrix} & \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \vdots \\ \\ \leftarrow v_n \end{array} \end{array}$$

rem: can add $f + g$ (ptwise), scalar mult $c \cdot f$ → $\{f : V \to \mathbb{R}\}$ a <u>vector space</u>.

[[duh, they're vectors]]

Dimension? $\underline{\ n\ }$

KEY TO SPEC. GH. THY.:

$$\mathcal{E}[f] := \tfrac{1}{2} \mathop{\mathbf{E}}_{u \sim v}\left[\left(f(u)-f(v)\right)^2\right]$$

$u \sim v$: a ~~rand~~ unif rand edge (u,v), both dirs. equally likely [[doesn't matter here but it's good for symm.]] [[helpful to think of edge as two directed edges]]

$\tfrac{1}{2}$: [[will be "natural"]]

def: Dirichlet form [[ugh]]. [[More natural names?:]] · "Local variance" · "Analytic bdry size" · "Laplacian quadratic form"

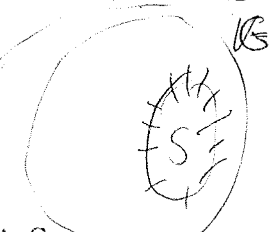
facts:
- $\mathcal{E}[f] \geq 0$ [[duh]]
- $\mathcal{E}[cf] = c^2\mathcal{E}[f]$, [[Quite like variance]]
- $\mathcal{E}[f+c] = \mathcal{E}[f]$ [[translation invariant]]

intuition: "small" $\Longleftrightarrow$ f "smooth", doesn't vary much on edges

key e.g.: $S \subseteq V$, $f = 1_S$ $\left(f(u) = \begin{cases} 1 & u \in S \\ 0 & u \notin S \end{cases}\right)$

Then $\mathcal{E}[f] = \tfrac{1}{2} \mathop{\mathbf{E}}_{u\sim v}\left[\left(1_S(u) - 1_S(v)\right)^2\right] = \tfrac{1}{2} \mathop{\mathbf{E}}_{u \sim v}\left[1[(u,v) \text{ "crosses" } S]\right]$, ("cuts")

$= \tfrac{1}{2}\{\text{frac. of edges on } \partial S\}$ [[bdry of S]]



[[Do you find $\tfrac{1}{2}$ annoying? It's cool. Think of $u \sim v$ as directed. If on bdry, $\tfrac{1}{2}$ chance going "in", $\tfrac{1}{2}$ chance "out"]]

$= \Pr_{u \sim v}[u \to v \text{ is stepping out of } S]$.

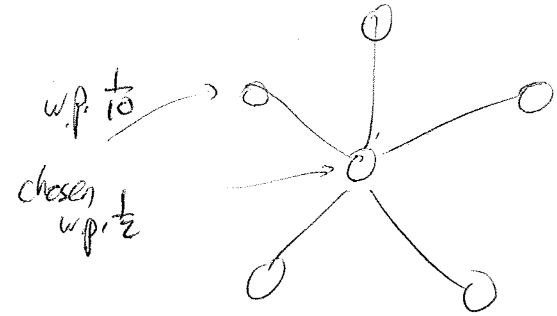
[[This is something we care about — finding small "cuts", partitioning graphs, finding "communities", etc.]]

[[We're often choosing unif. rand edge (u,v).
We should probably talk abt. choosing rand vertex, too.
Just unif random? No! Not "compatible" with our edge choice.]]

To choose rand vtx:

- choose edge $u \sim v$ unif.
- output u [[or v]].

call this dist. π — also yields π; $(u \sim v \equiv v \sim u.)$ [[reversible]]



(immed.) fact: $\pi[u]$ is proportional to $\deg(u)$. $\left(\pi[u] = \frac{\deg(u)}{2|E|}\right)$.

[[NB: $\pi > 0$ $\because$ no isolated vtcs]]

- gives weight/importance to vtcs.
- if G regular, π is uniform dist. [[this is why it's easiest to think of regular case]]

fact: $\left\{ \begin{array}{l} \cdot \text{ draw } u \sim \pi \\ \cdot \text{ let } v \text{ be unif. rand} \\ \quad \text{nbr of } u \end{array} \right\} \equiv$ draw unif rand edge $u \sim v$.

notat'n: $v \sim u$

Why? [[condit. on u, equally likely to have come fr. any nbr; Or, look at formula for π, mult by $\frac{1}{\deg(u)}$]]

$\Rightarrow$ distrib. of v is also π.

Cor: Let $t \in \mathbb{N}$. [[e.g., $t = 17$]].
Draw $u \sim \pi$.
Do rand. walk for t steps: $u \xrightarrow{t} v$.
Then v has distrib. π.

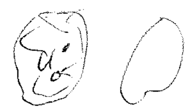
term: π is invariant / stationary distrib.

Q: Say $u_0 \in V$ <u>not</u> random. ⟦"Worst case"⟧

Do $u_0 \xrightarrow[\text{⟦std. rand walk⟧}]{t} v.$ ← ⟦random variable⟧.

As $t \to \infty$, does distribution of $v \longrightarrow \pi$?

A: Not if .... • $\underbrace{\text{G disconnected}}$,



• $\underbrace{\text{G bipartite}}$,

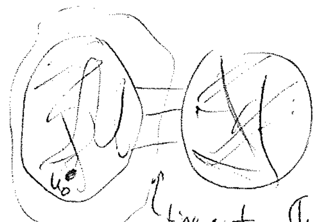
odd/even t problem

Otherwise yes! ⟦We'll see why later.
So it's also "limiting distrib." except in some freak cases.⟧

Q: How fast?

A: ⟦We'll see...⟧ <u>Spectral</u> ⟦eigenvalue⟧ considerations...

e.g.



⟦will take a "long time" to cross⟧

S: $\mathcal{E}[1_S]$ "small". ⟦We'll see...⟧

Fast convergence ⇔ $\mathcal{E}[f]$ "never small".
"always large"

⟦"Small"? Compared to what? $\mathcal{E}[cf] = c^2\mathcal{E}[f]$, so we have a "scaling issue".
Can just make f have huge values. Like variance, should normalize to 1...⟧

<u>not^n</u>: Let $f: V \to \mathbb{R}$. Say $u \sim \pi$. Then $f(u)$ is real rand vbl.

• mean: $\mathop{E}\limits_{u\sim\pi}[f(u)] =: E[f]$ ⟦←shorthand⟧. If $S \subseteq V$, $E[1_S] = \Pr\limits_{u\sim\pi}[u \in S]$

← "weight"/"volume"/"density" of S
⟦frac. of vtcs in S if regular.⟧

• $\mathrm{Var}[f] = \mathop{E}\limits_{u\sim\pi}[(f(u)-\mu)^2] = \mathop{E}\limits_{u\sim\pi}[f(u)^2] - E[f(u)]^2$

$= \frac{1}{2} \mathop{E}\limits_{\substack{u\sim\pi \\ v\sim\pi \text{ indep.}}}[(f(u)-f(v))^2]$ ⟦this is just totally normal fact abt. r.v.s: Var is $\frac{1}{2}E[(X-X')^2]$⟧

[[Pf: $= \frac{1}{2}E[f(u)^2] - \frac{1}{2}\cdot 2\,E[f(u)f(v)] + \frac{1}{2}E[f(v)^2]$

$E[f^2]$ $\quad -E[f]^2$, by indep.]]

Compare: [[our goal, actually]]

$$\mathcal{E}[f] = \frac{1}{2}\mathop{E}_{u\sim v}\left[(f(u)-f(v))^2\right]$$ "local var." (on edges)

$$Var[f] = \frac{1}{2}\mathop{E}_{\substack{u\sim\pi \\ v\sim\pi}\ \text{indep}}\left[(f(u)-f(v))^2\right]$$ "global var"

(*) [[$\mathcal{E}[f]$ always large comp'd to $Var[f]$ = definition of expander]]

[SEPARATE BOARD]

def^n: Let $f, g: V\to\mathbb{R}$. $\langle f,g\rangle_\pi := \mathop{E}_{u\sim\pi}[f(u)g(u)]$ [["correlation" of f,g; how "similar" they are]]

[[~~not nec disj~~ actually]]

$\left\langle \begin{bmatrix} \\ f \\ \\ \end{bmatrix}, \begin{bmatrix} \\ g \\ \\ \end{bmatrix} \right\rangle$ is normal "dot prod", scaled by $\frac{1}{n}$, <u>if G regular.</u> [[If not reg., not true, and this screws people up.]]

rem: It <u>is</u> a vec. space <u>inner product</u>:

- $\langle f,g\rangle = \langle g,f\rangle$
- $\langle c\cdot f+g, h\rangle = c\langle f,h\rangle + \langle g,h\rangle$
- $\langle f,f\rangle = \mathop{E}_{u\sim\pi}[f(u)^2] \geq 0$, equality $\Leftrightarrow f\equiv 0$.

not^n: $\|f\|_2^2$.

[[key]] remark: Let $S\subseteq V$, $f = 1_S$.

Then $\|f\|_1 := \mathop{E}_u[|f(u)|] = E[f(u)] = \Pr[u\in S] = $ "vol(S)".

$\|f\|_2^2 := \mathop{E}_u[f(u)^2] = E[f(u)] = \cdots$

"$\langle f,f\rangle$" $\quad \because f$ is 0/1-valued.

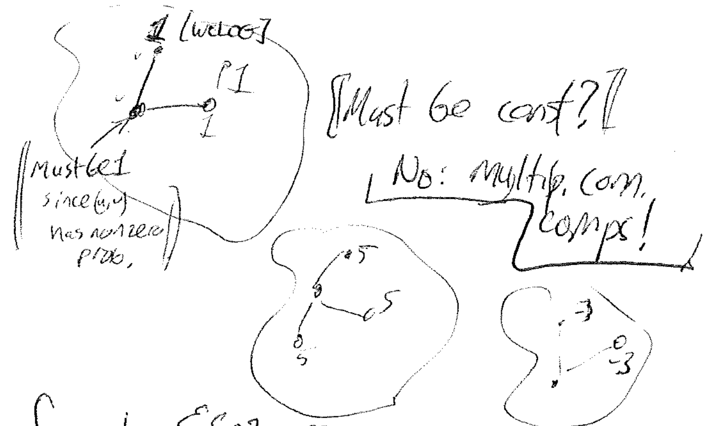
[[Back to ⊛]] [[We said fast cvgc. of rand walks $\Leftrightarrow$ $\mathcal{E}[f]$ "never small". How small can it be?]]

[[Certainly ≥ 0. Can it actually be 0? Sure, if $f \equiv 0$]]

$\mathcal{E}[f] = 0$ if $f \equiv 0$, but $\exists$? nontrivial such f?

[[Yes, $f \equiv 1$]]. $\mathcal{E}[\mathbb{1}] = 0$ [[or any const.]] Non-const. sol'ns?

[[Say we try to get $\mathcal{E}[f] = 0$.]]



[[Must be const?]] No: multip. conn. comps!

prop: $\mathcal{E}[f] = 0$ iff f is const. on each connected comp. of G.

& # conn comps = # lin. indep. f s.t. $\mathcal{E}[f] = 0$.

$\rightarrow$ If c.c's are $S_1, \ldots, S_\ell$, $\mathbb{1}_{S_1}, \ldots \mathbb{1}_{S_\ell}$ are lin indep.

[[and in general, all f s.t. $\mathcal{E}[f] = 0$ are $\sum_{i=1}^{\ell} c_i \mathbb{1}_{S_i}$]].

[Key idea of spectral gh theory is "robustification" of this. G has k "mostly disconnected" comps. iff $\exists k$ lin. indep f with $\mathcal{E}[f]$ "small"] [[Sparsest Cut approx algs.]]

Maximizing $\mathcal{E}[f]$? [[Need scaling constr., $\because$]] $\mathcal{E}[cf] = c^2 \mathcal{E}[f]$

[[Natural choice, given ⊛]]. Max $\mathcal{E}[f]$ s.t. $\mathrm{Var}[f] \underset{\leq}{\neq} 1$ [[same, $\because$ if not 1, mult by const > 1]]

or [[sometimes we consider]] s.t. $\|f\|_2^2 = \mathbf{E}[f^2] \underset{\leq}{\neq} 1$.

Same thing! [[Why?]] $\equiv$ min $\mathbf{E}[f^2]$ s.t. $\mathcal{E}[f] \geq 1$. $\mathcal{E}[f-c] = \mathcal{E}[f]$

Choose c to min $\mathbf{E}[(f-c)^2]$; cond'n becomes $\mathrm{Var}[f]$

[[Max $\mathcal{E}[f]$?]] Intuition: embed $V \to$ line so edges "far apart"!

Q: for what kind of G can you be most successful?

A: bipartite?

max $\mathcal{E}[f]$ s.t. $E[f^2] \le 1$?

If G bipartite, $V = (V_1, V_2)$, let $f = 1_{V_1} - 1_{V_2} = \begin{cases} 1 & \text{if } u \in V_1 \\ -1 & \text{if } u \in V_2 \end{cases}$.

$E[f^2] = 1$, $\quad \mathcal{E}[f] = 2 \leftarrow \frac{1}{2} \underset{u \sim v}{E}[(\underbrace{f(u)}_{\pm 1} - \underbrace{f(v)}_{\mp 1})^2]$

[[best possible?]]

prop: $\mathcal{E}[f] \le 2\|f\|_2^2$ always

pf: $\frac{1}{2} \underset{u \sim v}{E}[(f(u) - f(v))^2] \underset{\text{expand}}{=} \underbrace{\frac{1}{2} \underset{u \sim \pi}{E}[f(u)^2] + \frac{1}{2} \underset{v \sim \pi}{E}[f(v)^2]}_{\|f\|_2^2} - \cancel{\frac{1}{2}} 2 \cdot \underbrace{\underset{u \sim v}{E}[f(u)f(v)]}_{\text{in abs. val.}}$

$\underset{C-S}{\le} \sqrt{\underset{u}{E}[f(u)^2]} \sqrt{\underset{v}{E}[f(v)^2]} = \|f\|_2^2$

$= 2\|f\|_2^2$. $\square$

ex: equal possible $\Leftrightarrow$ G bip.

[Eq. almost possible iff G "close to" bipartite?

$\hookrightarrow$ Max-Cut approx algs.]]

Enter eigenvalues....