Lecture 3: Understanding & Measuring 1 qubit

(in computing we have concept of...)

(implemented on a physical computer/chip)

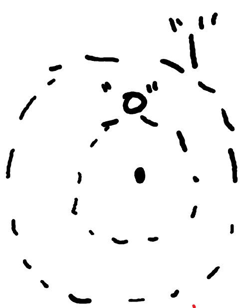
logical bit	physical bit
0	low voltage
1	high voltage

(e.g.; or magnetization/not)

(Maybe using 1 million electrons to store 1 bit. We're into miniaturization. Could we make a bit out of 1 particle, like an electron or photon?)

(Understanding properties of such subatomic particles is the domain of Q.M.)

(Need a particle "property" with 2 possibilities, which we can then call 0/1.)



(hydrogen atom has 1 electron, say it can be in 2 levels)

electron "spin"

up : "0"

down : "1"

(I don't really know what this "is". Somehow measured using magnetism.)

photon polarization

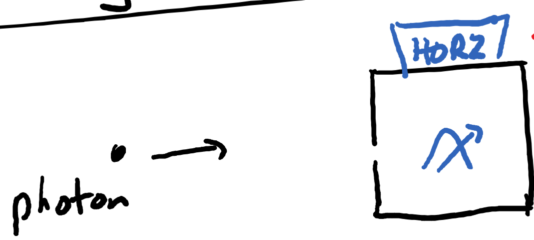
horizontal $\leftrightarrow$: "0"

vertical $\updownarrow$: "1"

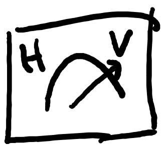
(You've heard of this. Pol. filters for cameras, polarizing sunglasses, 3-d glasses at movies)

(Take photons & polarization, since kinda familiar. Take my word for it, one can build this machine...)

measuring device



(Digital readout which says "HORZ" or "VERT" depending on if photon's polarization measured to be ↔ or ↕)



(← Traditional diagram: readout via a needle pointing)

(Great, so use HORZ=0, VERT=1, start building computer? Not so fast....)

Q.M. Law #1: If a "quantum system"/"particle" can be in of of two basic states $|0\rangle$ or $|1\rangle$, it can also be in a superposition state, meaning:

(will explain weird brackets later)

α "amplitude" on $|0\rangle$,

β "amplitude" on $|1\rangle$,

where α, β are numbers satisfying

$$|\alpha|^2 + |\beta|^2 = 1.$$

("amplitude": just some word)

$\boxed{\text{"Qubit"}}$

E.g., a photon may have state

" .8 amplitude on $|0\rangle$, (horz. polarization)
.6 amplitude on $|1\rangle$, (vert. ") "

(check: $.8^2 + .6^2 = .64 + .36 = 1$ ✓)

OR

" .8 ampl. on $|0\rangle$, $-.6$ ampl. on $|1\rangle$ "

(Negative amplitudes totally possible. These are not probabilities)

OR

" 1 amplitude on $|0\rangle$, 0 amplitude on $|1\rangle$. "

OR

" i amplitude on $|0\rangle$, 0 amplitude on $|1\rangle$ "

Yes, amplitudes are complex numbers!

Check: $|i|^2 + |0|^2 = 1 + 0 = 1$. ✓

(However, because we're already introducing a lot of wackiness, I'll mostly stick to possibly negative real numbers in all my examples.)

(You might think "wait, you told me that measuring device for polarization exists... and it never reads out "MIXED", just "HORZ" or "VERT". So what's the deal?)

Q.M. Law #2: For particle with
amplitude α on $|0\rangle$,
" β on $|1\rangle$,

if you measure it then...

(Deep & mysterious question: what exactly constitutes a "measurement". For our purposes, as the quote goes, physicists "know it when they see it".)

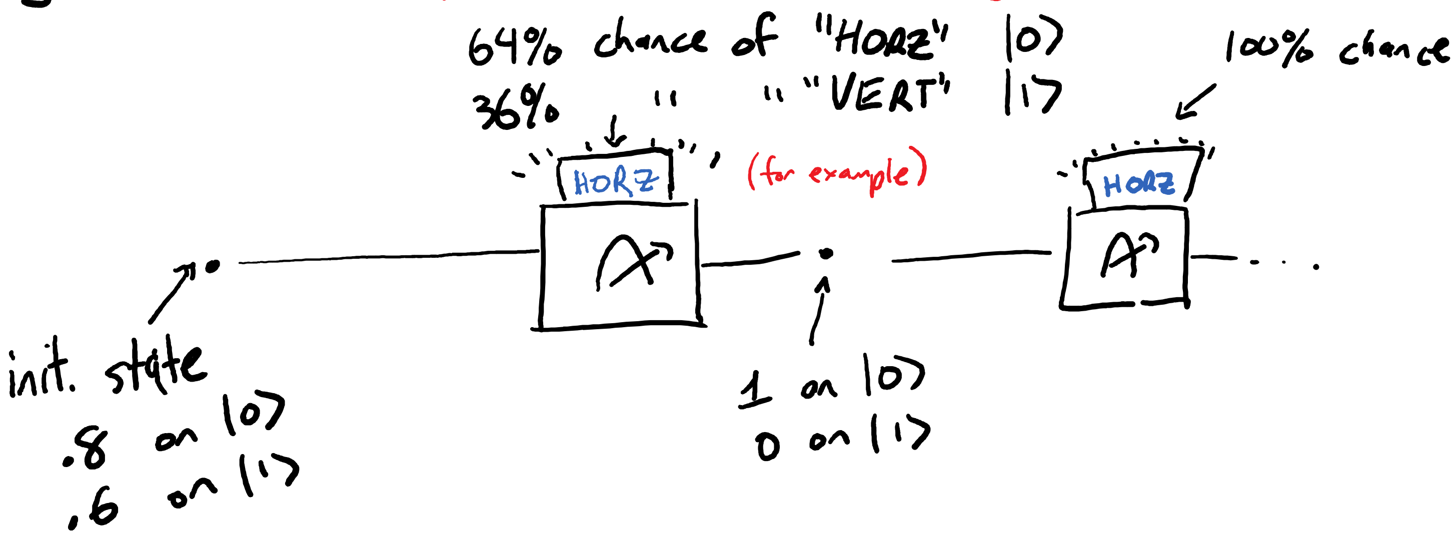
with probability $|\alpha|^2$, readout shows "$|0\rangle$"
with probability $|\beta|^2$, " " "$|1\rangle$",
$+$
$= 1$ ✓ (makes sense)

(BTW, in the prev. lecture, we were calling these two amplitudes "$f(0)$" and "$f(1)$".)

And: if readout is "$|0\rangle$", particle's state changes to "1 ampl. on $|0\rangle$,
0 " " $|1\rangle$",

if readout is '$|1\rangle$', ——
—— "0 ampl. on $|0\rangle$
1 " " $|1\rangle$".

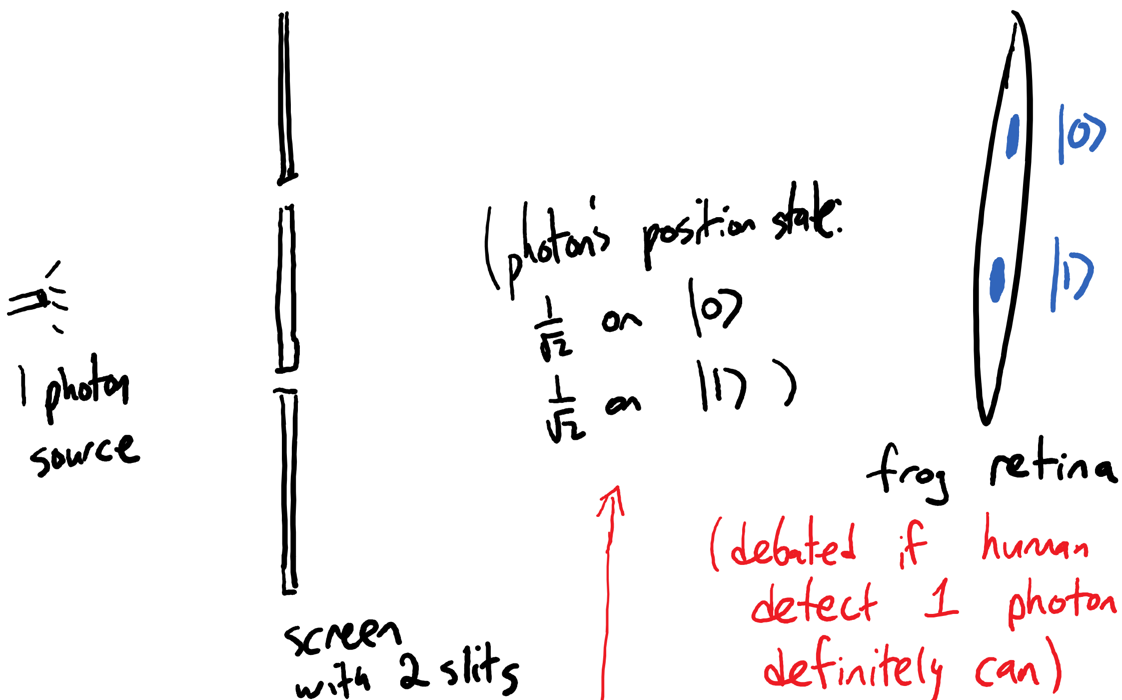
e.g.: (this is a typical "quantum circuit diagram")



<u>Same</u> if init. state is ".8 on $|0\rangle$, −.6 on $|1\rangle$".

(But this <u>is</u> a fundamentally different state. As we'll see, there <u>are</u> physical devices that have different detection behaviors for these two states.)

(Laws 1 & 2 are true for <u>any</u> particle property that has 2 basic states. For polarization, doesn't seem so bad. For, say, "position", can seem very weird.)

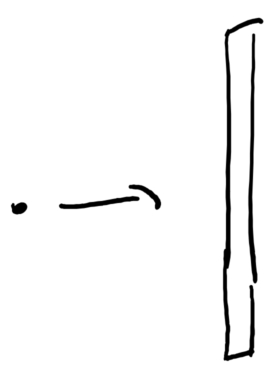


(photon's position state:
$\frac{1}{\sqrt{2}}$ on $|0\rangle$
$\frac{1}{\sqrt{2}}$ on $|1\rangle$)

(for simplicity, say retina has 2 sensitive positions; can call them 0 & 1)

(debated if human retina can detect 1 photon; frog retina definitely can)

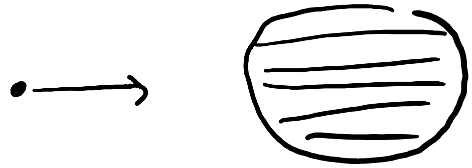
(Weird. Frog retina is the measuring device of Q.M. Law 2.
Photon goes thru... both slits?
Well... that's how it is. You can do the physical experiment.)



(Similar story for photon thru a thin slice of glass... State becomes some ampl. on reflection, some ampl. on transmission!)

(Can actually illustrate using polarizing filters.
I hesitate a bit because you have to think a bit carefully when thinking about them as meas. devices)

"horz. filter"



: ① measures : photon pol. state
↳ $|0\rangle$ or $|1\rangle$
$\leftrightarrow$ $\updownarrow$

② · if now $\underset{\leftrightarrow}{|0\rangle}$: photon flies thru

· else if now $\underset{\updownarrow}{|1\rangle}$: photon converted to heat

(If you fire laser pointer at it, can consider the millions of photons to be "randomly" polarized. Then 50% fly through. (It's a filter! Used in photography.) And those that do are all horz. polarized, Useful: it's like we now have a bunch of photons "initialized" to $|0\rangle$.)

(Also exists a "vertical filter", where ② is reversed. What's cute: you can obtain it by physically rotating horz filter 90°. 3-d glasses have one for each eye.)

(What if you rotate 45°? We'll see....!)

Particle with 3 basic states

(Perfectly possible.  3 energy levels, or "spin" of a
3 positions. "deuterium nucleus".)

$|1\rangle, |2\rangle, |3\rangle$: a "qutrit"

With 4 basic states .... $\underline{\text{a "qudit"}}$ with dimension $d=4$.

Most commonly: joint basic state of 2 qubits

e.g. 2 photons:

↔, ↔	$\|1\rangle$	$\|00\rangle$
↔, ↕	$\|2\rangle$	$\|01\rangle$
↕, ↔	$\|3\rangle$	$\|10\rangle$
↕, ↕	$\|4\rangle$	$\|11\rangle$

(We use the math convention of $1, 2, 3, \dots, d$.
Might have been more elegant to use the python convention of $0, 1, 2, \dots, d-1$, because for $d=2$ we always use 0 & 1.
Oh well.)

Q.M. Law 1: general state is
"amplitude α_1 on $|1\rangle$, α_2 on $|2\rangle$, ... α_d on $|d\rangle$, such that
$|\alpha_1|^2 + \dots + |\alpha_d|^2 = 1$"

(vs. last lecture's notation: N-dim. state, ampls $f(0), \dots, f(N-1)$)

Q.M. Law 2: measurement : $\rightarrow$ readout "$|i\rangle$" with prob. $|\alpha_i|^2$, and then state becomes 1 ampl. on $|i\rangle$, 0 on rest.

(Time to begin the math properly!)
(For a qudit we need to track a list of d numbers whose squared magnitudes sum to 1. That's nothing more than...)

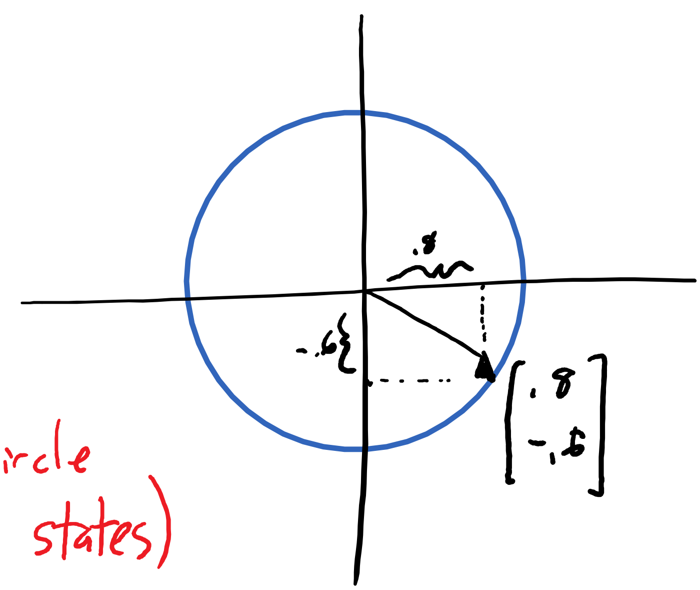
$$\text{qudit state} \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix} \in \mathbb{C}^d.$$

d-dim. column vector $\rightarrow \vec{v}$ (Nontraditional notation in Q.M. We'll change it soon.)

$|\alpha_1|^2 + \cdots + |\alpha_d|^2 = \|\vec{v}\|^2 = 1$: a unit vector

e.g.: qubit state ".8 amplitude on $|0\rangle$, $-.6$ amplitude on $|1\rangle$"

$$\equiv \begin{bmatrix} .8 \\ -.6 \end{bmatrix}$$



(all blue points, unit circle correspond to qubit states)

(Actually, that picture is only appropriate for real amplitudes :( )

Recall: for complex $z = x + iy$,

$$|z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z \cdot z^*$$

aka $\bar{z}$

(complex conjugate)

(squared magnitude of complex vector is sum of squares of all real & imaginary parts)

CONFUSING:
- One qubit $\to$ 2 (complex) numbers
- One complex # $\to$ 2 real numbers

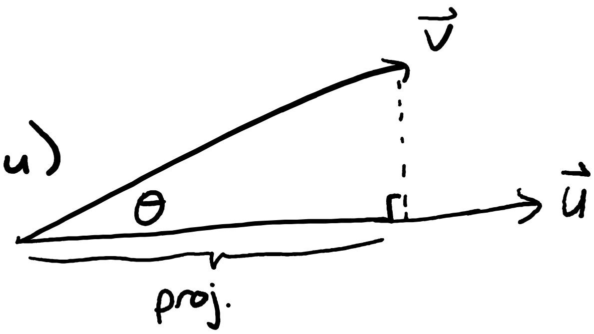
(we like to draw both of these in the 2-d plane)
(technically bogus for qubits: they need 4 real #'s)
(BUT we love to draw qubits in the plane so much, we'll almost always do so. We'll mainly only concern ourselves with real ampls. (till Shor's Alg.) and we'll try not to illustrate 1 complex # in plane.)

Unit vector: $\|\vec{v}\|^2 = 1$

$\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$ where $\langle \vec{u}, \vec{v} \rangle$ is inner (dot) product

Meanings of $\langle \vec{u}, \vec{v} \rangle$, $\quad \vec{u}, \vec{v} \in \mathbb{R}^d$

1. (len. of u) $\cdot$ (len of v's projection on u)



(best way to think of it;
0 if u and v are perp.,
$(\text{length})^2$ if $u = v$)

2. $\|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$. Just $\cos\theta$ if $\vec{u}, \vec{v}$ are unit (e.g. quantum states)

3. $u_1 v_1 + u_2 v_2 + \dots + u_d v_d = [u_1 \cdots u_d] \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = u^T v$

(sum of squares of coords when $u = v$)

Complex case: (We really want $\langle v, v \rangle = \|v\|^2 = \sum_i |v_i|^2$ to hold.)

$$\langle \vec{u}, \vec{v} \rangle \overset{\text{def}}{=} u_1^* v_1 + u_2^* v_2 + \dots + u_d^* v_d$$

$$= [u_1^* \; u_2^* \cdots u_d^*] \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$$

(these formulas still ok in the real case)

$u^\dagger$ (u-dagger, conjugate transpose)

Dirac's Bra-Ket notation

High school: $\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}3\\5\\-2\end{bmatrix} = 3\vec{i} + 5\vec{j} - 2\vec{k}$$

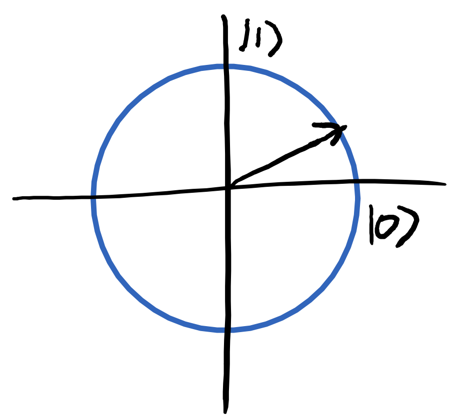
College: $e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$ ... $e_d = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$

Quantum: $\quad = |1\rangle \qquad = |2\rangle \quad \cdots \quad = |d\rangle$

(I <u>love</u> this notation. Used to hate it!
I use it whenever I do linalg., even if no quantum!)

exception: $d = 2$: $\begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$, $\begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$

qubit state $\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$



(Advantage 1: often deal with sparse vectors.
Adv. 2: no need to cram # into subscript.)

notation: $|\text{Blah}\rangle$

any name

$|\cdot\rangle$ signifies type = column vector

Called a "ket".

not$^{\underline{n}}$: $\langle\text{Blah}|$ denotes its conjugate transpose, $|\text{Blah}\rangle^\dagger$, a row vector.

Called a "bra".

(Bra-ket = bracket. Haha? Thanks, Dirac.)

not$^{\underline{n}}$: $\langle\vec{u},\vec{v}\rangle$ is $u^\dagger v$

(matrix mult)

$= \langle u|\cdot|v\rangle$

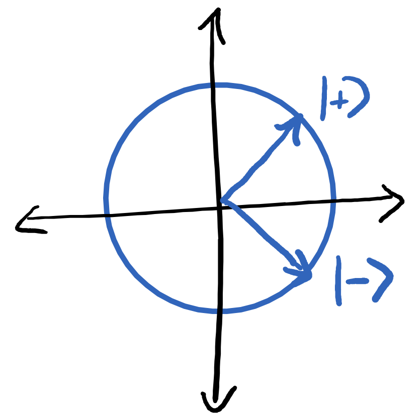
$=$ "$\langle u|v\rangle$"

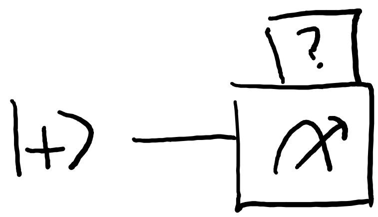
(genuinely convenient notational shorthand) example qubit: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(so famous/important, has its own name:)

$= \text{"}|+\rangle\text{"}$

also: "$|-\rangle$" $\stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

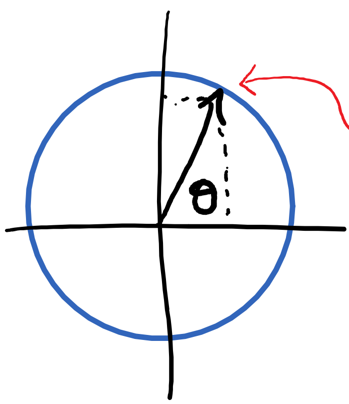




"$|0\rangle$" w.prob $\frac{1}{2}$

"$|1\rangle$" ~ $\frac{1}{2}$

(Same if you feed in $|-\rangle$. Tho these are different states.)



$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta \cdot |0\rangle + \sin\theta\,|1\rangle$

Pr[measuring $|0\rangle$]

$= (\cos\theta)^2$ ⊛

(important formula)

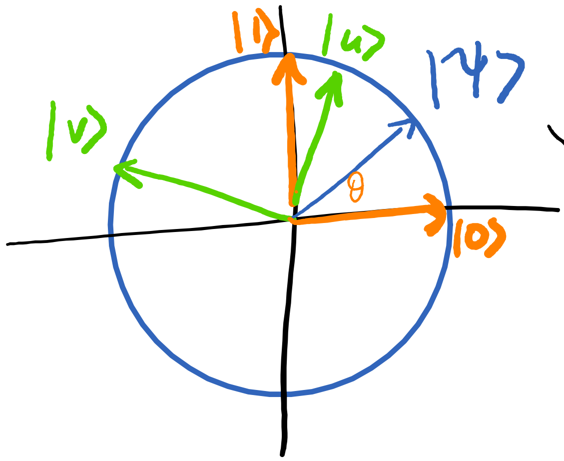
$|\psi\rangle = \underbrace{?}_{\langle 0|\psi\rangle}|0\rangle + \underbrace{?}_{\langle 1|\psi\rangle}|1\rangle$

$\langle 0|\psi\rangle$ (len of $|\psi\rangle$ projected on $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$)

$\langle 1|\psi\rangle$

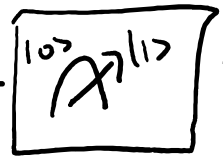
Pr[meas. $|0\rangle$] $= |\langle 0|\psi\rangle|^2$.

Measuring in a different basis



(Say you have a qubit in this state)

("standard" measuring device:)



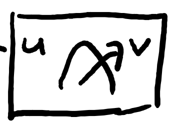
- readout "$|0\rangle$" with prob. $|\langle 0|\psi\rangle|^2 = (\cos\theta)^2$
- readout "$|1\rangle$" with prob. $|\langle 1|\psi\rangle|^2 = (\sin\theta)^2$

(In fact...)

For <u>any</u> orthonormal basis $\{|u\rangle, |v\rangle\}$, can build a
(perpendicular, unit)

"measuring device for this basis".

$|\psi\rangle$ —



:

- readout is "$|u\rangle$" with prob. $|\langle u|\psi\rangle|^2$,
  i.e., $\cos^2(\text{angle betw. } |u\rangle, |\psi\rangle)$
- ——— "$|v\rangle$" ——— $|\langle v|\psi\rangle|^2$,
  i.e. $\cos^2($ ——— $|v\rangle, |\psi\rangle)$
  $= \sin^2($ ——— $|u\rangle, |\psi\rangle)$.

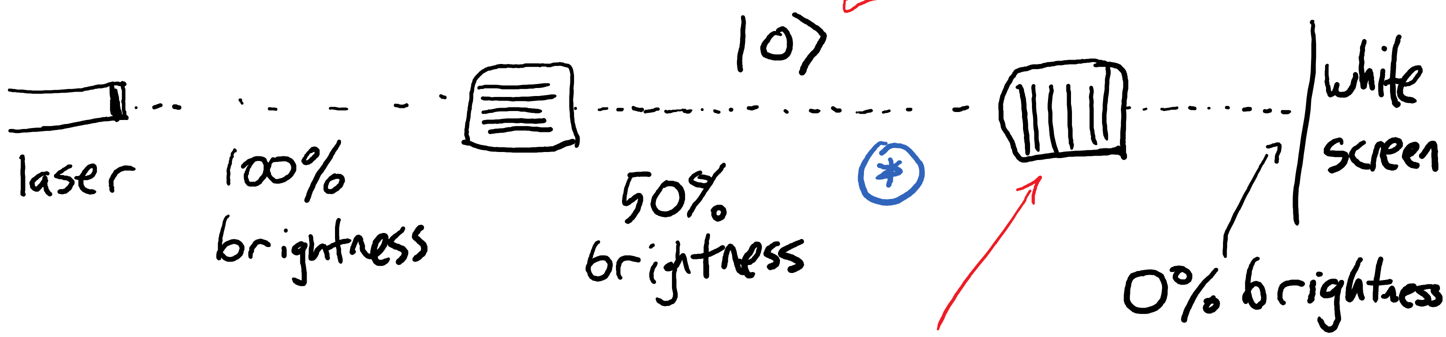
<u>E.g.</u>: Measuring in the $\{|+\rangle, |-\rangle\}$ basis. ← We love to do this. If your state is $f(0)|0\rangle + f(1)|1\rangle$, then it's $\hat{f}(0)|+\rangle + \hat{f}(1)|-\rangle$ and measuring in $\{|+\rangle, |-\rangle\}$ basis is like sampling from f's Fourier coeffs...)

- $|+\rangle$: → <u>always</u> reads out "$|+\rangle$"
- $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$: → reads out "$|+\rangle$" w.p. $\frac{1}{2}$, "$|-\rangle$" w.p. $\frac{1}{2}$.

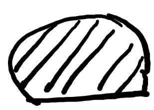
$$\stackrel{\checkmark}{=} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Fun: can* build a $\{|+\rangle, |-\rangle\}$ polarizing filter by physically rotating a horizontal one 45°.

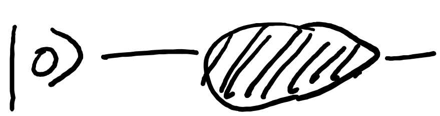
(Do actual experiment....) (all passing photons now HORZ $= |0\rangle$)



(Recall: measures in std. basis. Then: if $|0\rangle \to$ heat
if $|1\rangle \to$ pass thru.)

Now interpose  filter ($\{|+\rangle, |-\rangle\}$ measurer) at ⊛.

(Childlike intuition: Currently no light at the end. Surely after another impediment, still no light.)



with prob. $\frac{1}{2}$, measures $|+\rangle$, passes thru

$\frac{1}{2}$, measures $|-\rangle$, heat.

So 25% brightness now!

with prob $\frac{1}{2}$, $|0\rangle$ measured: heat

$\frac{1}{2}$, $|1\rangle$ measured: passes thru.

12.5% brightness at end!!

(A weak example of so-called "Quantum Anti-Zeno Effect," explored on Homework 2, Related to the "Elitzur-Vaidman Bomb", which we'll start with on Thursday!)