Reasoning with Quantified Boolean Formulas

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Automated Reasoning and Satisfiability
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What are QBF?

- **Quantified Boolean formulas (QBF)** are

  formulas of propositional logic + quantifiers

- *Examples*:
  - $(x \lor \overline{y}) \land (\overline{x} \lor y)$ (propositional logic)

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  - $\exists x \forall y (x \lor \overline{y}) \land (\overline{x} \lor y)$
    Is there a value for x such that for all values of y the formula is true?

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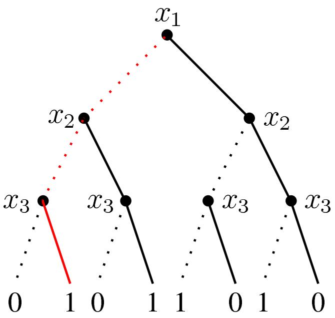
- *Examples*:
  - $(x \vee \overline{y}) \wedge (\overline{x} \vee y)$ (propositional logic)
  - $\exists x \forall y (x \vee \overline{y}) \wedge (\overline{x} \vee y)$
    Is there a value for x such that for all values of y the formula is true?
  - $\forall y \exists x (x \vee \overline{y}) \wedge (\overline{x} \vee y)$
    For all values of y, is there a value for x such that the formula is true?

SAT vs. QSAT aka NP-complete vs. PSPACE-complete

SAT

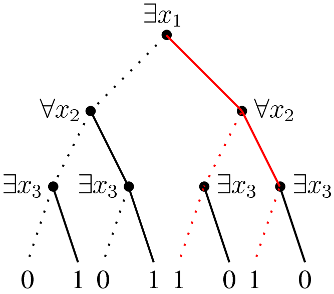
$\phi(x_1, x_2, x_3)$



Is there a satisfying assignment?

QBF

$\exists x_1 \forall x_2 \exists x_3 \phi(x_1, x_2, x_3)$



Is there a satisfying assignment **tree**?

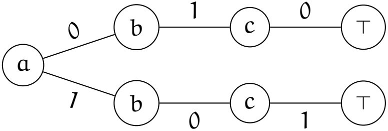
Small Example QSAT Problems

Consider the formula $\forall a\, \exists b, c. (a \lor b) \land (\overline{a} \lor c) \land (\overline{b} \lor \overline{c})$

Small Example QSAT Problems

Consider the formula $\forall a\,\exists b, c. (a \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee \overline{c})$

A model is:



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Consider the formula $\forall a\, \exists b, c. (a \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee \overline{c})$

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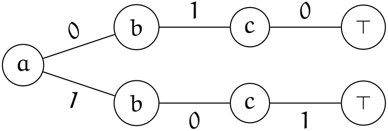

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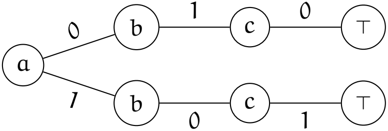
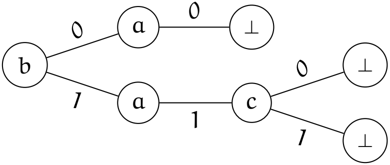


Consider the formula $\exists b\, \forall a\, \exists c. (a \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee \overline{c})$

A counter-model is:



The quantifier prefix frequently determines the truth of a QBF.

The Two Player Game Interpretation of QSAT

Interpretation of QSAT as *two player game* for a QBF $\exists x_1 \forall a_1 \exists x_2 \forall a_2 \cdots \exists x_n \forall a_n \psi$:

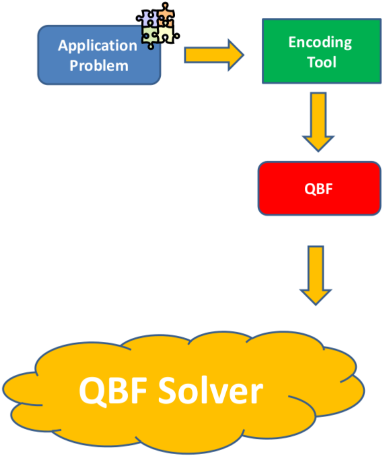
- Player A (existential player) tries to satisfy the formula by assigning existential variables
- Player B (universal player) tries to falsify the formula by assigning universal variables

- Player A and Player B make alternately an assignment of the variables in the outermost quantifier block
- Player A wins: formula is satisfiable, i.e., there is a strategy for assigning the existential variables such that the formula is always satisfied
- Player B wins: formula is unsatisfiable

Promises of QBF

- QSAT is the prototypical problem for *PSPACE*.
- QBFs are suitable as *host language* for the encoding of many application problems like
  - verification
  - artificial intelligence
  - knowledge representation
  - game solving
- In general, QBF allow more succinct encodings then SAT

Application of a QBF Solver



QBF Solver returns

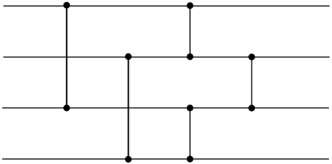
1. yes/no
2. witnesses

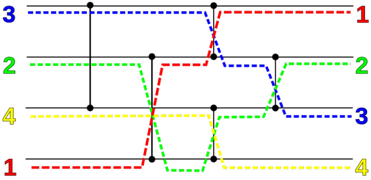
Example of ∃∀∃: Synthesis

Given an input-output specification, does there exists a circuit that satisfies the input-output specification.

QBF solving can be used to find the smallest sorting network:

- (∃) Does there exists a sorting network of k wires,
- (∀) such that for all input variables of the network
- (∃) the output $O_i \leq O_{i+1}$

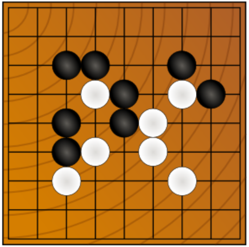




Example of $\forall\exists\ldots\forall\exists$: Games

Many games, such as Go and Reversi, can be naturally expressed as a QBF problem.

Boolean variables $a_{i,k}$, $b_{j,k}$ express that the existential player places a piece on row i and column j at his kth turn. Variables $c_{i,k}$, $d_{j,k}$ are used for the universal player.



Go

Reversi

The QBF problem is of the form

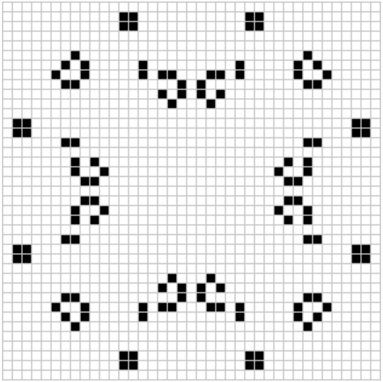
$$\forall c_{i,1}, d_{j,1} \exists a_{i,1}, b_{j,1} \ldots \forall c_{i,n}, d_{j,n} \exists a_{i,n}, b_{j,n} . \psi$$

Outcome "satisfiable": the second player (existential) can always prevent that the first player (universal) wins.

Illustrating Example ∀∃: Conway's Game of Life

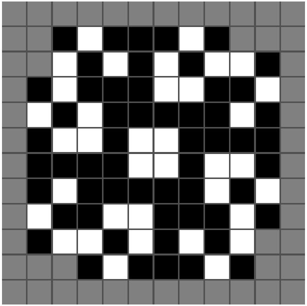
Conway's Game of Life is an infinite 2D grid of cells that are either alive or dead using the following update rules:

- Any alive cell with fewer than two alive neighbors dies;
- Any alive cell with two or three live neighbors lives;
- Any alive cell with more than three alive neighbors dies;
- Any dead cell with exactly three alive neighbors becomes alive.



Game of Life is very popular: over 1,100 wiki articles

Garden of Eden in Conway's Game of Life



A Garden of Eden (GoE) is a state that can only exist as initial state.

Let $T(x, y)$ denote the CNF formula that encodes the transition relation from a state to its successor using variables x that describe the current state and variables y the successor state.

A QBF that encodes the GoE problem is simply

$$\forall y \exists x . T(x, y)$$

The smallest Garden of Eden known so far (shown above) was found using a QBF solver. [Hartman et al. 2013]

The Language of QBF

The language of **quantified Boolean formulas** $\mathcal{L}_{\mathcal{P}}$ over a set of propositional variables $\mathcal{P}$ is the smallest set such that

- if $v \in \mathcal{P} \cup \{\top, \bot\}$ then $v \in \mathcal{L}_{\mathcal{P}}$ (variables, constants)
- if $\phi \in \mathcal{L}_{\mathcal{P}}$ then $\overline{\phi} \in \mathcal{L}_{\mathcal{P}}$ (negation)
- if ϕ and $\psi \in \mathcal{L}_{\mathcal{P}}$ then $\phi \wedge \psi \in \mathcal{L}_{\mathcal{P}}$ (conjunction)
- if ϕ and $\psi \in \mathcal{L}_{\mathcal{P}}$ then $\phi \vee \psi \in \mathcal{L}_{\mathcal{P}}$ (disjunction)
- if $\phi \in \mathcal{L}_{\mathcal{P}}$ then $\exists v \phi \in \mathcal{L}_{\mathcal{P}}$ (existential quantifier)
- if $\phi \in \mathcal{L}_{\mathcal{P}}$ then $\forall v \phi \in\mathcal{L}_{\mathcal{P}}$ (universal quantifier)

Some Notes on Variables and Truth Constants

- $\top$ stands for *top*
  - always true
  - empty conjunction
- $\bot$ stands for *bottom*
  - always false
  - empty disjunction
- *literal*: variable or negation of a variable
  - examples: $l_1 = v$, $l_2 = \overline{w}$
  - var(l) $= v$ if $l = v$ or $l = \overline{v}$
  - complement of literal l: $\bar{l}$
- $var(\phi)$: set of variables occurring in QBF ϕ

Some QBF Terminology

Let $Qv\psi$ with $Q \in \{\forall, \exists\}$ be a subformula in a QBF ϕ, then

- ψ is the *scope* of v
- Q is the *quantifier binding* of v
- *quant*$(v) = Q$
- *free variable* w in ϕ: w has no quantifier binding in ϕ
- *bound variable* w in QBF ϕ: w has quantifier binding in ϕ
- *closed QBF*: no free variables

Example

$$\overbrace{\forall a}^{\textit{bound var } a} \quad \underbrace{(a \land \overbrace{x}^{\textit{free variable}} \lor \overbrace{\overbrace{\forall y \exists z}^{\textit{bound vars } y,\, z} \quad \overbrace{((y \lor \bar{z}) \land (\bar{y} \lor z))}^{\textit{scope of } y,\, z}}^{\text{closed QBF}})}_{\textit{scope of } a}$$

Prenex Conjunctive Normal Form (PCNF)

A QBF ϕ is in **prenex conjunctive normal form** iff

- ϕ is in *prenex normal form* $\phi = Q_1 v_1 \dots Q_n v_n \psi$
- matrix ψ is in *conjunctive normal form*, i.e.,

$$\psi = C_1 \wedge \cdots \wedge C_m$$

where C_i are clauses, i.e., disjunctions of literals.

Example

$$\underbrace{\forall x \exists y}_{\textit{prefix}} \underbrace{((x \vee \overline{y}) \wedge (\overline{x} \vee y))}_{\textit{matrix in CNF}}$$

Some Words on Notation

If convenient, we write

- a conjunction of clauses as a set, i.e.,

$$C_1 \wedge \ldots \wedge C_m = \{C_1, \ldots, C_m\}$$

- a clause as a set of literals, i.e.,

$$l_1 \vee \ldots \vee l_k = \{l_1, \ldots, l_k\}$$

- $var(\phi)$ for the variables occurring in ϕ
- $var(l)$ for the variable of a literal, i.e.,

$$var(l) = x \text{ iff } l = x \text{ or } l = \bar{x}$$

Example

$$\underbrace{\forall x \exists y}_{\textit{prefix}}\underbrace{((x \vee \bar{y}) \wedge (\bar{x} \vee y))}_{\textit{matrix in CNF}} \approx \underbrace{\forall x \exists y}_{\textit{prefix}}\underbrace{\{\{x, \bar{y}\}, \{\bar{x}, y\}\}}_{\textit{matrix in CNF}}$$

Semantics of QBFs

A **valuation function** $\mathcal{I}$: $\mathcal{L}_{\mathcal{P}} \rightarrow \{\mathcal{T}, \mathcal{F}\}$ for closed QBFs is defined as follows:

- $\mathcal{I}(\top) = \mathcal{T}$; $\mathcal{I}(\bot) = \mathcal{F}$
- $\mathcal{I}(\overline{\psi}) = \mathcal{T}$ iff $\mathcal{I}(\psi) = \mathcal{F}$
- $\mathcal{I}(\phi \vee \psi) = \mathcal{T}$ iff $\mathcal{I}(\phi) = \mathcal{T}$ or $\mathcal{I}(\psi) = \mathcal{T}$
- $\mathcal{I}(\phi \wedge \psi) = \mathcal{T}$ iff $\mathcal{I}(\phi) = \mathcal{T}$ and $\mathcal{I}(\psi) = \mathcal{T}$
- $\mathcal{I}(\forall v.\psi) = \mathcal{T}$ iff $\mathcal{I}(\psi[\bot/v]) = \mathcal{T}$ and $\mathcal{I}(\psi[\top/v]) = \mathcal{T}$
- $\mathcal{I}(\exists v.\psi) = \mathcal{T}$ iff $\mathcal{I}(\psi[\bot/v]) = \mathcal{T}$ or $\mathcal{I}(\psi[\top/v]) = \mathcal{T}$

```
Boolean split (QBF $\phi$)

switch($\phi$)
  case $\top$: return true;
  case $\bot$: return false;
  case $\overline{\psi}$: return (not split($\psi$));
  case $\psi' \wedge \psi''$: return split($\psi'$) && split($\psi''$);
  case $\psi' \vee \psi''$: return split($\psi'$) || split($\psi''$);
  case $QX\psi$:
    select $x \in X$; $X' = X \backslash \{x\}$;
    if ($Q == \forall$)
      return (split($QX'\psi[\top/x]$) &&
              split($QX'\psi[\bot/x]$));
    else
      return (split($QX'\psi[\top/x]$) ||
              split($QX'\psi[\bot/x]$));
```

Some Simplifications

The following rewritings are *equivalence preserving*:

1. $\overline{\top} \Rightarrow \bot$; $\quad \overline{\bot} \Rightarrow \top$;
2. $\top \wedge \phi \Rightarrow \phi$; $\quad \bot \wedge \phi \Rightarrow \bot$; $\quad \top \vee \phi \Rightarrow \top$; $\quad \bot \vee \phi \Rightarrow \phi$;
3. $(Qx\,\phi) \Rightarrow \phi$, $Q \in \{\forall, \exists\}$, x does not occur in ϕ;

Example

$$\begin{aligned}\forall ab\exists x\forall c\exists yz\forall d\{\{a, b, \overline{c}\}, \{a, \overline{b}, \overline{\top}\},\\ \{c, y, d, \bot\}, \{x, y, \overline{\bot}\}, \{x, c, d, \top\}\}\end{aligned}$$

$$\approx$$

$$\forall abc\exists y\forall d\{\{a, b, \overline{c}\}, \{a, \overline{b}\}, \{c, y, d\}\}$$

```
Boolean splitCNF (Prefix P, matrix $\psi$)

if ($\psi == \top$): return true;
if ($\bot \in \psi$): return false;

$P = QXP', x \in X, X' = X \backslash \{x\}$;

if ($Q == \forall$)
    return (splitCNF($QX'P', \psi'$) &&
            splitCNF($QX'P', \psi''$));
else
    return (splitCNF($QX'P', \psi'$) ||
            splitCNF($QX'P', \psi''$));
where
$\psi'$: take clauses of $\psi$, delete clauses with $x$, delete literals $\bar{x}$
$\psi''$: take clauses of $\psi$, delete clauses with $\bar{x}$, delete literals $x$
```

Unit Clauses

A clause C is called **unit** in a formula ϕ iff

- C contains exactly one existential literal
- the universal literals of C are to the right of the existential literal in the prefix

The existential literal in the unit clause is called *unit literal*.

Example

$\forall ab \exists x \forall c \exists y \forall d \{\{a, b, \bar{x}, \bar{c}\}, \{a, \bar{b}\}, \{c, y, d\}, \{x, y\}, \{x, c, d\}, \{y\}\}$

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Unit literals: x, y

Unit Literal Elimination

Let ϕ be a QBF with unit literal l and let ϕ' be a QBF obtained from ϕ by

- removing all clauses containing l
- removing all occurrences of $\bar{l}$

Then

$$\phi \approx \phi'$$

Example

$\forall ab \exists x \forall c \exists y \forall d \{\{a, b, \bar{x}, \bar{c}\}, \{a, \bar{b}\}, \{c, y, d\}, \{x, y\}, \{x, c, d\}, \{y\}\}$

After unit literal elimiation: $\forall abc \{\{a, b, \bar{c}\}, \{a, \bar{b}\}\}$

Pure Literals

A literal l is called **pure** in a formula ϕ iff

- l occurs in ϕ
- the complement of l, i.e., $\bar{l}$ does not occur in ϕ

Example

$$\forall ab \exists x \forall c \exists yz \forall d \{\{a, b, \overline{c}\}, \{a, \overline{b}\}, \{c, y, d\}, \{x, y\}, \{x, c, d\}\}$$

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Pure: a, d, x, y

Pure Literal Elimination

Let ϕ be a QBF with pure literal l and let ϕ' be a QBF obtained from ϕ by

- removing all clauses with l if $\mathrm{quant}(l) = \exists$
- removing all occurrences of l if $\mathrm{quant}(l) = \forall$

Then

$$\phi \approx \phi'$$

Example

$\forall ab \exists x \forall c \exists yz \forall d \{\{a, b, \bar{c}\}, \{a, \bar{b}\}, \{c, y, d\}, \{x, y\}, \{x, c, d\}\}$

After Pure Literal Elimination: $\forall b \{\{b\}, \{\bar{b}\}\}$

Universal Reduction (UR)

- Let $\Pi.\psi$ be a QBF in PCNF and $C \in \psi$.
- Let $l \in C$ with
  - $\mathrm{quant}(l) = \forall$
  - forall $k \in C$ with $\mathrm{quant}(k) = \exists$ $k <_\Pi l$, i.e., all existential variables k of C are to the left of l in Π.
- Then l may be removed from C.
- $C \backslash \{l\}$ is called the *universal reduct* of C.

Example

$$\forall ab \exists x \forall c \exists yz \forall d \{\{a, b, x, \bar{c}\}, \{a, \bar{b}, x\}, \{c, y, d\}, \{x, y\}, \{x, c, d\}\}$$

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After Universal Reduction:

$$\forall ab \exists x \forall c \exists yz \forall d \{\{a, b, x\}, \{a, \bar{b}, x\}, \{c, y\}, \{x, y\}, \{x\}\}$$

```
Boolean splitCNF2 (Prefix P, matrix $\psi$)

$(P, \psi) = simplify(P, \psi)$;

if ($\psi == \bot$): return true;
if ($\bot \in \psi$): return false;

$P = QXP'$, $x \in X$, $X' = X \backslash \{x\}$;

if ($Q == \forall$)
    return (splitCNF2($QX'P', \psi'$) &&
            splitCNF2($QX'P', \psi''$));
else
    return (splitCNF2($QX'P', \psi'$) ||
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where
$\psi'$ : take clauses of $\psi$, delete clauses with $x$, delete $\overline{x}$
$\psi''$ : take clauses of $\psi$, delete clauses with $\overline{x}$, delete $x$
```

Resolution for QBF

Q-Resolution: propositional resolution + universal reduction.

Definition

Let C_1, C_2 be clauses with existential literal $l \in C_1$ and $\bar{l} \in C_2$.

1. Tentative Q-resolvent:
   $C_1 \bowtie C_2 := (\,UR(C_1) \cup UR(C_2)) \setminus \{l, \bar{l}\}$.
2. If $\{x, \overline{x}\} \subseteq C_1 \bowtie C_2$ then no Q-resolvent exists.
3. Otherwise, Q-resolvent $C := (C_1 \bowtie C_2)$.

- Q-resolution is a sound and complete calculus.
- Universals as pivot are also possible.

Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \vee y) \wedge (\overline{x} \vee \overline{y})$

Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \lor y) \land (\bar{x} \lor \bar{y})$

Truth Table

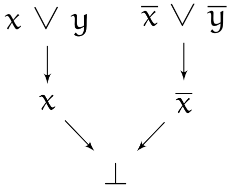
x	y	ψ
0	0	0
0	1	1
1	0	1
1	1	0

unsat

Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \vee y) \wedge (\overline{x} \vee \overline{y})$

Q-Resolution Proof



Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \lor y) \land (\overline{x} \lor \overline{y})$

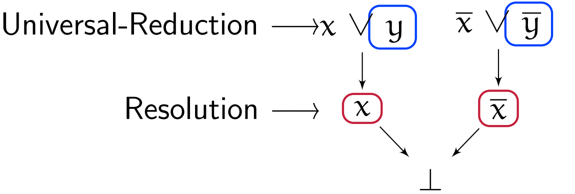
Q-Resolution Proof

$$\begin{tikzcd}
\text{Universal-Reduction} \arrow[r] & x \lor \boxed{y} \arrow[d] & & \overline{x} \lor \boxed{\overline{y}} \arrow[d] \\
& x \arrow[dr] & & \overline{x} \arrow[dl] \\
& & \bot &
\end{tikzcd}$$

Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \vee y) \wedge (\bar{x} \vee \bar{y})$

Q-Resolution Proof



Universal-Reduction ⟶ $x \vee \boxed{y}$ $\bar{x} \vee \boxed{\bar{y}}$

Resolution ⟶ $\boxed{x}$ $\boxed{\bar{x}}$

$\bot$

Q-Resolution Small Example

Exclusive OR (XOR): QBF $\psi = \exists x \forall y (x \lor y) \land (\overline{x} \lor \overline{y})$

Truth Table

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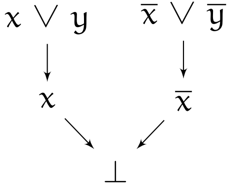
Q-Resolution Proof



$$\longrightarrow \quad y = x \quad \Rightarrow \quad \psi = 0$$

Q-Resolution Small Example

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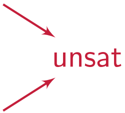

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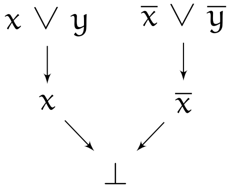
Q-Resolution Proof



$x \vee y$ $\overline{x} \vee \overline{y}$

x $\overline{x}$

$\perp$

$$\longrightarrow \quad y = x \quad \Rightarrow \quad \psi = 0$$

$$\longrightarrow \quad f_y(x) = x \quad \text{(counter model)}$$

Q-Resolution Large Example

Input Formula

$$\exists a \forall b \exists c d \forall e \exists f g . (\overline{a} \vee \overline{g}) \wedge (b \vee f \vee g) \wedge (c \vee \overline{e} \vee \overline{f}) \wedge (d \vee \overline{e} \vee \overline{f}) \wedge (\overline{c} \vee \overline{d} \vee e) \wedge (a \vee f)$$

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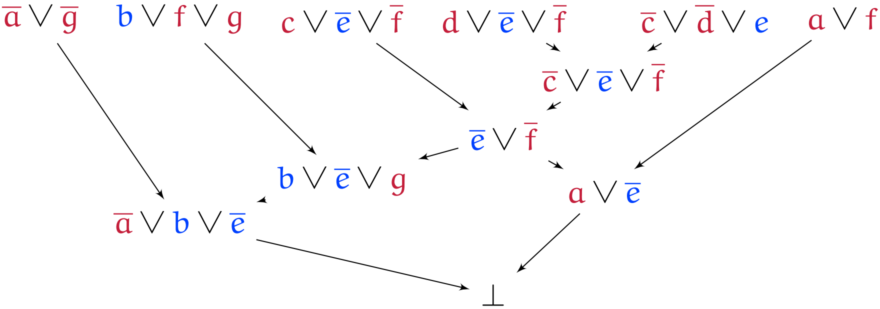
$$\exists a \forall b \exists c d \forall e \exists f g . (\overline{a} \lor \overline{g}) \land (b \lor f \lor g) \land (c \lor \overline{e} \lor \overline{f}) \land (d \lor \overline{e} \lor \overline{f}) \land (\overline{c} \lor \overline{d} \lor e) \land (a \lor f)$$

Q-Resolution Proof DAG



$\overline{a} \lor \overline{g}$ $\quad$ $b \lor f \lor g$ $\quad$ $c \lor \overline{e} \lor \overline{f}$ $\quad$ $d \lor \overline{e} \lor \overline{f}$ $\quad$ $\overline{c} \lor \overline{d} \lor e$ $\quad$ $a \lor f$

$\overline{c} \lor \overline{e} \lor \overline{f}$

$\overline{e} \lor \overline{f}$

$b \lor \overline{e} \lor g$ $\quad$ $a \lor \overline{e}$

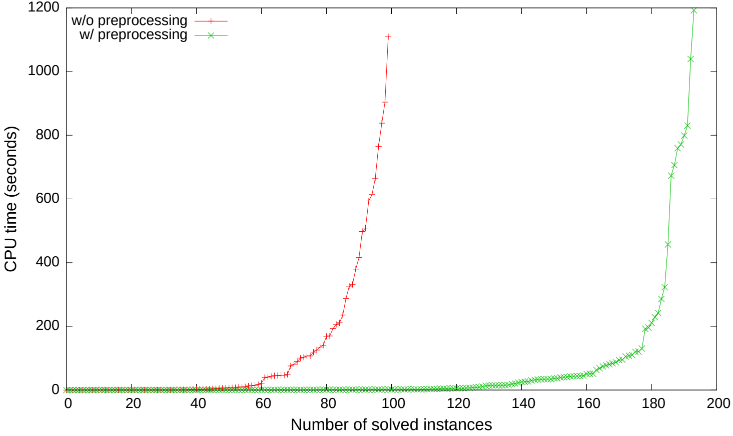
$\overline{a} \lor b \lor \overline{e}$

$\bot$

QBF Preprocessing

Preprocessing is crucial to solve most QBF instances efficiently.

Results of DepQBF w/ and w/o bloqqer on QBF Eval 2012 [1]



Quantified Blocked Clause

Definition (Quantified Blocking literal)

An existential literal l in a clause C of a QBF $\Pi.\varphi$ blocks C with respect to $\Pi.\varphi$ if for every clause $D \in F_{\bar{l}}$, there exists a literal $k \neq l$ with $k \leq_{\Pi} l$ such that $k \in C$ and $\bar{k} \in D$.

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Reasoning with Quantified Boolean Formulas

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http://www.cs.cmu.edu/~mheule/15816-f23/

Automated Reasoning and Satisfiability
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