Proof Systems and Proof Complexity

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`https://cmu.zoom.us/j/93095736668`

Automated Reasoning and Satisfiability
September 28, 2020

Certificates

What makes a problem hard?

Certificate angle: can one efficiently check an alleged solution?

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Is searching for the solution harder than verifying a given solution?

	4		3					
						7	9	
			6					
			1	4		5		
9							1	
2								6
				7	2			
	5					8		
				9				

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Intuition: yes!

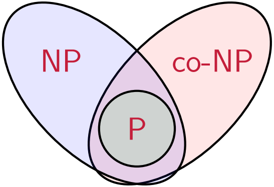
However, many problems for which we can efficiently check a solution turn out to be easy in practice.

1	4	*7*	3	*8*	*9*	*2*	*6*	*5*
5	*8*	*6*	*2*	*1*	*4*	7	9	*3*
3	*9*	*2*	6	*5*	*7*	*1*	*8*	*4*
8	*7*	*3*	1	4	*6*	5	*2*	*9*
9	*6*	*4*	*7*	*2*	*5*	*3*	1	*8*
2	*1*	*5*	*9*	*3*	*8*	*4*	*7*	6
6	*3*	*8*	*5*	7	2	*9*	*4*	*1*
7	5	*9*	*4*	*6*	*1*	8	*3*	*2*
4	*2*	*1*	*8*	9	*3*	*6*	*5*	*7*

Certificates and Complexity

Complexity classes of decision problems:

P : efficiently computable answers.
NP : efficiently checkable yes-answers.
co-NP : efficiently checkable no-answers.



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Solving the P $\stackrel{?}{=}$ NP question is worth \$1,000,000 [Clay MI '00].

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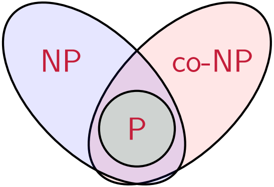

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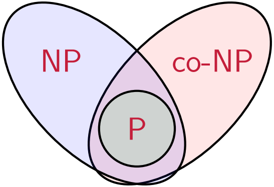

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What about co-NP?

How to find short proofs for interesting problems efficiently?

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Challenges

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Certifying Satisfiability and Unsatisfiability

- Certifying satisfiability of a formula is easy:

$$(x \lor y) \land (\overline{x} \lor \overline{y}) \land (\overline{y} \lor \overline{z})$$

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- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.
  - ➥ Checking whether every assignment falsifies the formula is costly.
  - More compact certificates of unsatisfiability are desirable.
    - ➥ Proofs

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    ... but can be of exponential size with respect to a formula.

- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \vee x \qquad \overline{x} \vee D}{C \vee D}$$

  - C_m is the empty clause (containing no literals), denoted by $\perp$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

- Example: $F = (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{z}) \wedge (x \vee \overline{y}) \wedge (\overline{u} \vee y) \wedge (u)$
- Resolution proof:
  $(\overline{x} \vee \overline{y} \vee z), (\overline{z}), (\overline{x} \vee \overline{y}), (x \vee \overline{y}), (\overline{y}), (\overline{u} \vee y), (\overline{u}), (u), \bot$

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$$\dfrac{\dfrac{\overline{u} \lor y \qquad \dfrac{\dfrac{\overline{x} \lor \overline{y} \lor z \qquad \overline{z}}{\overline{x} \lor \overline{y}} \qquad x \lor \overline{y}}{\overline{y}}}{\overline{u}} \qquad u}{\bot}$$

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$$

- Drawbacks of resolution:
  - For many seemingly simple formulas, there are only resolution proofs of exponential size.
  - State-of-the-art solving techniques are not succinctly expressible.

Clausal Proofs

Reduce the size of the proof by only storing added clauses

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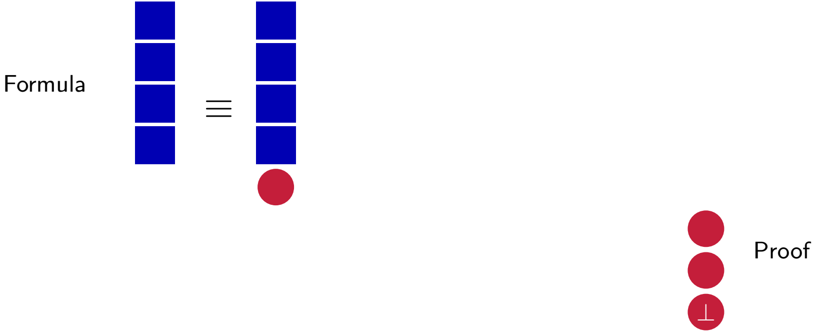

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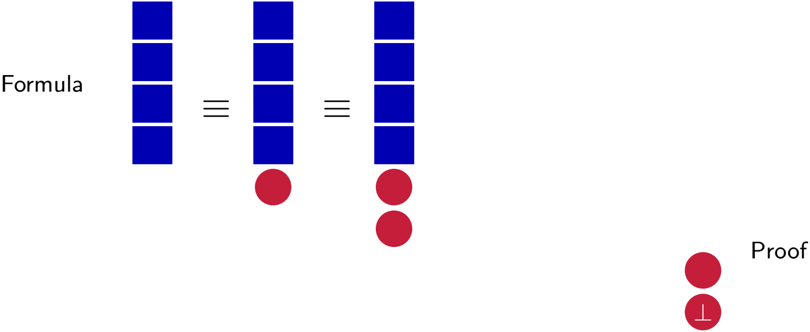

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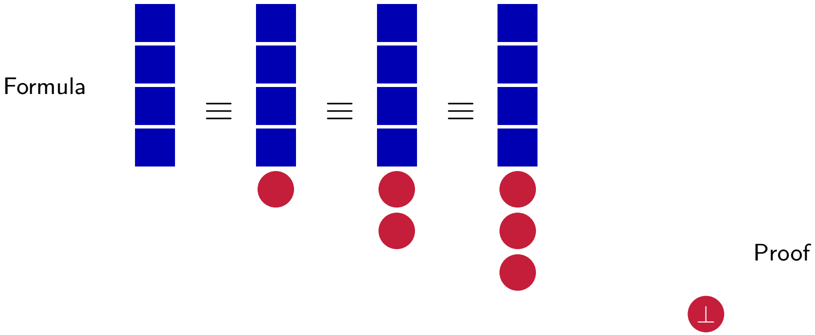

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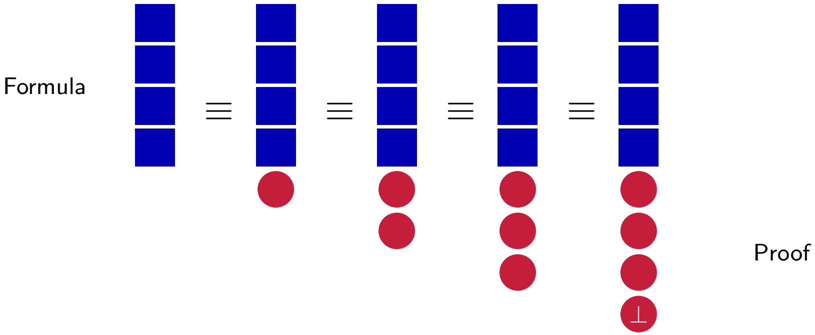

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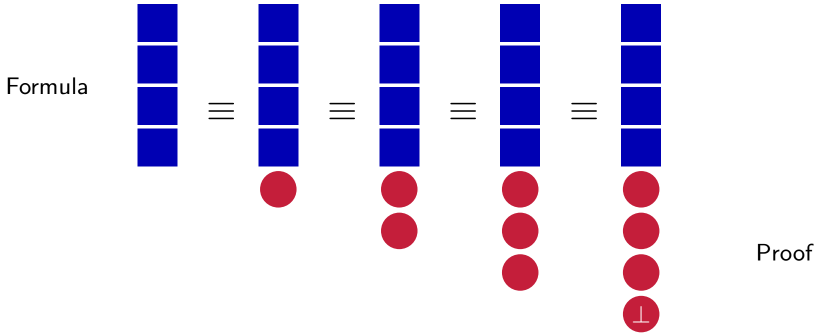

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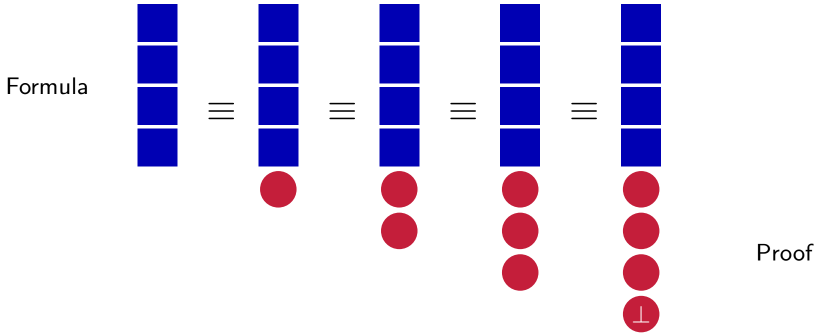

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➥ Idea: Only add clauses that fulfill an efficiently checkable redundancy criterion.

Reverse Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let F be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by F via UP (denoted by $F \vdash_1 C$) if UP on $F|_\alpha$ results in a conflict.

Example

$$F = (a \lor b \lor \overline{c}) \land (\overline{a} \lor \overline{b} \lor c) \land (b \lor c \lor \overline{d}) \land (\overline{b} \lor \overline{c} \lor d) \land (a \lor c \lor d) \land (\overline{a} \lor \overline{c} \lor \overline{d}) \land (\overline{a} \lor b \lor d) \land (a \lor \overline{b} \lor \overline{d})$$

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$$\begin{aligned} F = \; & (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee \bar{b} \vee c) \wedge (b \vee c \vee \bar{d}) \wedge (\bar{b} \vee \bar{c} \vee d) \; \wedge \\ & (a \vee c \vee d) \wedge (\bar{a} \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee b \vee d) \wedge (a \vee \bar{b} \vee \bar{d}) \end{aligned}$$

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$$\dfrac{\dfrac{(a \lor c \lor d) \qquad (b \lor c \lor \overline{d})}{(a \lor b \lor c)} \qquad (a \lor b \lor \overline{c})}{(a \lor b)}$$

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Challenges

Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \vee x \qquad \overline{x} \vee D}{C \vee D}\ \text{(RES)} \qquad\qquad \frac{A \qquad A \rightarrow B}{B}\ \text{(MP)}$$

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➥ Inference rules reason about the presence of facts.
- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ➥ This leads to interference-based proof systems.

Early work on reasoning beyond resolution

The early SAT decision procedures used the Pure Literal rule [Davis and Putnam 1960; Davis, Logemann and Loveland 1962]:

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Extended Resolution (ER) [Tseitin 1966]

- Combines resolution with the Extension rule:

$$\frac{x \notin F \qquad \overline{x} \notin F}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)} \text{ (ER)}$$

- Equivalently, adds the definition $x := \mathrm{AND}(a, b)$
- Can be considered the first interference-based proof system
- Is very powerful: No known lower bounds

Short Proofs of Pigeon Hole Formulas [Cook 1967]

Can $n{+}1$ pigeons be in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \leq p \leq n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \leq h \leq n,\ 1 \leq p < q \leq n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

Resolution proofs of PHP_n are exponential [Haken 1985]

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However, these proofs require introducing new variables:

- Hard to find such proofs automatically
- Existing ER approaches produce exponentially large proofs
- How to get rid of this hurdle? First approach: blocked clauses...

Blocked Clauses [Kullmann 1999]

Definition (Block Clause)

A clause $(C \vee x)$ is a blocked on x w.r.t. a CNF formula F if for every clause $(D \vee \overline{x}) \in F$, resolvent $C \vee D$ is a tautology.

Definition (Blocked clause)

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{c}$.
Third clause is blocked by c

Theorem

Adding or removing a blocked clause preserves satisfiability.

Blocked Clause Addition and Blocked Clause Elimination

The Blocked Clause proof system (BC) combines the resolution rule with the addition of blocked clauses.

- BC generalizes ER [Kullmann 1999]
- Recall
$$\frac{x \notin F \qquad \overline{x} \notin F}{(x \lor \overline{a} \lor \overline{b}) \land (\overline{x} \lor a) \land (\overline{x} \lor b)} \text{ (ER)}$$
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Blocked clause elimination used in preprocessing and inprocessing

- Simulates many circuit optimization techniques
- Removes redundant Pythagorean Triples

DRAT: An Interference-Based Proof System

- DRAT is a popular interference-based proof system
- DRAT allows adding RATs (defined below) to a formula.
  - It can be efficiently checked if a clause is a RAT.
  - RATs are not necessarily implied by the formula.
  - But RATs are redundant: their addition preserves satisfiability.
- DRAT also allows clause deletion
  - Initially introduced to check proofs more efficiently
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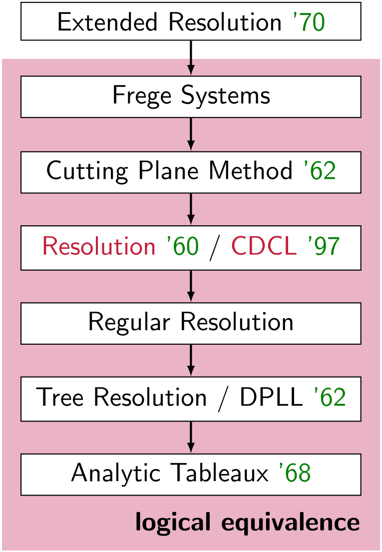
Definition (Resolution Asymmetric Tautology)

A clause $(C \lor x)$ is a resolution asymmetric tautology (RAT) on x w.r.t. a CNF formula F if for every clause $(D \lor \overline{x}) \in F$, $C \lor D$ is implied by F via unit-propagation, i.e., $F \vdash_1 C \lor D$.

Proof Search in Strong Proof Systems

Existence of Short Proofs



Proof Search in Strong Proof Systems

Existence of Short Proofs

- Extended Resolution '70
- Frege Systems
- Cutting Plane Method '62
- Resolution '60 / CDCL '97
- Regular Resolution
- Tree Resolution / DPLL '62
- Analytic Tableaux '68

logical equivalence

Finding Short Proofs

satisfiability equivalence

- Propagation Redundancy '17
- Set Propagation Red. / SDCL '17
- Resolution Asymmetric Taut. '12
- Blocked Clauses '99
- Extended Resolution '70

Express solving techniques compactly [Järvisalo, Heule, and Biere '12]

Short proofs without new variables [Heule, Kiesl, and Biere '17A]

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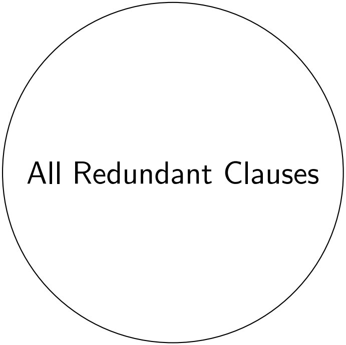
Redundant Clauses

- Strong proof systems allow addition of many redundant clauses.



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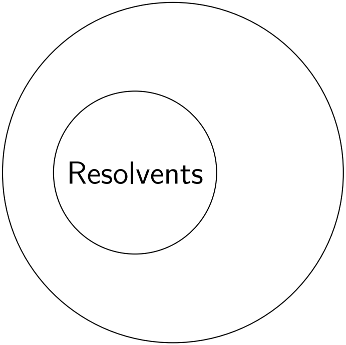

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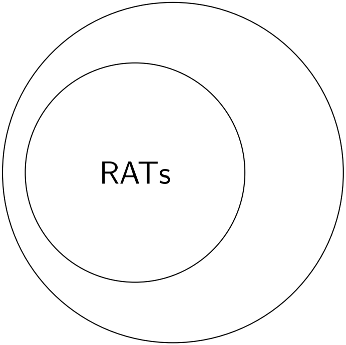

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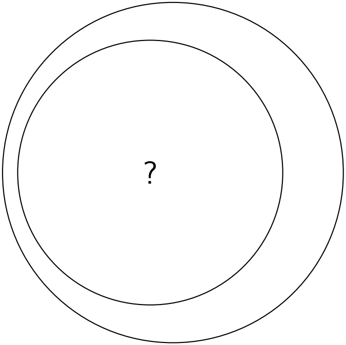
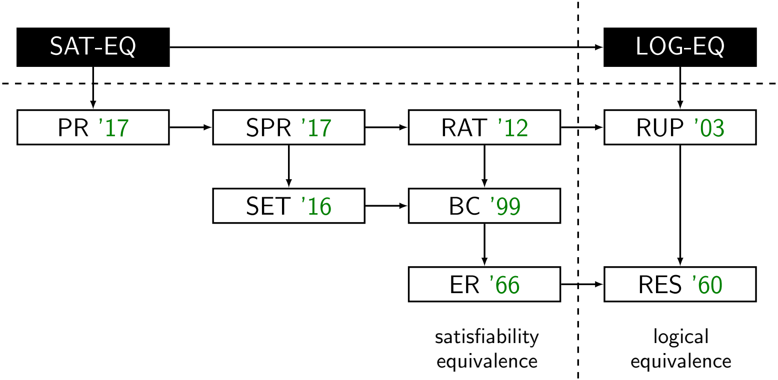


- Are stronger redundancy notions still efficiently checkable?

New Propositional Proof Systems

- We introduced new clause-redundancy notions:
  - Propagation-redundant (PR) clauses
  - Set-propagation-redundant (SPR) clauses
  - Literal-propagation-redundant (LPR) clauses
- LPR clauses coincide with RAT.
- SPR clauses strictly generalize RATs.
- PR clauses strictly generalize SPR clauses.
- The redundancy notions provide the basis for new proof systems.

New Proof Systems for Propositional Logic



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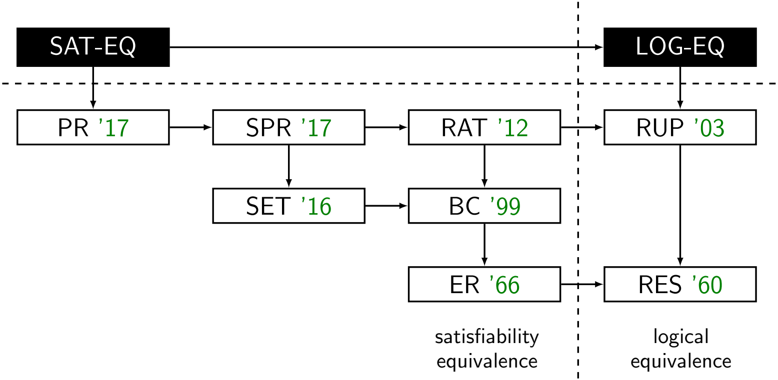

RAT simulates PR [Heule and Biere 2018]

ER simulates RAT [Kiesl, Rebola-Pardo, Heule 2018]

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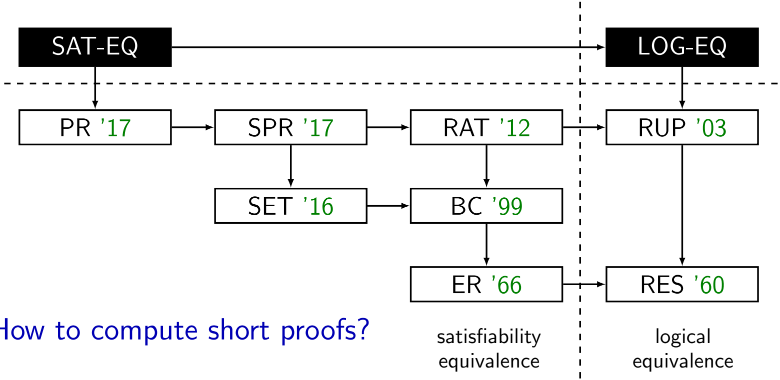


How to compute short proofs?

RAT simulates PR [Heule and Biere 2018]

ER simulates RAT [Kiesl, Rebola-Pardo, Heule 2018]

Stronger Proof Systems: What Are They Good For?

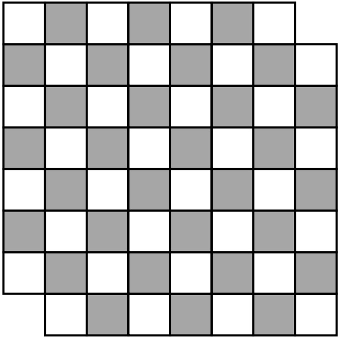
- The new proof systems can give short proofs of formulas that are considered hard.

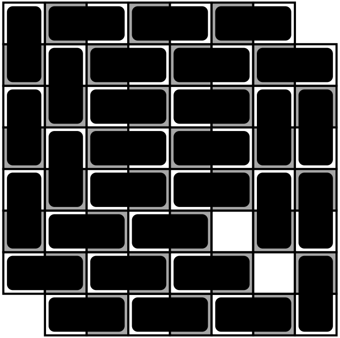
- We have short SPR and PR proofs for the well-known pigeon hole formulas (linear in the size of the input).
  - Pigeon hole formulas have only exponential-size resolution proofs.
  - If the addition of new variables via definitions is allowed, there are polynomial-size proofs.

- Strong proof systems do not require new variables.
  - ➥ Search space of possible clauses is finite.
  - ➥ Makes search for such clauses easier.

Mutilated Chessboards: “A Tough Nut to Crack” [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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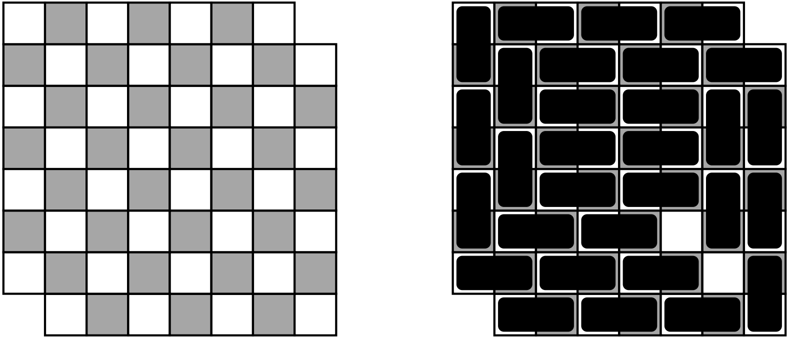
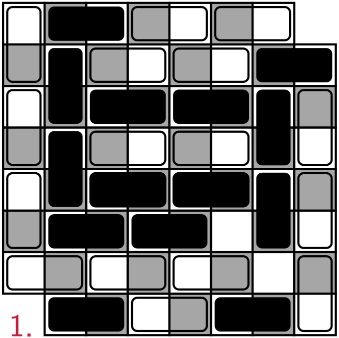


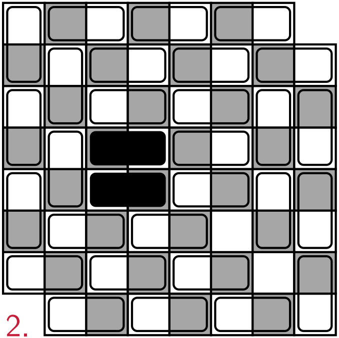
Easy to refute based on the following two observations:

- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Without Loss of Satisfaction

One of the crucial techniques in SAT solvers is to generalize a conflicting state and use it to constrain the problem.


1.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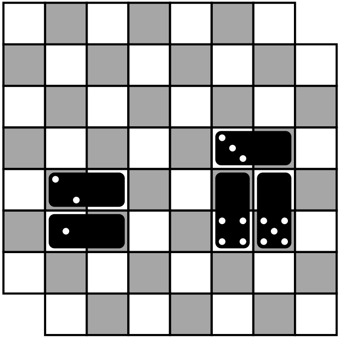
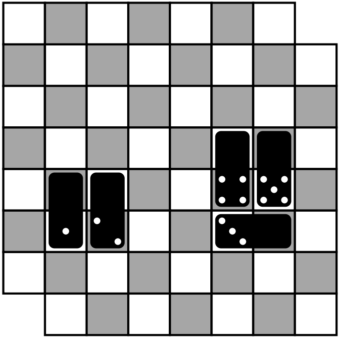
The used proof system can have a big impact on the size:

1. Resolution can only reduce the 30 dominos to 14 (left); and
2. "Without loss of satisfaction" can reduce them to 2 (right).

Mutilated Chessboards: An alternative proof

Satisfaction-Driven Clause Learning (SDCL) is a new solving paradigm that finds proofs in the PR proof system [HKB '17]





SDCL can detect that the above two patterns can be blocked

- This reduces the number of explored states exponentially
- We produced SPR proofs that are linear in the formula size

Redundancy as an Implication

A formula G is at least as satisfiable as a formula F if $F \models G$.

Given a formula F and assignment α, we denote with $F|_\alpha$ the reduced formula after removing from F all clauses satisfied by α and all literals falsified by α.

Theorem

Let F be a formula, C a clause, and α the smallest assignment that falsifies C. Then, C is redundant w.r.t. F iff there exists an assignment ω such that 1) ω satisfies C; and 2) $F|_\alpha \models F|_\omega$.

This is the strongest notion of redundancy. However, entailment ($\models$) cannot be checked in polynomial time (assuming $P \neq NP$), unless bounded.

Checking Redundancy Using Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let F be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by F via UP (denoted by $F \vdash_1 C$) if UP on $F|_\alpha$ results in a conflict.
- Implied by UP is used in SAT solvers to determine redundancy of learned clauses and therefore $\vdash_1$ is a natural restriction of $\models$.
- We bound $F|_\alpha \models F|_\omega$ by $F|_\alpha \vdash_1 F|_\omega$.
- Example:
  $F = (x \lor y \lor z) \land (\overline{x} \lor y \lor z) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor \overline{y} \lor z)$
  and $G = (z)$. Observe that $F \models G$, but that $F \nvdash_1 G$.

Hand-crafted PR Proofs of Pigeon Hole Formulas

We manually constructed PR proofs of the famous pigeon hole formulas and the two-pigeons-per-hole family.

- The proofs consist only of binary and unit clauses.
- Only original variables appear in the proof.
- All proofs are linear in the size of the formula.

➥ The PR proofs are smaller than Cook's ER proofs.

- All resolution proofs of these formulas are exponential in size.

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Challenges

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is SET or PR w.r.t. a formula F is an NP-complete problem.

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Key Idea: While solving a formula F, check whether the positive reduct of F and the current assignment α is satisfiable. In that case, prune the branch α.

The Positive Reduct: An Example

Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. The positive reduct of F and α, denoted by $p(F, \alpha)$, is the formula that contains C and all $assigned(D, \alpha)$ with $D \in F$ and D is satisfied by α.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

Let $C_1 = (\overline{x})$, so $\alpha_1 = x$.
The positive reduct $p(F, \alpha_1) = (\overline{x}) \wedge (x) \wedge (x)$ is unsatisfiable.

Let $C_2 = (\overline{x} \vee \overline{y})$, so $\alpha_2 = x\,y$.
The positive reduct $p(F, \alpha_2) = (\overline{x} \vee \overline{y}) \wedge (x \vee y) \wedge (x \vee \overline{y})$ is satisfiable.

Autarkies

A non-empty assignment α is an autarky for formula F if every clause $C \in F$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $F := (x \lor y) \land (x \lor \overline{y}) \land (\overline{y} \lor \overline{z})$.
Assignment $\alpha_1 = \overline{z}$ is an autarky:
$(x \lor y) \land (x \lor \overline{y}) \land (\overline{y} \lor \overline{z})$. Assignment $\alpha_2 = x\,\overline{y}\,z$ is an autarky: $(x \lor y) \land (x \lor \overline{y}) \land (\overline{y} \lor \overline{z})$.

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Given an assignment α, $F|_\alpha$ denotes a formula F without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula F.
Then, F and $F|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula F if α_{aut} is an autarky for $F|_{\alpha_{\text{con}}}$.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \overline{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\overline{y}$ is a conditional autarky for F:

$$\alpha_{\text{aut}} = \overline{y} \text{ is an autarky for } F|_{\alpha_{\text{con}}} = (\overline{y} \vee \overline{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula F.
Then F and $F \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\overline{x} \vee \overline{y})$.

Learning PR clauses

Theorem

Given a formula F and an assignment α. Every satisfying assignment ω of $p(F, \alpha)$ is a conditional autarky of F.

Recall: Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. C is SET w.r.t. F if and only if $p(F, \alpha)$ is satisfiable.

Let assignment ω satisfy $p(F, \alpha)$. Removing all but one of the literals in C that are satisfied by ω results in a PR clause w.r.t. F.

Pseudo-Code of CDCL (formula F)

$\alpha := \emptyset$
forever do
    $\alpha :=$ Simplify (F, α)
    if $F|_\alpha$ contains a falsified clause **then**
        $C :=$ AnalyzeConflict $()$
        if C is the empty clause **then return** unsatisfiable
        $F := F \cup \{C\}$
        $\alpha :=$ BackJump (C, α)
    else
        $l :=$ Decide $()$
        if l is undefined **then return** satisfiable
        $\alpha := \alpha \cup \{l\}$

Pseudo-Code of SDCL (formula F)

```
α := ∅
forever do
   α := Simplify (F, α)
   if F|α contains a falsified clause then
      C := AnalyzeConflict ()
      if C is the empty clause then return unsatisfiable
      F := F ∪ {C}
      α := BackJump (C, α)
   else if p(F, α) is satisfiable then
      C := AnalyzeWitness ()
      F := F ∪ {C}
      α := BackJump (C, α)
   else
      l := Decide ()
      if l is undefined then return satisfiable
      α := α ∪ {l}
```

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Theoretical Challenges

Lower bounds for interference-based proof systems with new variables will be hard, but what about without new variables?

- Lower bound for BC w/o new variables? Pigeon-hole formulas?
- Lower bound for SET w/o new variables? Tseitin formulas?
- Lower bound for PR w/o new variables?!

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Can we design stronger proof systems that make it even easier to compute short proofs?

Practical Challenges

The current version of SDCL is just the beginning:

- Which heuristics allow learning short PR clauses?
- How to construct an AnalyzeWitness procedure?
- Can the positive reduct be improved?

Can local search be used to find short proofs of unsatisfiability?

Constructing positive reducts (or similar formulas) efficiently:

- Generating a positive reduct is more costly than solving them
- Can we design data-structures to cheaply compute them?