Applications for Automated Reasoning

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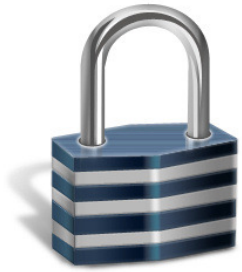
https://cmu.zoom.us/j/93095736668

Automated Reasoning and Satisfiability
September 9, 2020

Automated Reasoning Has Many Applications

formal verification

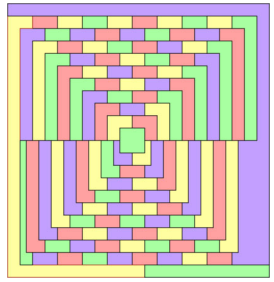


security

bioinformatics

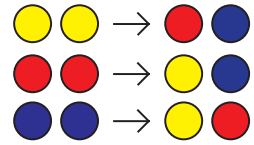
planning and scheduling

train safety



automated theorem proving

exploit generation



term rewriting termination

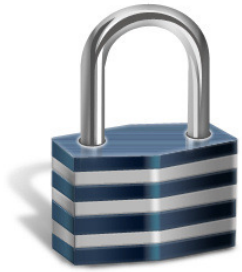
encode

automated reasoning

decode

Automated Reasoning Has Many Applications

formal verification



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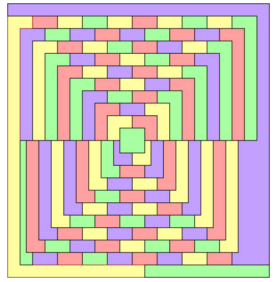

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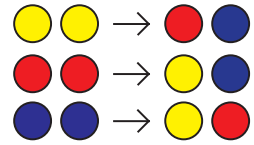

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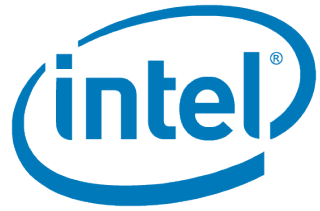
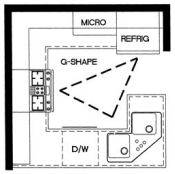
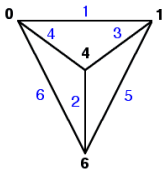


Encoding problems into SAT

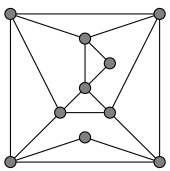


Architectural 3D Layout
[VSMM '07]
Henriette Bier

Edge-matching Puzzles
[LaSh '08]

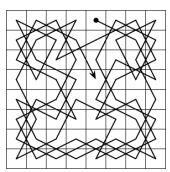


Graceful Graphs
[AAAI '10]
Toby Walsh

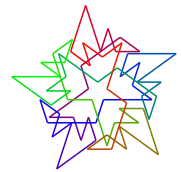


Clique-Width
[SAT '13, TOCL '15]
Stefan Szeider

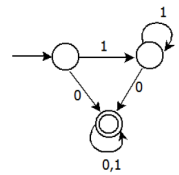
Firewall Verification
[SSS '16]
Mohamed Gouda



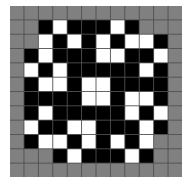
Open Knight Tours
Moshe Vardi



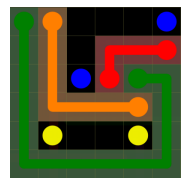
Van der Waerden numbers
[EJoC '07]



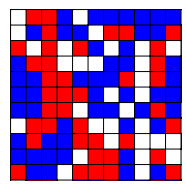
Software Model Synthesis
[ICGI '10, ESE '13]
Sicco Verwer



Conway's Game of Life
[EJoC '13]
Willem van der Poel



Connect the Pairs
Donald Knuth



Pythagorean Triples
[SAT '16, CACM '17]
Victor Marek

Collatz conjecture [Open]
Scott Aaronson

Equivalence Checking

Bounded Model Checking

Graphs and Symmetry Breaking

Arithmetic Operations

Equivalence Checking

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Equivalence checking introduction

Given two formulae, are they equivalent?

Applications:

- Hardware and software optimization
- Software to FPGA conversion

Equivalence checking example

original C code

```
if(!a && !b) h();
else if(!a) g();
else f();
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if(a) f();
else {
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⇒

optimized C code

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if(a) f();
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if(a) f();
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```

Are these two code fragments equivalent?

Equivalence checking encoding (1)

1. represent procedures as Boolean variables

original C code :=

```
if ā ∧ b̄ then h
else if ā then g
else f
```

optimized C code :=

```
if a then f
else if b then g
else h
```

Equivalence checking encoding (1)

1. represent procedures as Boolean variables

original C code :=	**optimized C code** :=
`if` $\overline{\mathtt{a}} \wedge \overline{\mathtt{b}}$ `then h`	`if a then f`
`else if` $\overline{\mathtt{a}}$ `then g`	`else if b then g`
`else f`	`else h`

2. compile code into Conjunctive Normal Form

   compile (`if x then y else z`) $\equiv (\overline{x} \vee y) \wedge (x \vee z)$

Equivalence checking encoding (1)

1. represent procedures as Boolean variables

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$\texttt{if } \overline{\texttt{a}} \wedge \overline{\texttt{b}} \texttt{ then h}$
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optimized C code :=

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2. compile code into Conjunctive Normal Form

compile $(\texttt{if x then y else } z) \equiv (\overline{\texttt{x}} \vee \texttt{y}) \wedge (\texttt{x} \vee z)$

3. check equivalence of Boolean formulae

compile (**original C code**) $\Leftrightarrow$ *compile* (**optimized C code**)

Equivalence checking encoding (2)

compile (**original C code**):

$$\texttt{if } \overline{a} \wedge \overline{b} \texttt{ then } h \texttt{ else if } \overline{a} \texttt{ then } g \texttt{ else } f \quad \equiv$$
$$\overline{((\overline{a} \wedge \overline{b}) \vee h)} \wedge ((\overline{a} \wedge \overline{b}) \vee (\texttt{if } \overline{a} \texttt{ then } g \texttt{ else } f)) \quad \equiv$$
$$(a \vee b \vee h) \wedge ((\overline{a} \wedge \overline{b}) \vee ((a \vee g) \wedge (\overline{a} \vee f)))$$

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\end{aligned}$$

compile (**optimized C code**):

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$$(a \vee b \vee h) \wedge ((\overline{a} \wedge \overline{b}) \vee ((a \vee g) \wedge (\overline{a} \vee f)))$$

$$\Updownarrow$$

$$(\overline{a} \vee f) \wedge (a \vee ((\overline{b} \vee g) \wedge (b \vee h)))$$

Checking (in)equivalence

Reformulate it as a satisfiability (SAT) problem:
Is there an assignment to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{f}$, $\mathfrak{g}$, *and* $\mathfrak{h}$, *which results in different evaluations of the compiled codes?*

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Is the Boolean formula

$$\mathfrak{x} \leftrightarrow ((\mathfrak{a} \vee \mathfrak{b} \vee \mathfrak{h}) \wedge ((\overline{\mathfrak{a}} \wedge \overline{\mathfrak{b}}) \vee ((\mathfrak{a} \vee \mathfrak{g}) \wedge (\overline{\mathfrak{a}} \vee \mathfrak{f})))) \wedge$$
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satisfiable?

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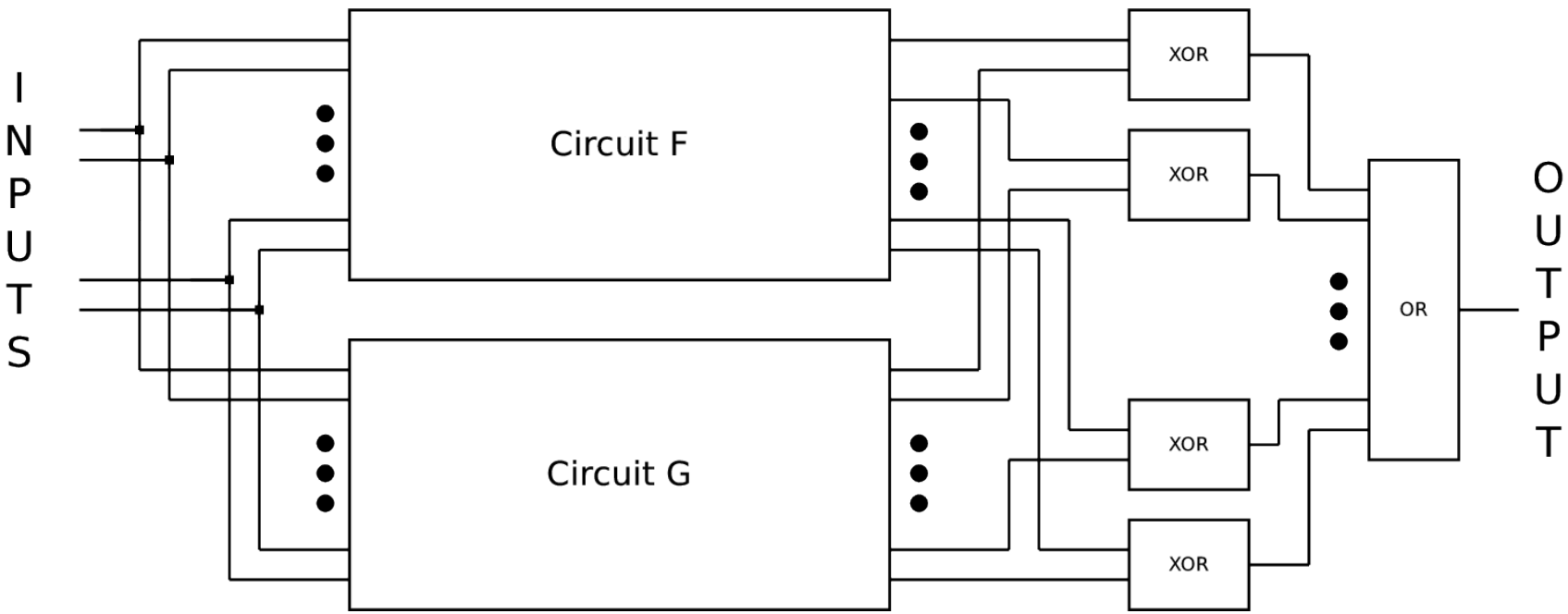
Note: by concentrating on counterexamples we moved from co-NP to NP (not really important for applications)

Equivalence Checking via Miters

Equivalence checking is mostly used to validate whether two hardware designs (circuits) are functionally equivalent.

Given two circuits, a miter is circuit that tests whether there exists an input for both circuits such that the output differs.



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Bounded Model Checking (BMC)

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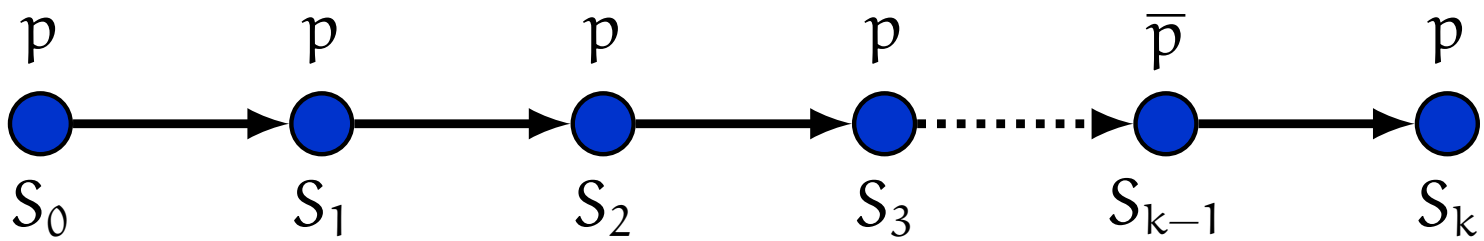
Is there a state reachable in k steps, which satisfies $\overline{p}$?



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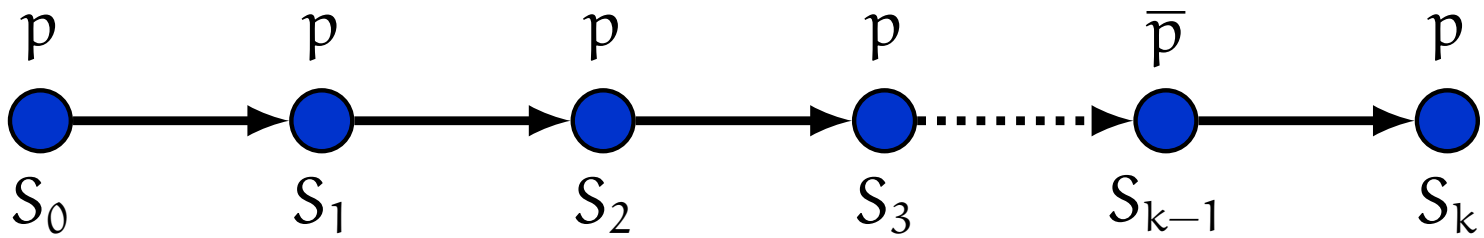


Turing award 2007 for Model Checking
Edmund M. Clarke, E. Allen Emerson and Joseph Sifakis

BMC Encoding (1)

Three components:

- I The description of the initial state
- T The transition of a state into the next state
- P The (safety) property

The reachable states in k steps are captured by:

$$\mathrm{I}(S_0) \wedge \mathrm{T}(S_0, S_1) \wedge \cdots \wedge \mathrm{T}(S_{\mathrm{k}-1}, S_{\mathrm{k}})$$

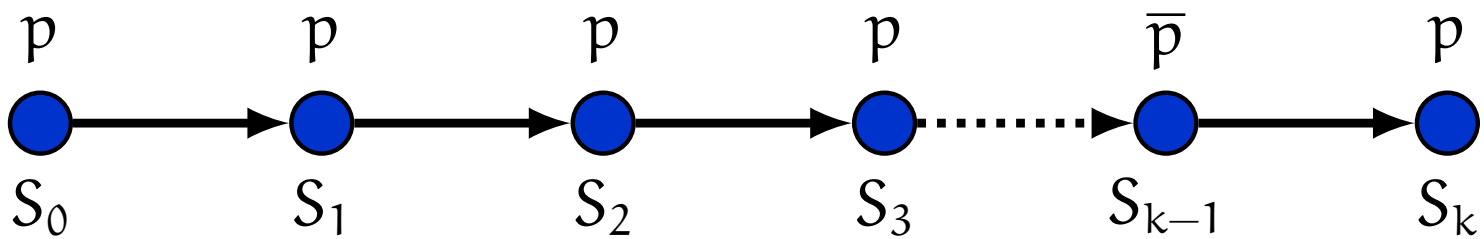
The property p fails in one of the k steps by:

$$\overline{\mathrm{P}}(S_0) \vee \overline{\mathrm{P}}(S_1) \vee \cdots \vee \overline{\mathrm{P}}(S_{\mathrm{k}})$$

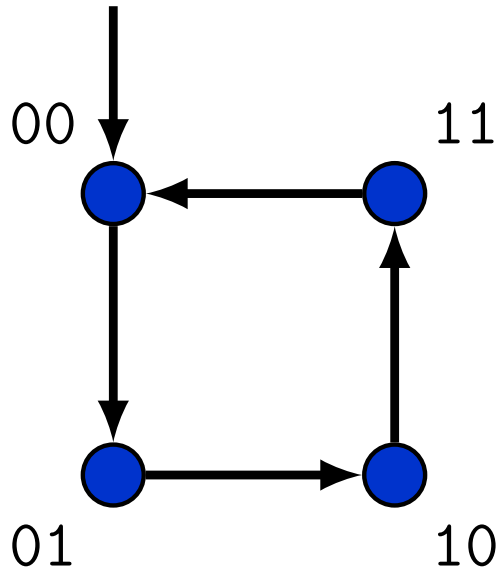
BMC Encoding (2)

The safety property p is valid up to step k
if and only if $\mathcal{F}(k)$ is unsatisfiable:

$$\mathcal{F}(k) = I(S_0) \wedge \bigwedge_{i=0}^{k-1} T(S_i, S_{i+1})) \wedge \bigvee_{i=0}^{k} \overline{P}(S_i)$$



Bounded Model Checking Example: Two-bit counter



Initial state I: $l_0 = 0, r_0 = 0$

Transition T: $l_{i+1} = l_i \oplus r_i,$
$r_{i+1} = \bar{r}_i$

Property P: $\bar{l}_i \vee \bar{r}_i$

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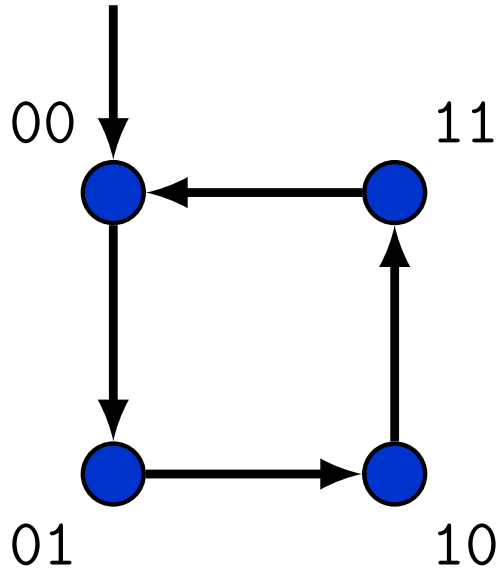

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$$\mathcal{F}(2) = (\bar{l}_0 \wedge \bar{r}_0) \wedge \left(\begin{array}{c} l_1 = l_0 \oplus r_0 \wedge r_1 = \bar{r}_0 \wedge \\ l_2 = l_1 \oplus r_1 \wedge r_2 = \bar{r}_1 \end{array} \right) \wedge \left(\begin{array}{c} (l_0 \wedge r_0) \vee \\ (l_1 \wedge r_1) \vee \\ (l_2 \wedge r_2) \end{array} \right)$$

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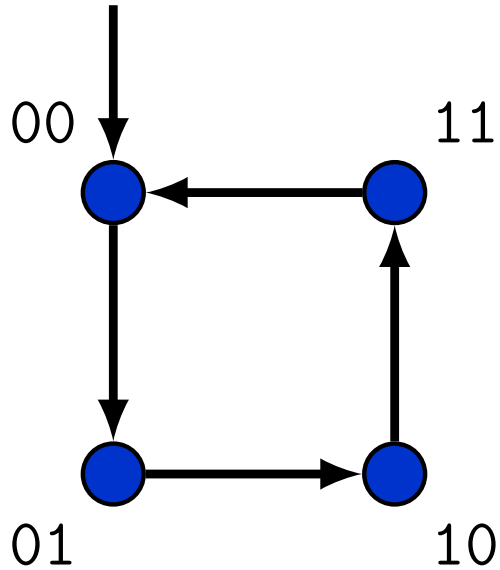

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For $k = 2$, $\mathcal{F}(k)$ is unsatisfiable; for $k = 3$ it is satisfiable

Equivalence Checking

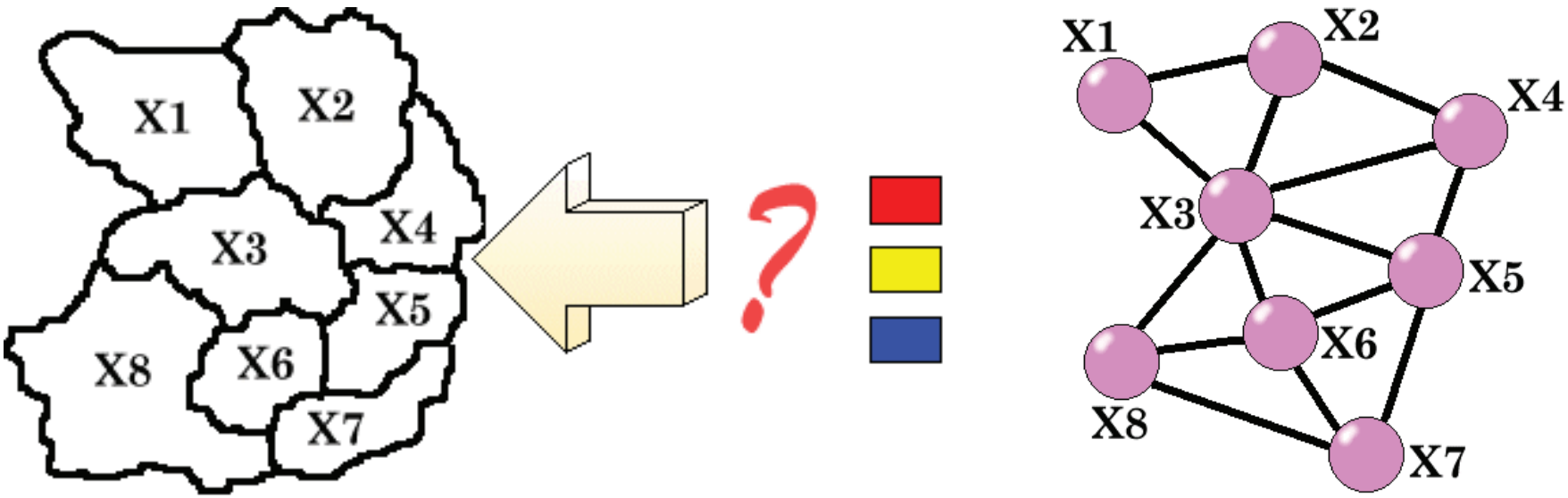
Bounded Model Checking

Graphs and Symmetry Breaking

Arithmetic Operations

Graph coloring

Given a graph $G(V, E)$, can the vertices be colored with k colors such that for each edge $(v, w) \in E$, the vertices v and w are colored differently.



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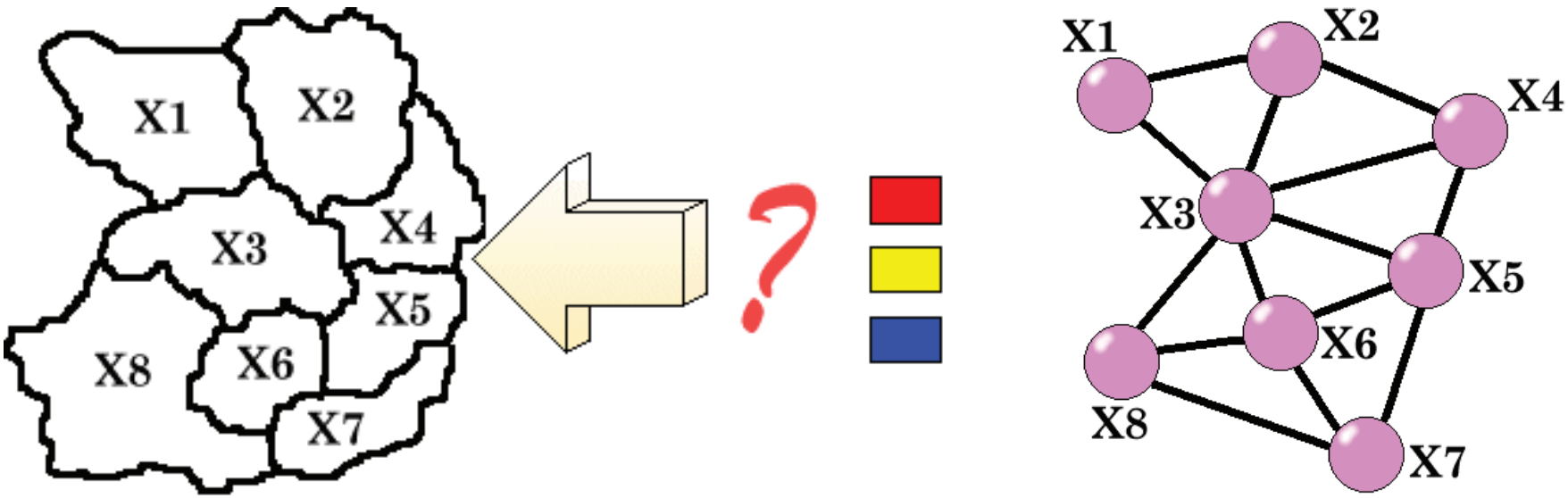


Problem: Many symmetries!!!

Graph coloring encoding

Variables	Range	Meaning
$x_{v,i}$	$i \in \{1,\ldots,c\}$ $v \in \{1,\ldots,\lvert V\rvert\}$	node v has color i

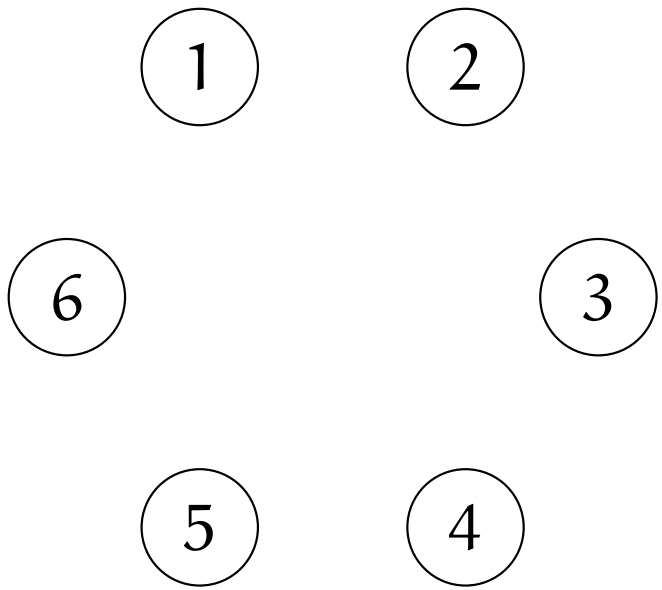
Clauses	Range	Meaning
$(x_{v,1} \vee x_{v,2} \vee \cdots \vee x_{v,c})$	$v \in \{1,\ldots,\lvert V\rvert\}$	v is colored
$(\overline{x}_{v,s} \vee \overline{x}_{v,t})$	$s \in \{1,\ldots,c-1\}$ $t \in \{s+1,\ldots,c\}$	v has at most one color
$(\overline{x}_{v,i} \vee \overline{x}_{w,i})$	$(v,w) \in E$	v and w have a different color
???	???	breaking symmetry

Unavoidable Subgraphs and Ramsey Numbers

A connected undirected graph G is an unavoidable subgraph of clique K of order n if any red/blue edge-coloring of the edges of K contains G either in red or in blue.

Ramsey Number $R(k)$: What is the smallest n such that any graph with n vertices has either a clique or a co-clique of size k?

$$\begin{aligned} & R(3) = 6 \\ & R(4) = 18 \\ 43 \leq \quad & R(5) \leq 49 \end{aligned}$$



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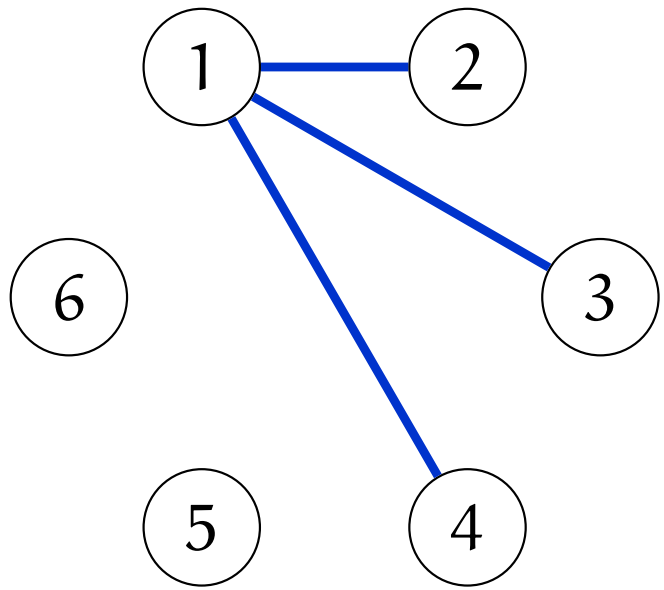

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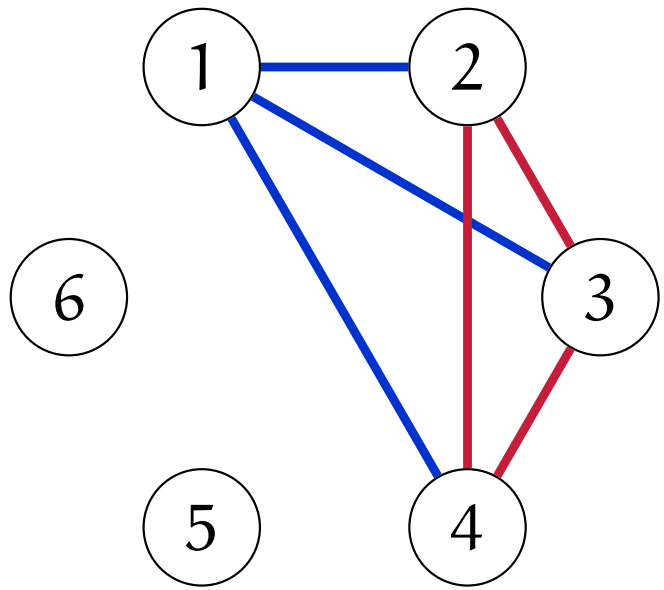

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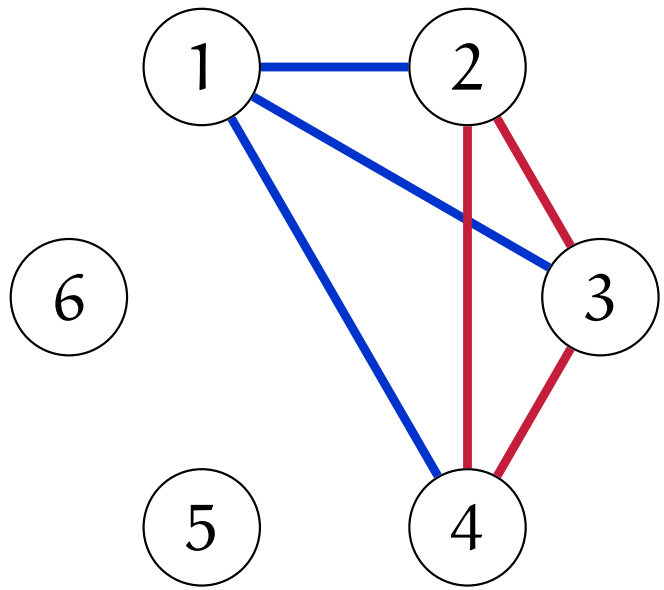

SAT solvers can determine that $R(4) = 18$ in 1 second using symmetry breaking; w/o symmetry breaking it requires weeks.

Example formula: an unavoidable path of two edges

Consider the formula below — which expresses the statement whether path of two edges unavoidable in a clique of order 3:

$$F := \overbrace{(x \vee y)}^{C_1} \wedge \overbrace{(x \vee z)}^{C_2} \wedge \overbrace{(y \vee z)}^{C_3} \wedge \overbrace{(\overline{x} \vee \overline{y})}^{C_4} \wedge \overbrace{(\overline{x} \vee \overline{z})}^{C_5} \wedge \overbrace{(\overline{y} \vee \overline{z})}^{C_6}$$

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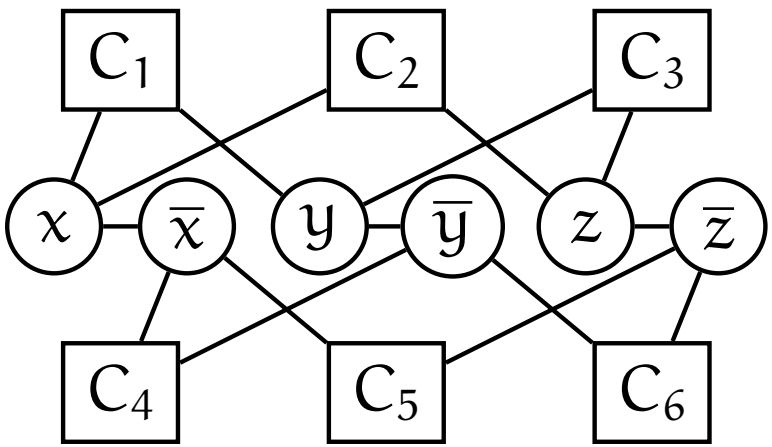
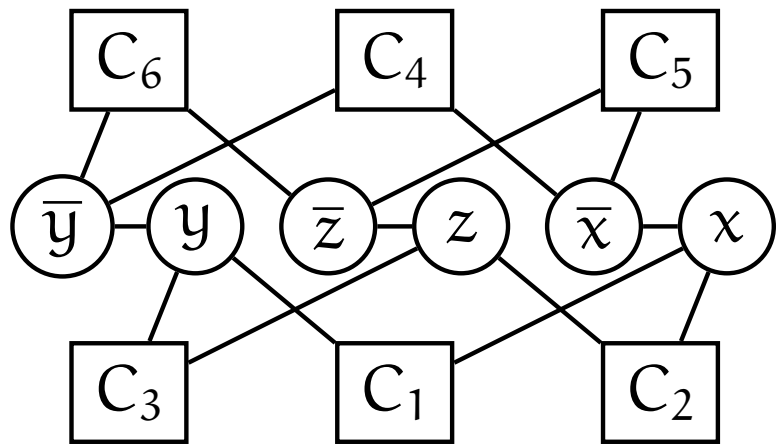
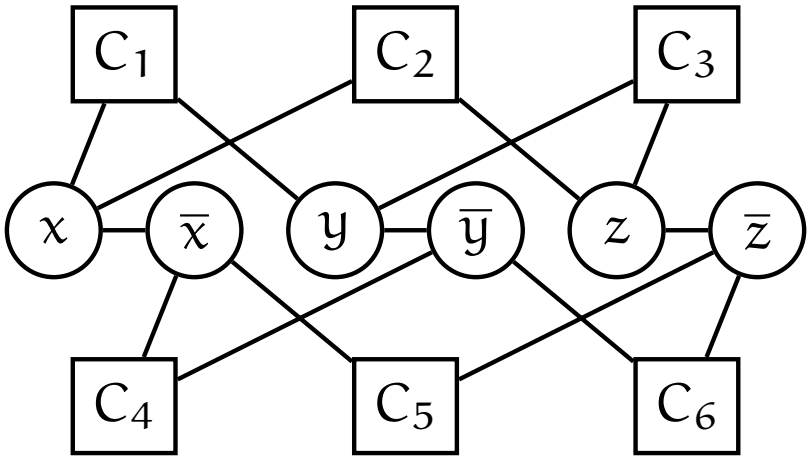
A clause-literal graph has a vertex for each clause and literal, and edges for each literal occurrence connecting the literal and clause vertex. Also, two complementary literals are connected.



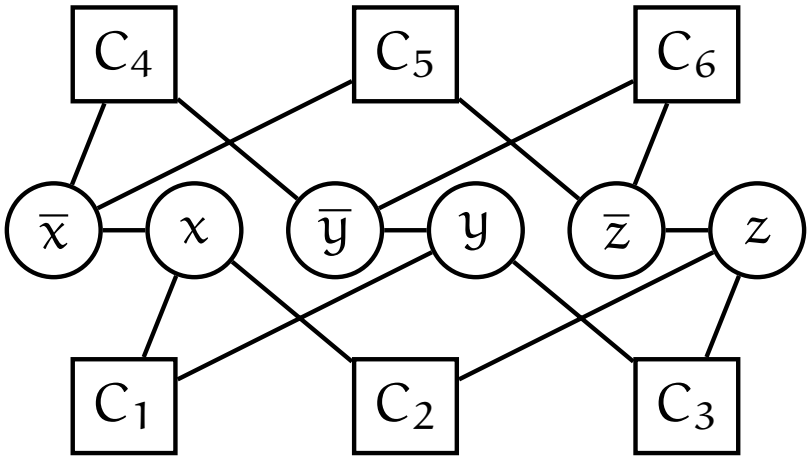


Symmetry: $(x,y,z)(\overline{y},\overline{z},\overline{x})$ is an edge-preserving bijection

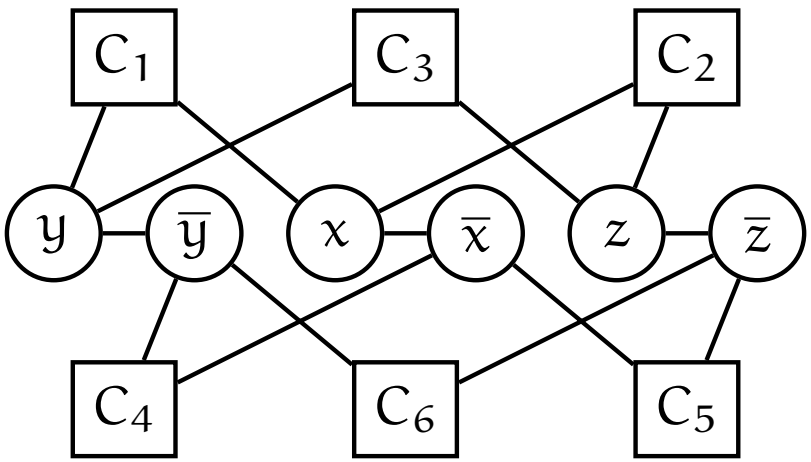
Three Symmetries of the Example Formula



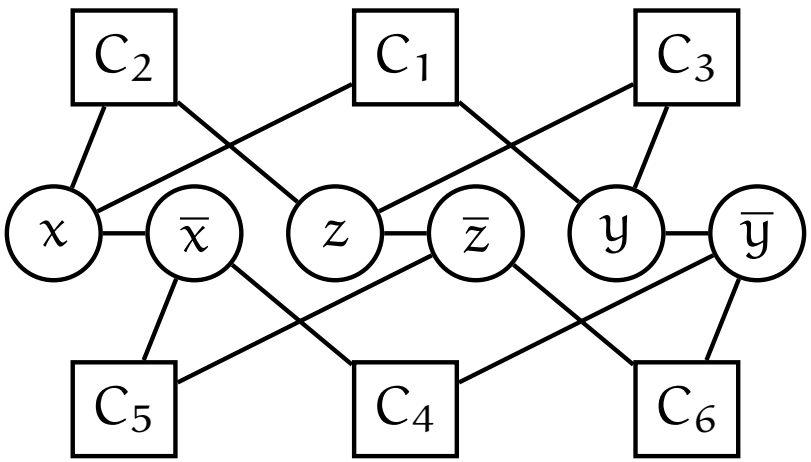
identity symmetry



$(x, y, z, C_1, C_2, C_3, C_4, C_5, C_6)$
$(\overline{x}, \overline{y}, \overline{z}, C_4, C_5, C_6, C_1, C_2, C_3)$



$(x, y, C_2, C_5, C_3, C_6)$
$(y, x, C_3, C_6, C_2, C_5)$



$(y, z, C_1, C_4, C_2, C_5)$
$(z, y, C_2, C_5, C_1, C_4)$

Convert Symmetries into Symmetry-Breaking Predicates

A symmetry $\sigma = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ of a CNF formula F is an edge-preserving bijection of the clause-literal graph of F, that maps literals x_i onto p_i and $\overline{x}_i$ onto $\overline{p}_i$ with $i \in \{1, \ldots, n\}$

Given a CNF formula F. Let α be a satisfying truth assignment for F and σ a symmetry for F, then $\sigma(\alpha)$ is also a satisfying truth assignment for F.

Symmetry $\sigma = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ for F can be broken by adding a symmetry-breaking predicate:
$x_1, \ldots, x_n \leq p_1, \ldots, p_n$.

$$(\overline{x}_1 \vee p_1) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee p_2) \wedge (p_1 \vee \overline{x}_2 \vee p_2) \wedge$$
$$(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee p_3) \wedge (\overline{x}_1 \vee p_2 \vee \overline{x}_3 \vee p_3) \wedge$$
$$(p_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee p_3) \wedge (p_1 \vee p_2 \vee \overline{x}_3 \vee p_3) \wedge \ldots$$

Symmetry Breaking in Practice

In practice, symmetry breaking is mostly used as a preprocessing technique.

A given CNF formula is first transformed into a clause-literal graph. Symmetries are detected in the clause-literal graph. An efficient tool for this is `saucy`.

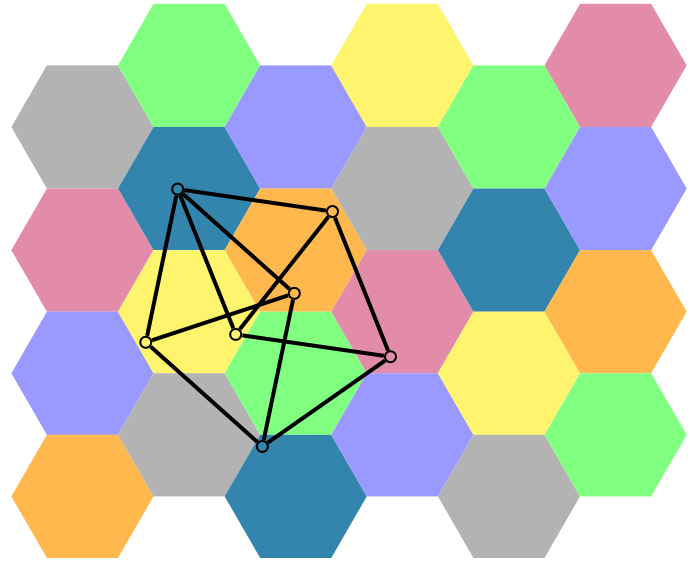
The symmetries can broken by adding symmetry-breaking predicates to the given CNF.

Many hard problems for resolution, such as pigeon hole formulas, can be solved instantly after symmetry-breaking predicates are added.

Chromatic Number of the Plane [Nelson '50]

How many colors are required to color the plane such that each pair of points that are exactly 1 apart are colored differently?

- The Moser Spindle graph shows the lower bound of 4
- A colored tiling of the plane shows the upper bound of 7
- Lower bound of 5 [DeGrey '18] based on a 1581-vertex graph



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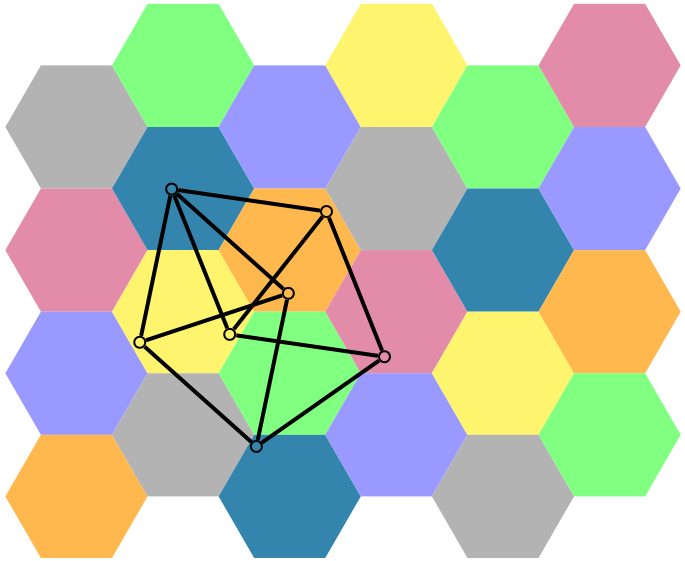


Quantamagazine Physics Mathematics

WIRED

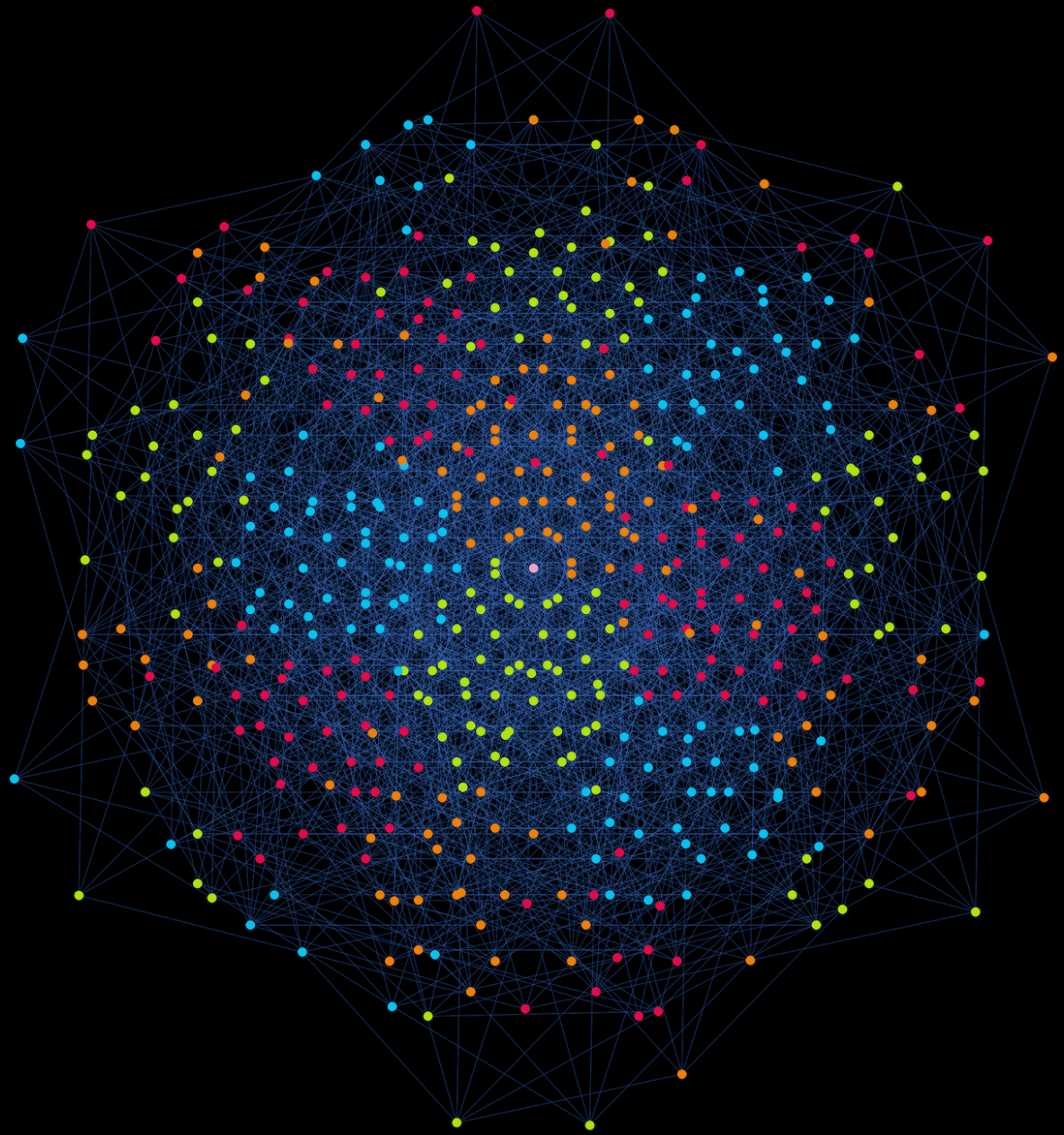
Marijn Heule, a computer scientist at the University of Texas, Austin, found one with just 874 vertices. Yesterday he lowered this number to 826 vertices.

We found smaller graphs with SAT:

- 874 vertices on April 14, 2018
- 803 vertices on April 30, 2018
- 610 vertices on May 14, 2018

marijn@cmu.edu

Record by Proof Minimization: 529 Vertices [Heule 2019]



Equivalence Checking

Bounded Model Checking

Graphs and Symmetry Breaking

Arithmetic Operations

Arithmetic operations: Introduction

How to encode arithmetic operations into SAT?

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Efficient encoding using electronic circuits

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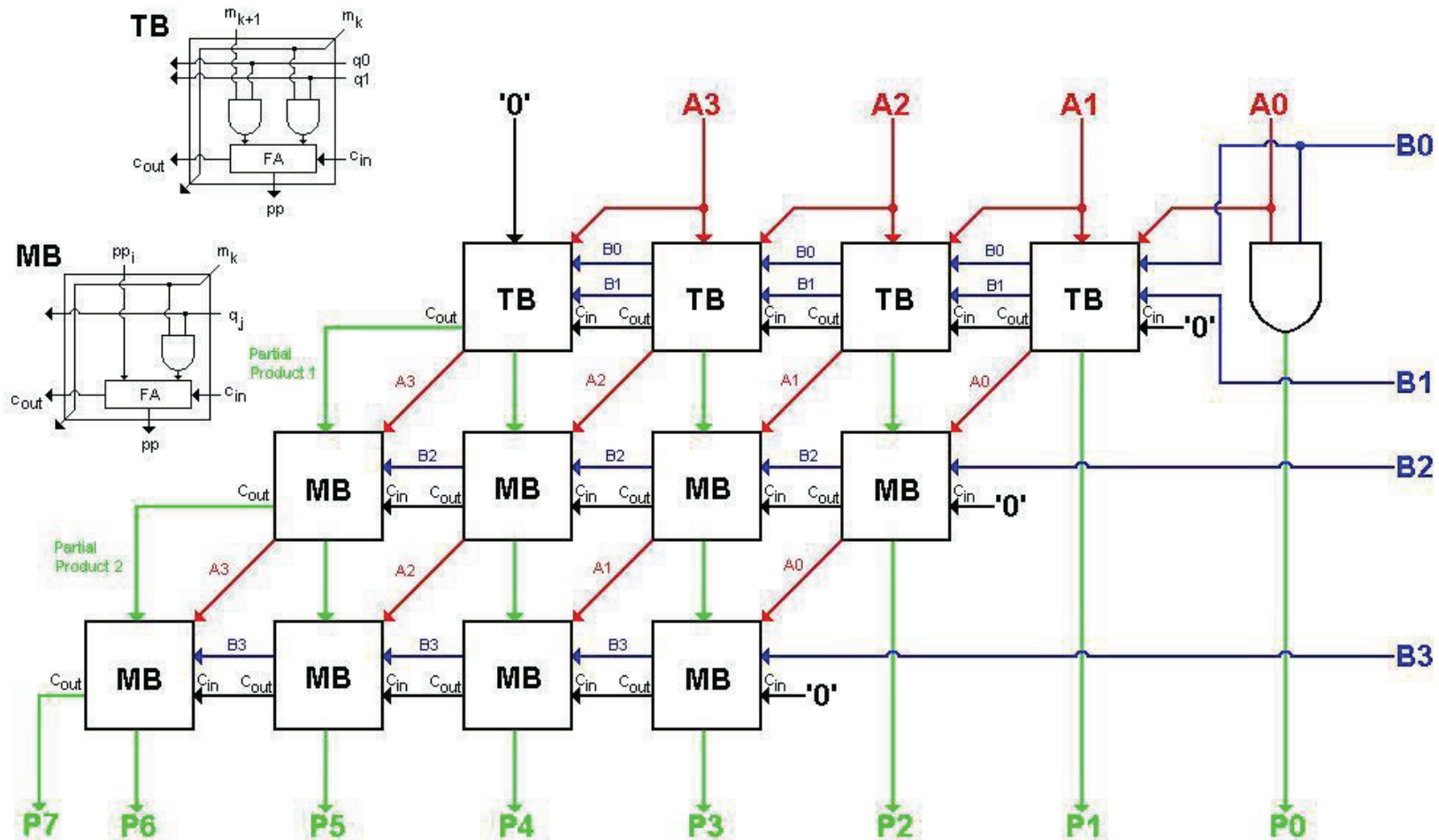
How to encode arithmetic operations into SAT?

Efficient encoding using electronic circuits

Applications:

- factorization (not competitive)
- term rewriting

Arithmetic operations: 4x4 Multiplier circuit



Arithmetic operations: Multiplier encoding

1. Multiplication $m_{i,j} = x_i \times y_j = \textsc{And}\ (x_i, y_j)$

$(m_{i,j} \vee \overline{x}_i \vee \overline{y}_j) \wedge (\overline{m}_{i,j} \vee x_i) \wedge (\overline{m}_{i,j} \vee y_j)$

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2. Carry out $c_{out} = 1$ if and only if $p_{in} + m_{i,j} + c_{in} > 1$

$(c_{out} \vee \overline{p}_{in} \vee \overline{m}_{i,j}) \wedge (c_{out} \vee \overline{p}_{in} \vee \overline{c}_{in}) \wedge (c_{out} \vee \overline{m}_{i,j} \vee \overline{c}_{in}) \wedge$
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3. Parity out p_{out} of variables p_{in}, $m_{i,j}$ and c_{in}

$$(p_{out} \vee \overline{p}_{in} \vee \overline{m}_{i,j} \vee \overline{c}_{in}) \wedge \quad (p_{out} \vee p_{in} \vee m_{i,j} \vee \overline{c}_{in}) \wedge$$
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Arithmetic operations: Is 27 prime?

$$
\begin{array}{cccccccc}
 & & & x_3 & x_2 & x_1 & x_0 & \\
 & & & x_3y_0 & x_2y_0 & x_1y_0 & x_0y_0 & y_0 \\
 & & x_3y_1 & x_2y_1 & x_1y_1 & x_0y_1 & & y_1 \\
 & x_3y_2 & x_2y_2 & x_1y_2 & x_0y_2 & & & y_2 \\
x_3y_3 & x_2y_3 & x_1y_3 & x_0y_3 & & & & y_3 \\
\hline
0 & 0 & 1 & 1 & 0 & 1 & 1 &
\end{array}
$$

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Prime: $(x_1 \vee x_2 \vee x_3) \wedge (y_1 \vee y_2 \vee y_3)$

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Prime: $(x_1 \vee x_2 \vee x_3) \wedge (y_1 \vee y_2 \vee y_3)$

Arithmetic operations: Is 29 prime?

$$
\begin{array}{cccccccc}
 & & & x_3 & x_2 & x_1 & x_0 & \\
 & & & x_3y_0 & x_2y_0 & x_1y_0 & x_0y_0 & y_0 \\
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\end{array}
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Arithmetic operations: Term rewriting

Given a set of rewriting rules,
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Example set of rules:

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$$bb\underline{aa} \to_R b\underline{bb}c \to_R ba\underline{cc} \to_R b\underline{aa}b \to_R \underline{bb}cb \to_R$$
$$a\underline{cc}b \to_R aa\underline{bb} \to_R a\underline{aa}c \to_R ab\underline{cc} \to_R abab$$

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$$a\underline{cc}b \to_R aa\underline{bb} \to_R a\underline{aa}c \to_R ab\underline{cc} \to_R abab$$

Strongest rewriting solvers use SAT (e.g. AProVE)

Example solved by Hofbauer, Waldmann (2006)

Arithmetic operations: Term rewriting proof outline

Proof termination of:

- $aa \to_R bc$
- $bb \to_R ac$
- $cc \to_R ab$

Proof outline:

- Interpret a,b,c by linear functions $[a], [b], [c]$ from $\mathbb{N}^4$ to $\mathbb{N}^4$
- Interpret string concatenation by function composition
- Show that if $[uaav]\ (0,0,0,0) = (x_1, x_2, x_3, x_4)$ and $[ubcv]\ (0,0,0,0) = (y_1, y_2, y_3, y_4)$ then $x_1 > y_1$
- Similar for $bb \to ac$ and $cc \to ab$
- Hence every rewrite step gives a decrease of $x_1 \in \mathbb{N}$, so rewriting terminates

Arithmetic operations: Term rewriting linear functions

The linear functions:

$$[a](\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[b](\vec{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$[c](\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Checking decrease properties using linear algebra

Arithmetic operations: Solving Mathematical Challenges

Recent articles in Quanta Magazine:

- **Computer Search Settles 90-Year-Old Math Problem** August 19, 2020
- **Computer Scientists Attempt to Corner the Collatz Conjecture** August 26, 2020
- **How Close Are Computers to Automating Mathematical Reasoning?** August 27, 2020

Arithmetic operations: Collatz

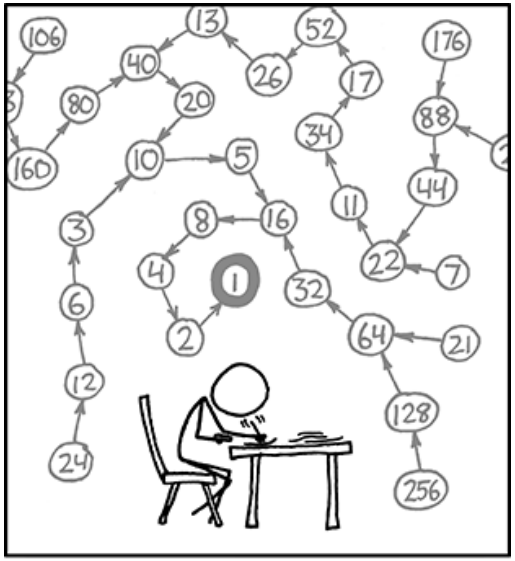
Resolving foundational algorithm questions

$$\mathrm{Col}(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

`while(n > 1) n = Col(n);` terminates?

Find a non-negative function $\mathrm{fun}(n)$ s.t.

$$\forall n > 1 : \mathrm{fun}(n) > \mathrm{fun}(\mathrm{Col}(n))$$



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

source: xkcd.com/710

Arithmetic operations: Collatz

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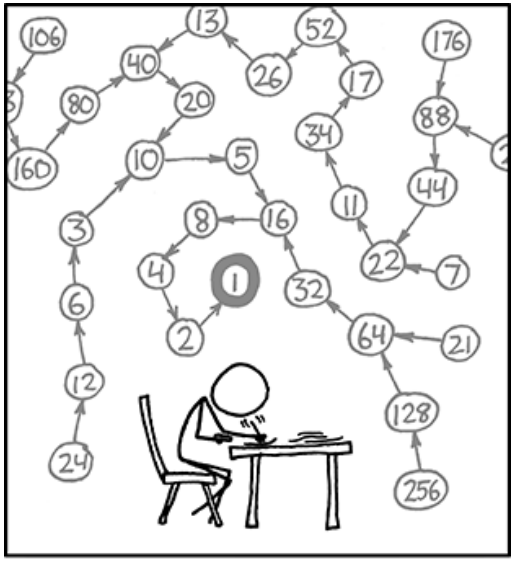

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source: xkcd.com/710

$\mathrm{fun}(3)$	$\mathrm{fun}(5)$	$\mathrm{fun}(8)$	$\mathrm{fun}(4)$	$\mathrm{fun}(2)$	$\mathrm{fun}(1)$
$\mathbf{t}(\mathbf{t}(\vec{0}))$	$\mathbf{t}(\mathbf{f}(\mathbf{t}(\vec{0})))$	$\mathbf{t}(\mathbf{f}(\mathbf{f}(\mathbf{f}(\vec{0}))))$	$\mathbf{t}(\mathbf{f}(\mathbf{f}(\vec{0})))$	$\mathbf{t}(\mathbf{f}(\vec{0}))$	$\mathbf{t}(\vec{0})$

Arithmetic operations: Collatz

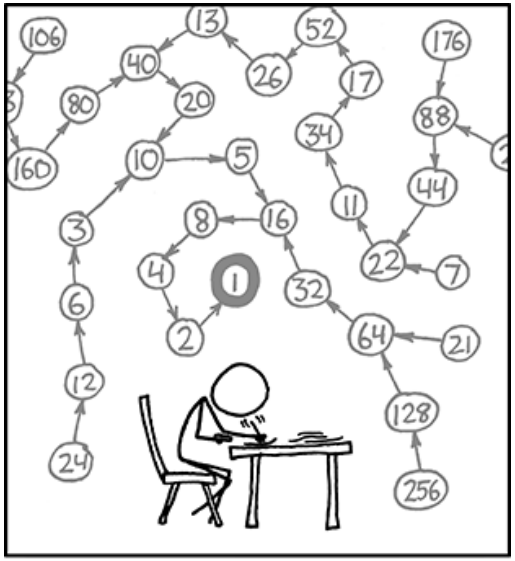
Resolving foundational algorithm questions

$$\mathrm{Col}(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

`while(n > 1) n = Col(n);` terminates?

Find a non-negative function $\mathrm{fun}(n)$ s.t.

$$\forall n > 1 : \mathrm{fun}(n) > \mathrm{fun}(\mathrm{Col}(n))$$



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

source: xkcd.com/710

$\mathrm{fun}(3)$	$\mathrm{fun}(5)$	$\mathrm{fun}(8)$	$\mathrm{fun}(4)$	$\mathrm{fun}(2)$	$\mathrm{fun}(1)$
$\mathbf{t}(\mathbf{t}(\vec{0}))$	$\mathbf{t}(\mathbf{f}(\mathbf{t}(\vec{0})))$	$\mathbf{t}(\mathbf{f}(\mathbf{f}(\mathbf{f}(\vec{0}))))$	$\mathbf{t}(\mathbf{f}(\mathbf{f}(\vec{0})))$	$\mathbf{t}(\mathbf{f}(\vec{0}))$	$\mathbf{t}(\vec{0})$
$\begin{pmatrix}5\\1\end{pmatrix}$	$\begin{pmatrix}4\\1\end{pmatrix}$	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$

using $\mathbf{t}(\vec{x}) = \begin{pmatrix}1&5\\0&0\end{pmatrix}\vec{x} + \begin{pmatrix}0\\1\end{pmatrix}$ and $\mathbf{f}(\vec{x}) = \begin{pmatrix}1&3\\0&0\end{pmatrix}\vec{x} + \begin{pmatrix}1\\0\end{pmatrix}$

Arithmetic Operations: Collatz as Rewriting System

Consider the following functions:

- Binary system: $\mathbf{f}(x) = 2x$, $\mathbf{t}(x) = 2x + 1$
- Ternary system: $p(x) = 3x$, $q(x) = 3x + 1$, $r(x) = 3x + 2$
- Start and end symbols: $c(x) = 1$, $d(x) = x$

D_1: $\mathbf{f}d \to_R d$
D_2: $\mathbf{t}d \to_R rd$

F_1: $\mathbf{f}p \to_R p\mathbf{f}$
F_2: $\mathbf{f}q \to_R p\mathbf{t}$
F_3: $\mathbf{f}r \to_R q\mathbf{f}$

T_1: $\mathbf{t}p \to_R q\mathbf{t}$
T_2: $\mathbf{t}q \to_R r\mathbf{f}$
T_3: $\mathbf{t}r \to_R r\mathbf{t}$

C_1: $cp \to_R c\mathbf{t}$
C_2: $cq \to_R c\mathbf{f}\mathbf{f}$
C_3: $cr \to_R c\mathbf{f}\mathbf{t}$

Interpretation using the functions above:

$D_1 : 2x \to x$

$D_2 : 2x + 1 \to 3x + 2 \quad (= (3(2x + 1) + 1)/2)$

$F_1 : 6x \to 6x$

$T_3 : 6x + 5 \to 6x + 5$

Arithmetic Operations: Collatz Rewriting Example

D_1: $\mathbf{f}d \to_R d$
D_2: $\mathbf{t}d \to_R rd$

F_1: $\mathbf{f}p \to_R p\mathbf{f}$
F_2: $\mathbf{f}q \to_R p\mathbf{t}$
F_3: $\mathbf{f}r \to_R q\mathbf{f}$

T_1: $\mathbf{t}p \to_R q\mathbf{t}$
T_2: $\mathbf{t}q \to_R r\mathbf{f}$
T_3: $\mathbf{t}r \to_R r\mathbf{t}$

C_1: $cp \to_R c\mathbf{t}$
C_2: $cq \to_R c\mathbf{f}\mathbf{f}$
C_3: $cr \to_R c\mathbf{f}\mathbf{t}$

$$\begin{array}{ccccccccccccccccccc}
c\underline{td} & \to & \underline{cr}d & \to & cf\underline{td} & \to & cf\underline{r}d & \to & \underline{cq}fd & \to & cff\underline{fd} & \to & cf\underline{fd} & \to & \underline{cf}d & \to & cd \\
 & & D_2 & & C_3 & & D_2 & & F_3 & & C_2 & & D_1 & & D_1 & & D_1 & & \\
3 & \to & 5 & \to & 5 & \to & 8 & \to & 8 & \to & 8 & \to & 4 & \to & 2 & \to & 1 & &
\end{array}$$

Arithmetic Operations: Collatz Rewriting Example

D_1: $\mathbf{f}d \to_R d$
D_2: $\mathbf{t}d \to_R rd$

F_1: $\mathbf{f}p \to_R p\mathbf{f}$
F_2: $\mathbf{f}q \to_R p\mathbf{t}$
F_3: $\mathbf{f}r \to_R q\mathbf{f}$

T_1: $\mathbf{t}p \to_R q\mathbf{t}$
T_2: $\mathbf{t}q \to_R r\mathbf{f}$
T_3: $\mathbf{t}r \to_R r\mathbf{t}$

C_1: $cp \to_R c\mathbf{t}$
C_2: $cq \to_R c\mathbf{ff}$
C_3: $cr \to_R c\mathbf{ft}$

$$\begin{array}{ccccccccccccccccccc}
c\underline{td} & \to & \underline{cr}d & \to & cf\underline{td} & \to & cf\underline{r}d & \to & \underline{cq}fd & \to & cfff\underline{d} & \to & cf\underline{fd} & \to & c\underline{fd} & \to & cd \\
 & & D_2 & & C_3 & & D_2 & & F_3 & & C_2 & & D_1 & & D_1 & & D_1 & & \\
3 & \to & 5 & \to & 5 & \to & 8 & \to & 8 & \to & 8 & \to & 4 & \to & 2 & \to & 1 & &
\end{array}$$

Can we prove termination of the Collatz rewriting system?

Arithmetic Operations: Collatz Rewriting Example

D_1: $\mathbf{f}d \to_R d$
D_2: $\mathbf{t}d \to_R rd$

F_1: $\mathbf{f}p \to_R p\mathbf{f}$
F_2: $\mathbf{f}q \to_R p\mathbf{t}$
F_3: $\mathbf{f}r \to_R q\mathbf{f}$

T_1: $\mathbf{t}p \to_R q\mathbf{t}$
T_2: $\mathbf{t}q \to_R r\mathbf{f}$
T_3: $\mathbf{t}r \to_R r\mathbf{t}$

C_1: $cp \to_R c\mathbf{t}$
C_2: $cq \to_R c\mathbf{ff}$
C_3: $cr \to_R c\mathbf{ft}$

$$\begin{array}{ccccccccccccccccc} c\underline{td} & \to & \underline{cr}d & \to & c\underline{ftd} & \to & c\underline{fr}d & \to & \underline{cq}fd & \to & cfff\underline{d} & \to & cf\underline{fd} & \to & c\underline{fd} & \to & cd \\ & D_2 & & C_3 & & D_2 & & F_3 & & C_2 & & D_1 & & D_1 & & D_1 & \\ 3 & \to & 5 & \to & 5 & \to & 8 & \to & 8 & \to & 8 & \to & 4 & \to & 2 & \to & 1 \end{array}$$

Can we prove termination of the Collatz rewriting system?

Farkas (n) {
  if $(n \equiv 1 \mod 3)$ **then** Farkas $((n-1)/3)$
  else if $(n \equiv 0 \mod 2)$ **then** Farkas $(n/2)$
  else Farkas $((3n+1)/2)$

Applications for Automated Reasoning

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http://www.cs.cmu.edu/~mheule/15816-f20/

https://cmu.zoom.us/j/93095736668

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