10-607 Computational Foundations for Machine Learning

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Perceptron Mistake Bound

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Lecture 4
Oct. 31, 2018

Reminders

- Homework A:
  - Out: Tue, Oct. 29
  - Due: Wed, Nov. 7 at 11:59pm

Q&A

THE PERCEPTRON ALGORITHM

Perceptron Algorithm: Example

Example:

$(-1,2)\ -$ ✗

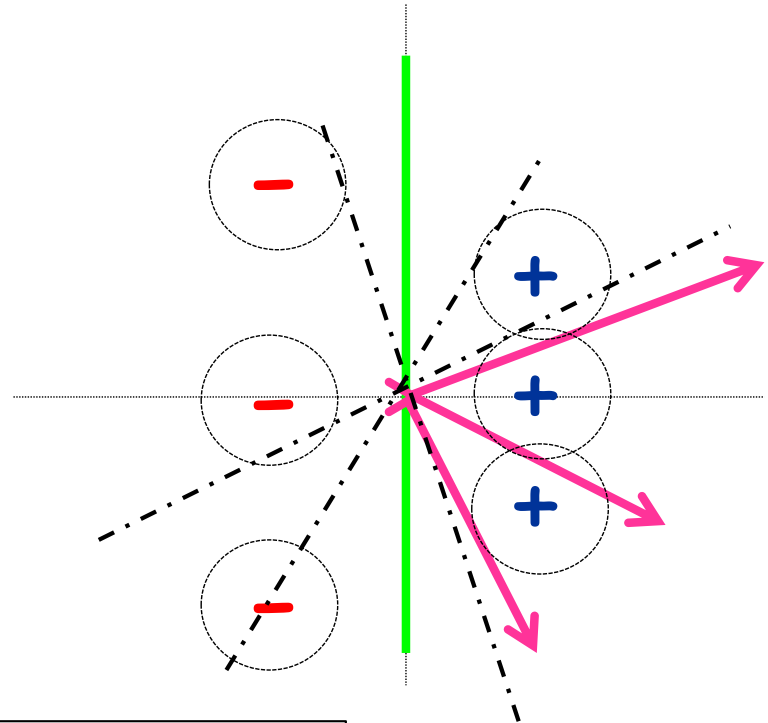
$(1,0)\ +$ ✓

$(1,1)\ +$ ✗

$(-1,0)\ -$ ✓

$(-1,-2)\ -$ ✗

$(1,-1)\ +$ ✓



$w_1 = (0,0)$

$w_2 = w_1 - (-1,2) = (1,-2)$

$w_3 = w_2 + (1,1) = (2,-1)$

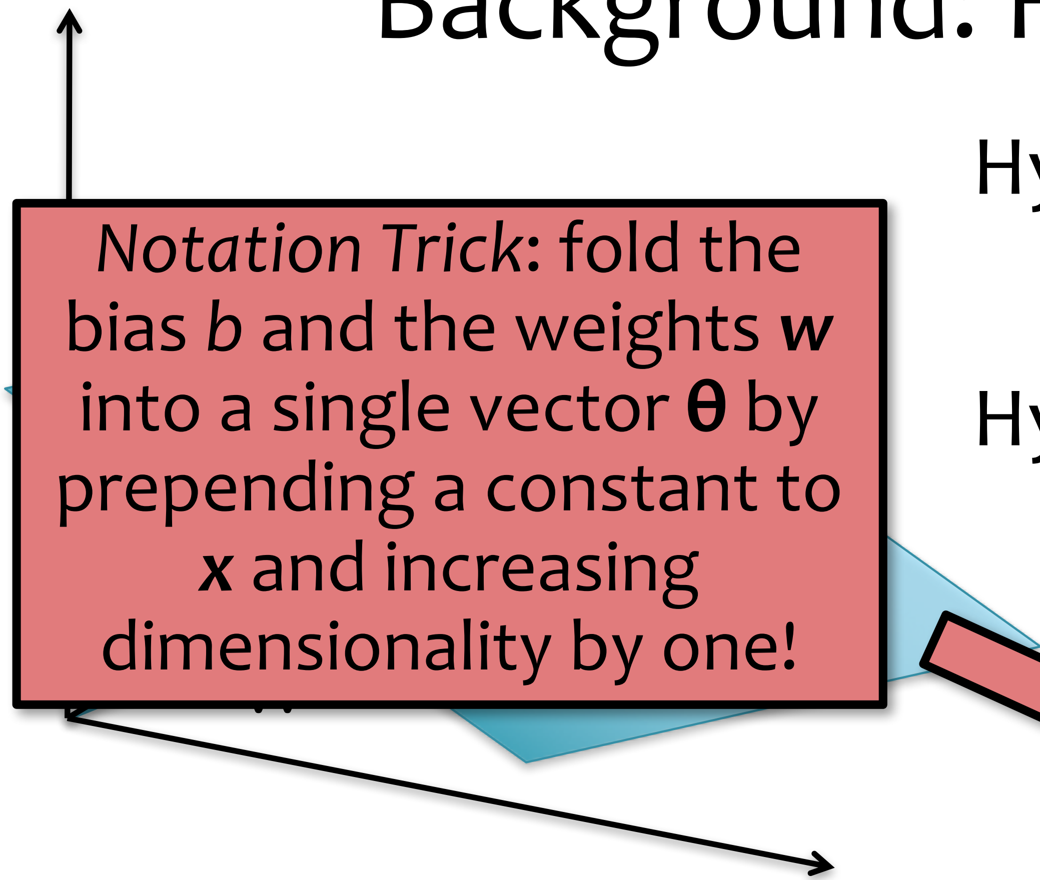
$w_4 = w_3 - (-1,-2) = (3,1)$

Perceptron Algorithm: (without the bias term)

- Set t=1, start with all-zeroes weight vector w_1.
- Given example x, predict positive iff $w_t \cdot x \geq 0$.
- On a mistake, update as follows:
  - Mistake on positive, update $w_{t+1} \leftarrow w_t + x$
  - Mistake on negative, update $w_{t+1} \leftarrow w_t - x$

Slide adapted from Nina Balcan

Background: Hyperplanes



Notation Trick: fold the bias *b* and the weights ***w*** into a single vector **θ** by prepending a constant to ***x*** and increasing dimensionality by one!

Hyperplane (Definition 1):

$$\mathcal{H} = \{\mathbf{x} : \mathbf{w}^T \mathbf{x} = b\}$$

Hyperplane (Definition 2):

$$\mathcal{H} = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} = 0 \text{ and } x_0 = 1\}$$

$$\boldsymbol{\theta} = [b, w_1, \ldots, w_M]^T$$

Half-spaces:

$$\mathcal{H}^+ = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} > 0 \text{ and } x_0 = 1\}$$

$$\mathcal{H}^- = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} < 0 \text{ and } x_0 = 1\}$$

(Online) Perceptron Algorithm

Data: Inputs are continuous vectors of length *M*. Outputs are discrete.

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots$$

where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \{+1, -1\}$

Prediction: Output determined by hyperplane.

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a \geq 0 \\ -1, & \text{otherwise} \end{cases}$$

Assume $\boldsymbol{\theta} = [b, w_1, \ldots, w_M]^T$ and $x_0 = 1$

Learning: Iterative procedure:

- **initialize** parameters to vector of all zeroes
- **while** not converged
  - **receive** next example $(\mathbf{x}^{(i)}, y^{(i)})$
  - **predict** $y' = h(\mathbf{x}^{(i)})$
  - **if** positive mistake: **add** $\mathbf{x}^{(i)}$ to parameters
  - **if** negative mistake: **subtract** $\mathbf{x}^{(i)}$ from parameters

(Online) Perceptron Algorithm

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$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots$$

where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \{+1, -1\}$

Prediction: Output determined

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

Assume $\boldsymbol{\theta} = [b, w_1, \ldots, w_M]$

Learning:

Algorithm 1 Perceptron Learning Alg

```
procedure PERCEPTRON(𝒟 = {(x
    θ ← 0                                      ▷ Initialize parameters
    for i ∈ {1, 2, ...} do                     ▷ For each example
        ŷ ← sign(θ^T x^(i))                    ▷ Predict
        if ŷ ≠ y^(i) then                      ▷ If mistake
            θ ← θ + y^(i) x^(i)                ▷ Update parameters
    return θ
```

Implementation Trick: same behavior as our "*add on positive mistake and subtract on negative mistake*" version, because $y^{(i)}$ takes care of the sign

(Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset, D. We call this the "batch" setting in contrast to the "online" setting that we've discussed so far.

Algorithm 1 Perceptron Learning Algorithm (Batch)

1: **procedure** PERCEPTRON($\mathcal{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), \ldots, (\mathbf{x}^{(N)}, y^{(N)})\}$)
2: $\quad \boldsymbol{\theta} \leftarrow \mathbf{0}$ ▷ Initialize parameters
3: $\quad$ **while** not converged **do**
4: $\quad\quad$ **for** $i \in \{1, 2, \ldots, N\}$ **do** ▷ For each example
5: $\quad\quad\quad \hat{y} \leftarrow \text{sign}(\boldsymbol{\theta}^T \mathbf{x}^{(i)})$ ▷ Predict
6: $\quad\quad\quad$ **if** $\hat{y} \neq y^{(i)}$ **then** ▷ If mistake
7: $\quad\quad\quad\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(i)} \mathbf{x}^{(i)}$ ▷ Update parameters
8: $\quad$ **return** $\boldsymbol{\theta}$

(Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset, D. We call this the “batch” setting in contrast to the “online” setting that we’ve discussed so far.

Discussion:

The Batch Perceptron Algorithm can be derived in two ways.

1. By extending the online Perceptron algorithm to the batch setting (as mentioned above)
2. By applying **Stochastic Gradient Descent (SGD)** to minimize a so-called **Hinge Loss** on a linear separator

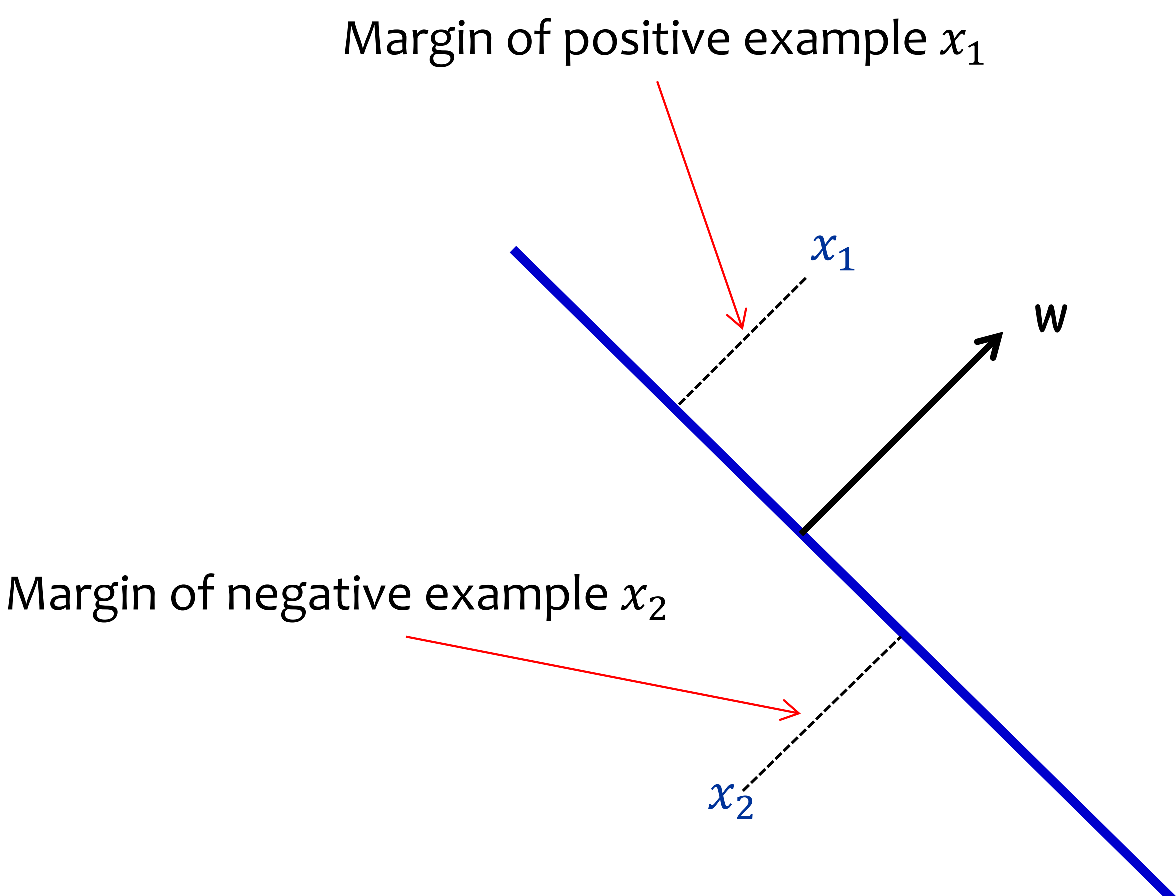
Extensions of Perceptron

- **Voted Perceptron**
  - generalizes better than (standard) perceptron
  - memory intensive (keeps around every weight vector seen during training, so each one can vote)
- **Averaged Perceptron**
  - empirically similar performance to voted perceptron
  - can be implemented in a memory efficient way (running averages are efficient)
- **Kernel Perceptron**
  - Choose a kernel K(x', x)
  - Apply the **kernel trick** to Perceptron
  - Resulting algorithm is **still very simple**
- **Structured Perceptron**
  - Basic idea can also be applied when **y** ranges over an exponentially large set
  - Mistake bound **does not** depend on the size of that set

ANALYSIS OF PERCEPTRON

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$ (or the negative if on wrong side)



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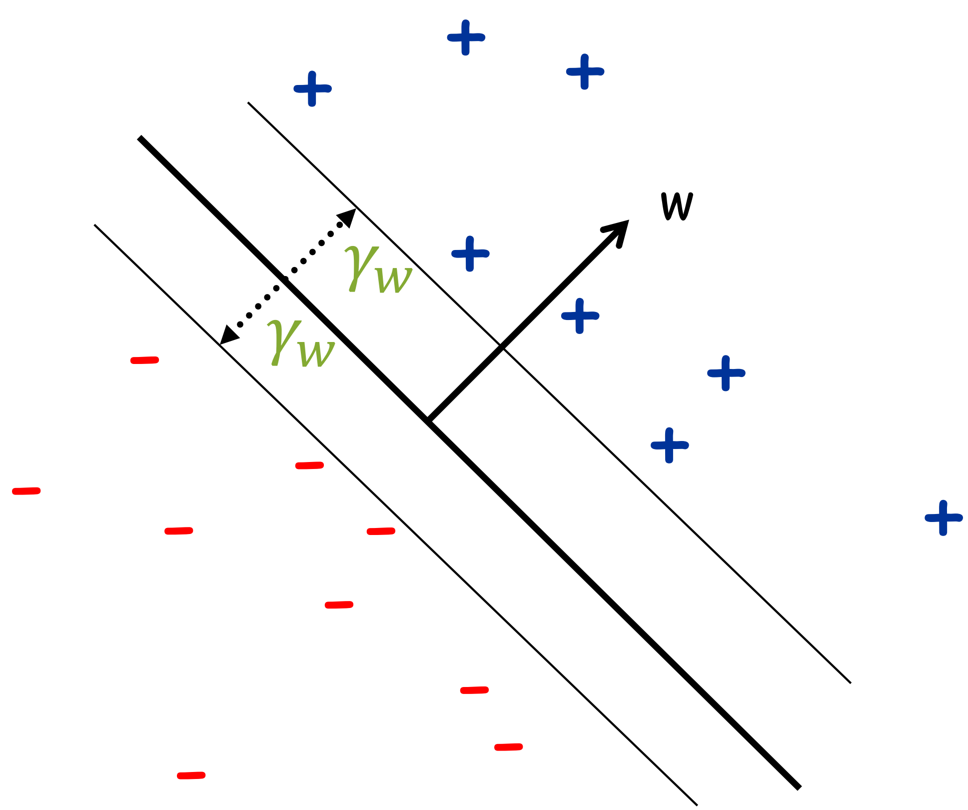
Definition: The margin γ_w of a set of examples S wrt a linear separator w is the smallest margin over points $x \in S$.



Slide from Nina Balcan

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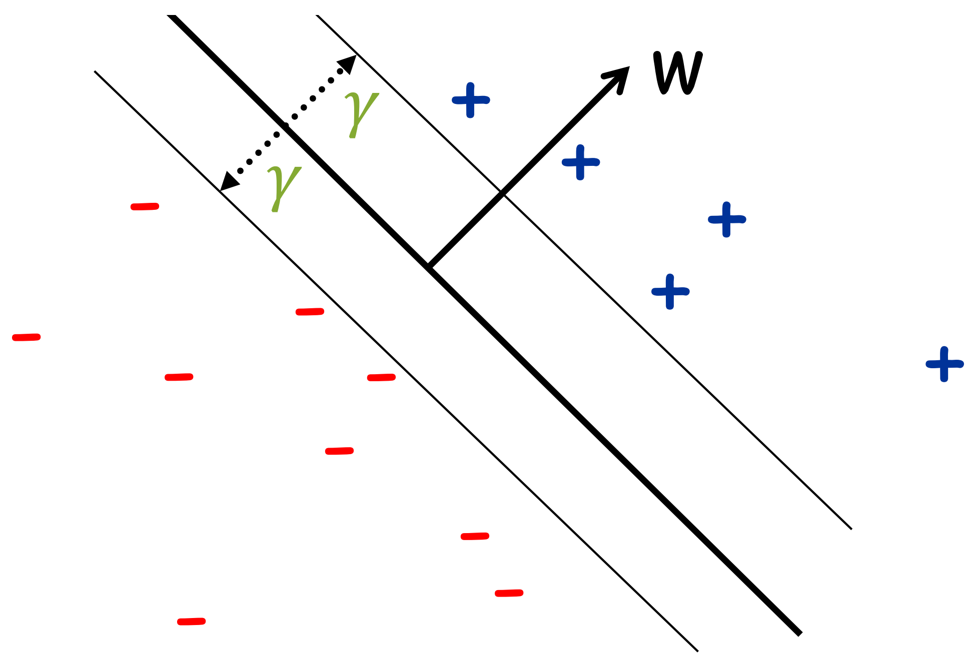
Definition: The margin γ_w of a set of examples S wrt a linear separator w is the smallest margin over points $x \in S$.

Definition: The margin γ of a set of examples S is the maximum γ_w over all linear separators w.

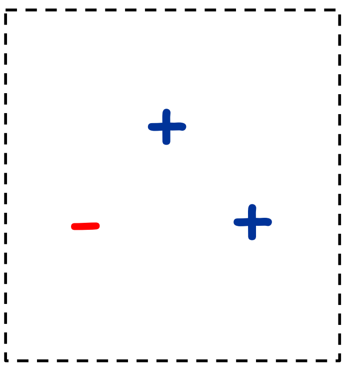


Slide from Nina Balcan

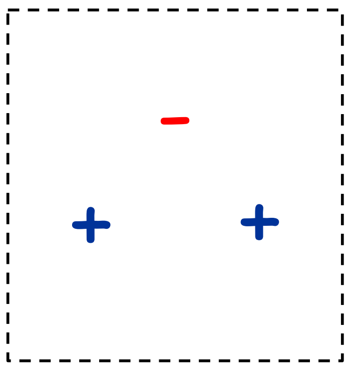
Linear Separability

Def: For a **binary classification** problem, a set of examples S is **linearly separable** if there exists a linear decision boundary that can separate the points

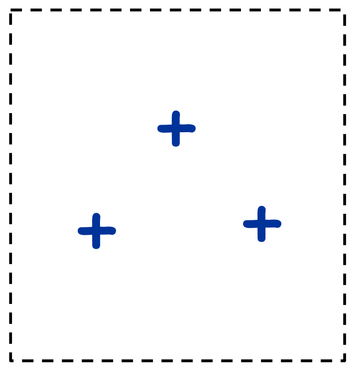
Case 1:



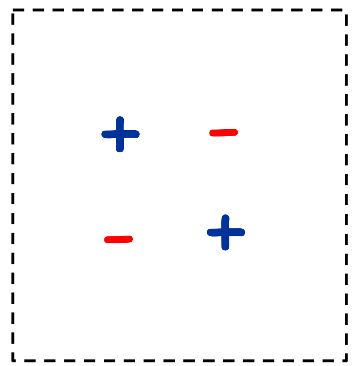
Case 2:



Case 3:



Case 4:

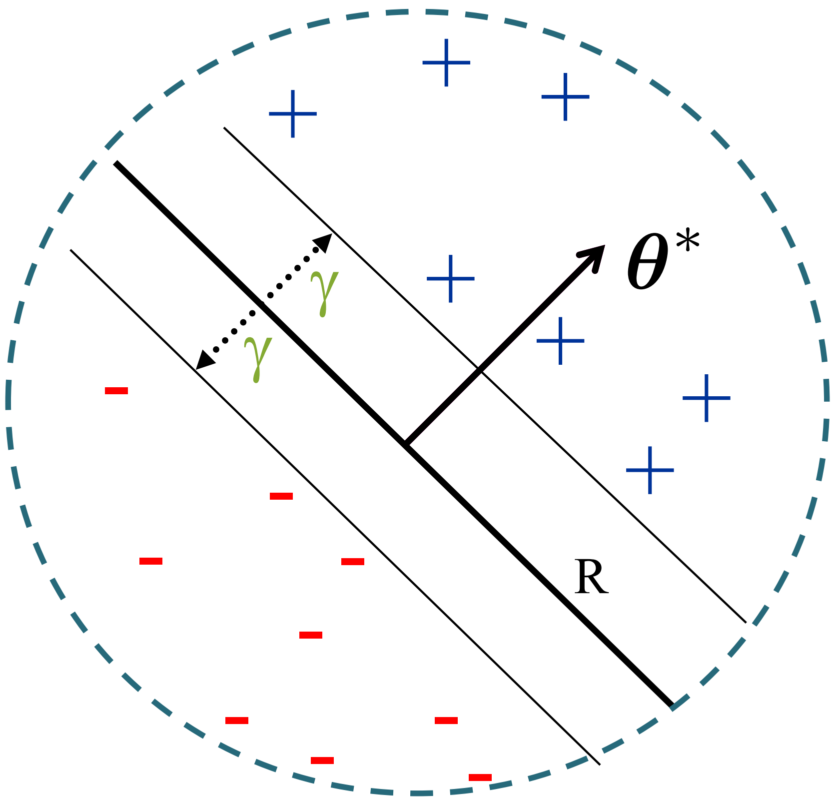


Analysis: Perceptron

Perceptron Mistake Bound

Guarantee: If data has margin γ and all points inside a ball of radius R, then Perceptron makes $\leq (R/\gamma)^2$ mistakes.

(Normalized margin: multiplying all points by 100, or dividing all points by 100, doesn't change the number of mistakes; algo is invariant to scaling.)



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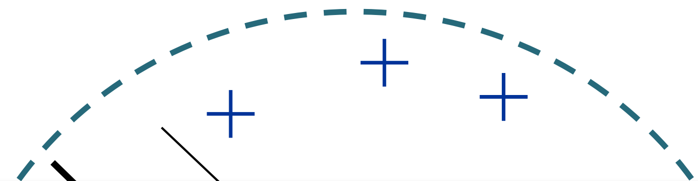


Def: We say that the (batch) perceptron algorithm has **converged** if it stops making mistakes on the training data (perfectly classifies the training data).

Main Takeaway: For **linearly separable** data, if the perceptron algorithm cycles repeatedly through the data, it will **converge** in a finite # of steps.

Slide adapted from Nina Balcan

Analysis: Perceptron

Perceptron Mistake Bound

Theorem 0.1 (Block (1962), Novikoff (1962)).
Given dataset: $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^{N}$.
Suppose:

1. *Finite size inputs:* $||x^{(i)}|| \leq R$
2. *Linearly separable data:* $\exists \boldsymbol{\theta}^*$ *s.t.* $||\boldsymbol{\theta}^*|| = 1$ *and* $y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)}) \geq \gamma, \forall i$

Then: The number of mistakes made by the Perceptron algorithm on this dataset is

$$k \leq (R/\gamma)^2$$

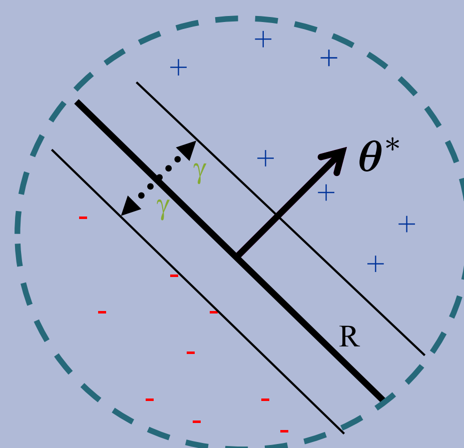


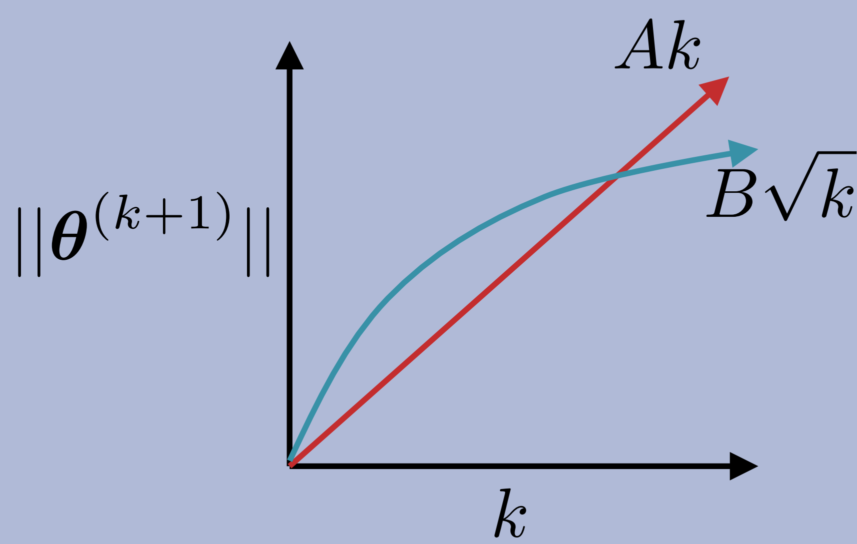
Figure from Nina Balcan

Analysis: Perceptron

Proof of Perceptron Mistake Bound:

We will show that there exist constants A and B s.t.

$$Ak \leq ||\boldsymbol{\theta}^{(k+1)}|| \leq B\sqrt{k}$$



Analysis: Perceptron

Theorem 0.1 (Block (1962), Novikoff (1962)).
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Then: The number of mistakes made by the Perceptron algorithm on this dataset is

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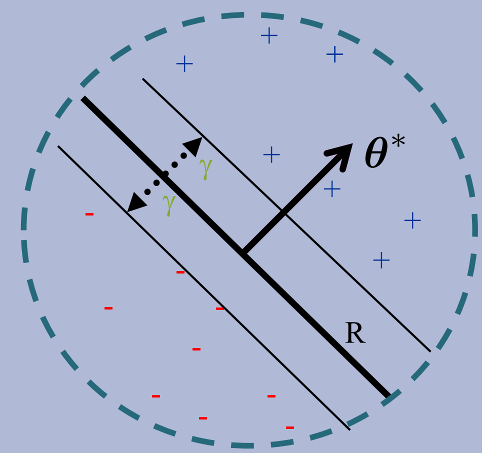

Algorithm 1 Perceptron Learning Algorithm (Online)

```
procedure PERCEPTRON(D = {(x^(1), y^(1)), (x^(2), y^(2)), ...})
    θ ← 0, k = 1                              ▷ Initialize parameters
    for i ∈ {1, 2, ...} do                    ▷ For each example
        if y^(i)(θ^(k) · x^(i)) ≤ 0 then      ▷ If mistake
            θ^(k+1) ← θ^(k) + y^(i) x^(i)     ▷ Update parameters
            k ← k + 1
    return θ
```

Analysis: Perceptron

Chalkboard:

- Proof of Perceptron Mistake Bound

Analysis: Perceptron

Proof of Perceptron Mistake Bound:

Part 1: for some A, $Ak \leq ||\boldsymbol{\theta}^{(k+1)}||$

$$\boldsymbol{\theta}^{(k+1)} \cdot \boldsymbol{\theta}^* = (\boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)})\boldsymbol{\theta}^*$$

by Perceptron algorithm update

$$= \boldsymbol{\theta}^{(k)} \cdot \boldsymbol{\theta}^* + y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)})$$

$$\geq \boldsymbol{\theta}^{(k)} \cdot \boldsymbol{\theta}^* + \gamma$$

by assumption

$$\Rightarrow \boldsymbol{\theta}^{(k+1)} \cdot \boldsymbol{\theta}^* \geq k\gamma$$

by induction on k since $\theta^{(1)} = \mathbf{0}$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}|| \geq k\gamma$$

since $\underbrace{||\mathbf{w}|| \times ||\mathbf{u}|| \geq \mathbf{w} \cdot \mathbf{u}}_{\text{Cauchy-Schwartz inequality}}$ and $||\theta^*|| = 1$

Analysis: Perceptron

Proof of Perceptron Mistake Bound:

Part 2: for some B, $||\boldsymbol{\theta}^{(k+1)}|| \leq B\sqrt{k}$

$$||\boldsymbol{\theta}^{(k+1)}||^2 = ||\boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)}||^2$$

by Perceptron algorithm update

$$= ||\boldsymbol{\theta}^{(k)}||^2 + (y^{(i)})^2||\mathbf{x}^{(i)}||^2 + 2y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)})$$

$$\leq ||\boldsymbol{\theta}^{(k)}||^2 + (y^{(i)})^2||\mathbf{x}^{(i)}||^2$$

since kth mistake $\Rightarrow y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)}) \leq 0$

$$= ||\boldsymbol{\theta}^{(k)}||^2 + R^2$$

since $(y^{(i)})^2||\mathbf{x}^{(i)}||^2 = ||\mathbf{x}^{(i)}||^2 = R^2$ by assumption and $(y^{(i)})^2 = 1$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}||^2 \leq kR^2$$

by induction on k since $(\theta^{(1)})^2 = 0$

$$\Rightarrow ||\boldsymbol{\theta}^{(k+1)}|| \leq \sqrt{k}R$$

Analysis: Perceptron

Proof of Perceptron Mistake Bound:

Part 3: Combining the bounds finishes the proof.

$$k\gamma \leq ||\boldsymbol{\theta}^{(k+1)}|| \leq \sqrt{k}R$$

$$\Rightarrow k \leq \underbrace{(R/\gamma)^2}$$

The total number of mistakes must be less than this

Analysis: Perceptron

What if the data is *not* linearly separable?

1. Perceptron will **not converge** in this case (it can't!)
2. However, Freund & Schapire (1999) show that by projecting the points (hypothetically) into a higher dimensional space, we can achieve a similar bound on the number of mistakes made on **one pass** through the sequence of examples

Theorem 2. *Let $\langle(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)\rangle$ be a sequence of labeled examples with $\|\mathbf{x}_i\| \leq R$. Let $\mathbf{u}$ be any vector with $\|\mathbf{u}\| = 1$ and let $\gamma > 0$. Define the deviation of each example as*

$$d_i = \max\{0, \gamma - y_i(\mathbf{u} \cdot \mathbf{x}_i)\},$$

and define $D = \sqrt{\sum_{i=1}^{m} d_i^2}$. Then the number of mistakes of the online perceptron algorithm on this sequence is bounded by

$$\left(\frac{R + D}{\gamma}\right)^2.$$

Summary: Perceptron

- Perceptron is a **linear classifier**
- **Simple learning algorithm**: when a mistake is made, add / subtract the features
- Perceptron will converge if the data are **linearly separable**, it will **not** converge if the data are **linearly inseparable**
- For linearly separable and inseparable data, we can **bound the number of mistakes** (geometric argument)
- **Extensions** support nonlinear separators and structured prediction