10-601 Introduction to Machine Learning

Machine Learning Department
School of Computer Science
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Reinforcement Learning:
Value Iteration & Policy Iteration

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Lecture 23
Apr. 8, 2020

Reminders

- 3 extra grace days
- Homework 7: HMMs
  - Out: Thu, Apr 02
  - Due: Fri, Apr 10 at 11:59pm
- Homework 8: HMMs
  - Out: Fri, Apr 10
  - Due: Wed, Apr 22 at 11:59pm

- Today's In-Class Poll
  - http://poll.mlcourse.org

MARKOV DECISION PROCESSES

Markov Decision Process

- For **supervised learning** the **PAC learning framework** provided assumptions about where our data came from:

$$\mathbf{x} \sim p^*(\cdot) \text{ and } y = c^*(\cdot)$$

- For **reinforcement learning** we assume our data comes from a **Markov decision process** (MDP)

Markov Decision Process

Whiteboard

- Components: states, actions, state transition probabilities, reward function
- Markovian assumption
- MDP Model
- MDP Goal: Infinite-horizon Discounted Reward
- deterministic vs. nondeterministic MDP
- deterministic vs. stochastic policy

Exploration vs. Exploitation

Whiteboard

- Explore vs. Exploit Tradeoff
- Ex: k-Armed Bandits
- Ex: Traversing a Maze

FIXED POINT ITERATION

Fixed Point Iteration for Optimization

- Fixed point iteration is a general tool for solving systems of equations
- It can also be applied to optimization.

$$J(\boldsymbol{\theta})$$

$$\frac{dJ(\boldsymbol{\theta})}{d\theta_i} = 0 = f(\boldsymbol{\theta})$$

$$0 = f(\boldsymbol{\theta}) \Rightarrow \theta_i = g(\boldsymbol{\theta})$$

$$\theta_i^{(t+1)} = g(\boldsymbol{\theta}^{(t)})$$

1. Given objective function:
2. Compute derivative, set to zero (call this function f).
3. Rearrange the equation s.t. one of parameters appears on the LHS.
4. Initialize the parameters.
5. For i in $\{1,...,K\}$, update each parameter and increment t:
6. Repeat #5 until convergence

Fixed Point Iteration for Optimization

- Fixed point iteration is a general tool for solving systems of equations
- It can also be applied to optimization.

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

1. Given objective function:
2. Compute derivative, set to zero (call this function f).
3. Rearrange the equation s.t. one of parameters appears on the LHS.
4. Initialize the parameters.
5. For i in $\{1,\ldots,K\}$, update each parameter and increment t:
6. Repeat #5 until convergence

Fixed Point Iteration for Optimization

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

We can implement our example in a few lines of python.

```
def f1(x):
    '''f(x) = x^2 - 3x + 2'''
    return x**2 - 3.*x + 2.

def g1(x):
    '''g(x) = \frac{x^2 + 2}{3}'''
    return (x**2 + 2.) / 3.

def fpi(g, x0, n, f):
    '''Optimizes the 1D function g by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    for i in range(n):
        print("i=%2d x=%.4f f(x)=%.4f" % (i, x, f(x)))
        x = g(x)
    i += 1
    print("i=%2d x=%.4f f(x)=%.4f" % (i, x, f(x)))
    return x

if __name__ == "__main__":
    x = fpi(g1, 0, 20, f1)
```

Fixed Point Iteration for Optimization

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

```
$ python fixed-point-iteration.py
i= 0 x=0.0000 f(x)=2.0000
i= 1 x=0.6667 f(x)=0.4444
i= 2 x=0.8148 f(x)=0.2195
i= 3 x=0.8880 f(x)=0.1246
i= 4 x=0.9295 f(x)=0.0755
i= 5 x=0.9547 f(x)=0.0474
i= 6 x=0.9705 f(x)=0.0304
i= 7 x=0.9806 f(x)=0.0198
i= 8 x=0.9872 f(x)=0.0130
i= 9 x=0.9915 f(x)=0.0086
i=10 x=0.9944 f(x)=0.0057
i=11 x=0.9963 f(x)=0.0038
i=12 x=0.9975 f(x)=0.0025
i=13 x=0.9983 f(x)=0.0017
i=14 x=0.9989 f(x)=0.0011
i=15 x=0.9993 f(x)=0.0007
i=16 x=0.9995 f(x)=0.0005
i=17 x=0.9997 f(x)=0.0003
i=18 x=0.9998 f(x)=0.0002
i=19 x=0.9999 f(x)=0.0001
i=20 x=0.9999 f(x)=0.0001
```

VALUE ITERATION

Definitions for Value Iteration

Whiteboard

- State trajectory
- Value function
- Bellman equations
- Optimal policy
- Optimal value function
- Computing the optimal policy
- Ex: Path Planning

RL Terminology

Question: Match each term (on the left) to the corresponding statement or definition (on the right)

Terms:

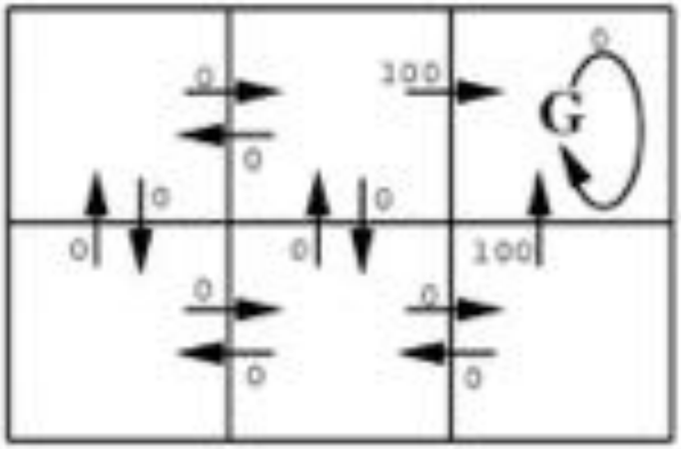
A. a reward function
B. a transition probability
C. a policy
D. state/action/reward triples
E. a value function
F. transition function
G. an optimal policy
H. Matt's favorite statement

Statements:

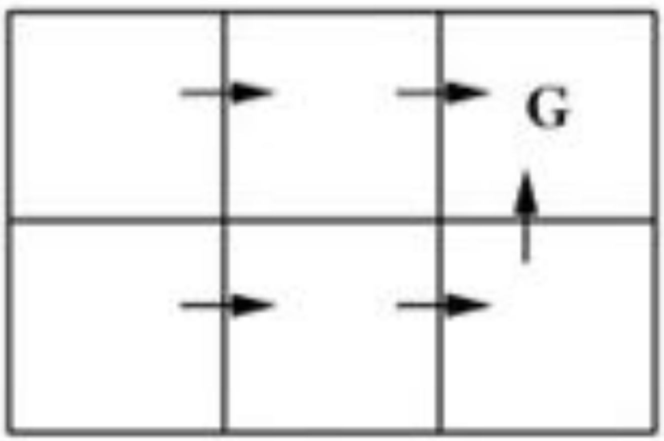
1. gives the expected future discounted reward of a state
2. maps from states to actions
3. quantifies immediate success of agent
4. is a deterministic map from state/action pairs to states
5. quantifies the likelihood of landing a new state, given a state/action pair
6. is the desired output of an RL algorithm
7. can be influenced by trading off between exploitation/exploration

Example: Path Planning

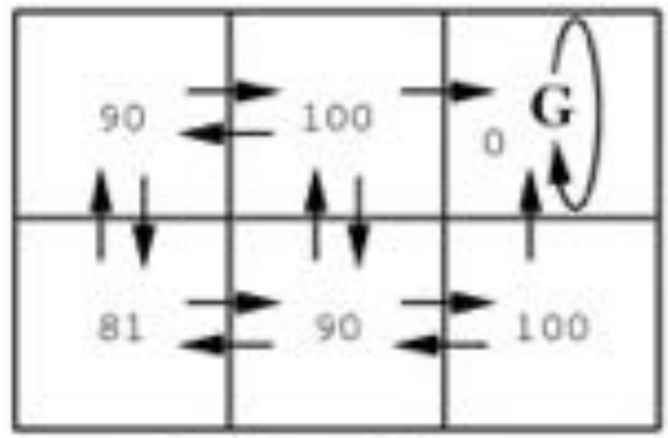
Example: Robot Localization



$r(s, a)$ (immediate reward) values



One optimal policy



$V^*(s)$ values

Figure from Tom Mitchell

Value Iteration

Whiteboard

- Value Iteration Algorithm
- Synchronous vs. Asychronous Updates

Value Iteration

Algorithm 1 Value Iteration

procedure VALUEITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
    Initialize value function $V(s) = 0$ or randomly
    while not converged **do**
        for $s \in \mathcal{S}$ **do**
            for $a \in \mathcal{A}$ **do**
                $Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$
            $V(s) = \max_a Q(s,a)$
    Let $\pi(s) = \operatorname{argmax}_a Q(s,a), \ \forall s$
    return π

Variant 1: with Q(s,a) table

Value Iteration

Algorithm 1 Value Iteration

procedure VALUEITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
  Initialize value function $V(s) = 0$ or randomly
  while not converged **do**
    for $s \in \mathcal{S}$ **do**
      $V(s) = \max_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$
  Let $\pi(s) = \text{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s'),\ \forall s$
  return π

Variant 2: without Q(s,a) table

Synchronous vs. Asynchronous Value Iteration

Algorithm 1 Asynchronous Value Iteration

procedure ASYNCHRONOUSVALUEITERATION($R(s,a)$, $p(\cdot|s,a)$)
Initialize value function $V(s)^{(0)} = 0$ or randomly
$t = 0$
while not converged **do**
for $s \in \mathcal{S}$ **do**
$V(s)^{(t+1)} = \max_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')^{(t)}$
$t = t + 1$
Let $\pi(s) = \text{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')$, $\forall s$
return π

asynchronous updates: compute and update V(s) for each state one at a time

Algorithm 1 Synchronous Value Iteration

procedure SYNCHRONOUSVALUEITERATION($R(s,a)$, $p(\cdot|s,a)$)
Initialize value function $V(s)^{(0)} = 0$ or randomly
$t = 0$
while not converged **do**
for $s \in \mathcal{S}$ **do**
$V(s)^{(t+1)} = \max_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')^{(t)}$
$t = t + 1$
Let $\pi(s) = \text{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')$, $\forall s$
return π

synchronous updates: compute all the fresh values of V(s) from all the stale values of V(s), then update V(s) with fresh values

Value Iteration Convergence

very abridged

Theorem 1 (Bertsekas (1989))
V converges to V^*, if each state is visited infinitely often

Holds for both asynchronous and sychronous updates

Theorem 2 (Williams & Baird (1993))
if $max_s |V^{t+1}(s) - V^t(s)| < \epsilon$
then $max_s |V^{t+1}(s) - V^*(s)| < \frac{2\epsilon\gamma}{1-\gamma}, \forall s$

Provides reasonable stopping criterion for value iteration

Theorem 3 (Bertsekas (1987))
greedy policy will be optimal in a finite number of steps (even if not converged to optimal value function!)

Often greedy policy converges well before the value function

Value Iteration Variants

Question:

True or False: The value iteration algorithm shown below is an example of **synchronous** updates

Algorithm 1 Value Iteration

procedure ValueIteration($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)

  Initialize value function $V(s) = 0$ or randomly

  while not converged **do**

    for $s \in \mathcal{S}$ **do**

      for $a \in \mathcal{A}$ **do**

        $Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$

      $V(s) = \max_a Q(s,a)$

  Let $\pi(s) = \text{argmax}_a Q(s,a), \; \forall s$

  return π

POLICY ITERATION

Policy Iteration

Algorithm 1 Policy Iteration

1: **procedure** POLICYITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
2: Initialize policy π randomly
3: **while** not converged **do**
4: Solve Bellman equations for fixed policy π

$$V^{\pi}(s) = R(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,\pi(s))V^{\pi}(s'), \ \forall s$$

5: Improve policy π using new value function

$$\pi(s) = \operatorname*{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V^{\pi}(s')$$

6: **return** π

Policy Iteration

Algorithm 1 Policy Iteration

procedure POLICYITERATION($R(s, a$ … n, $p(\cdot|s, a)$ transition probabilities)
  Initialize policy π randomly
  while not converged **do**
    Solve Bellman equations for fixed policy π

$$V^{\pi}(s) = R(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V^{\pi}(s'), \ \forall s$$

    Improve policy π using new value function

$$\pi(s) = \operatorname*{argmax}_{a} R(s, a) + \gamma \sum_{s' \in S} p(s'|s, a) V^{\pi}(s')$$

  return π

Compute value function for fixed policy is easy

System of |S| equations and |S| variables

Greedy policy w.r.t. current value function

Greedy policy might **remain the same** for a particular state if there is no better action

Policy Iteration Convergence

In-Class Exercise:
How many policies are there for a finite sized state and action space?

In-Class Exercise:
Suppose policy iteration is shown to improve the policy at every iteration. Can you bound the number of iterations it will take to converge?

Value Iteration vs. Policy Iteration

- Value iteration requires
  $O(|A|\ |S|^2)$
  computation per iteration

- Policy iteration requires
  $O(|A|\ |S|^2 + |S|^3)$
  computation per iteration
- In practice, policy iteration converges in fewer iterations

Algorithm 1 Value Iteration

procedure ValueIteration($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
  Initialize value function $V(s) = 0$ or randomly
  while not converged **do**
    for $s \in \mathcal{S}$ **do**
      for $a \in \mathcal{A}$ **do**
        $Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$
      $V(s) = \max_a Q(s,a)$
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Algorithm 1 Policy Iteration

procedure PolicyIteration($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
  Initialize policy π randomly
  while not converged **do**
    Solve Bellman equations for fixed policy π

$$V^\pi(s) = R(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,\pi(s))V^\pi(s'),\ \forall s$$

    Improve policy π using new value function

$$\pi(s) = \operatorname*{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V^\pi(s')$$

  return π

Learning Objectives

Reinforcement Learning: Value and Policy Iteration

You should be able to…

1. Compare the reinforcement learning paradigm to other learning paradigms
2. Cast a real-world problem as a Markov Decision Process
3. Depict the exploration vs. exploitation tradeoff via MDP examples
4. Explain how to solve a system of equations using fixed point iteration
5. Define the Bellman Equations
6. Show how to compute the optimal policy in terms of the optimal value function
7. Explain the relationship between a value function mapping states to expected rewards and a value function mapping state-action pairs to expected rewards
8. Implement value iteration
9. Implement policy iteration
10. Contrast the computational complexity and empirical convergence of value iteration vs. policy iteration
11. Identify the conditions under which the value iteration algorithm will converge to the true value function
12. Describe properties of the policy iteration algorithm