15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

UNDECIDABLE PROBLEMS

THURSDAY Feb 13

Definition: A Turing Machine is a 7-tuple
$T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

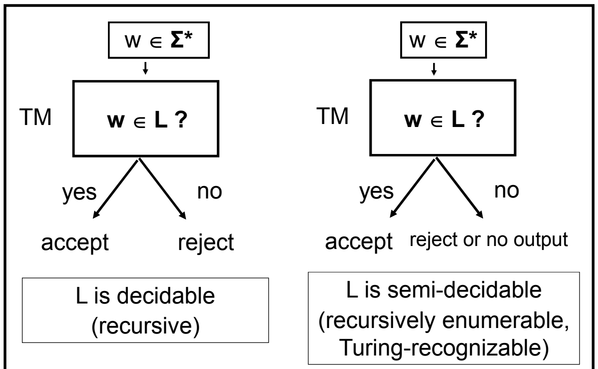
- **Q is a finite set of states**
- **Σ is the input alphabet, where $\square \notin \Sigma$**
- **Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$**
- **$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L,R\}$**
- **$q_0 \in Q$ is the start state**
- **$q_{accept} \in Q$ is the accept state**
- **$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$**

A TM recognizes a language if it accepts all and only those strings in the language

A language is called Turing-recognizable or recursively enumerable, (or r.e. or semi-decidable) if some TM recognizes it

A TM decides a language if it accepts all strings in the language and rejects all strings not in the language

A language is called decidable or recursive if some TM decides it



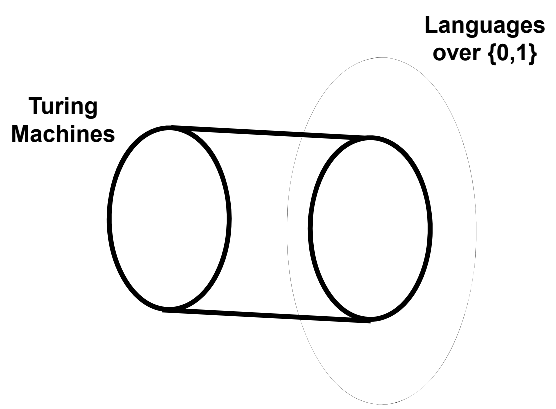
Theorem: L is decidable if both L and $\neg$L are recursively enumerable

There are languages over {0,1} that are not decidable

If we believe the Church-Turing Thesis, this is MAJOR: it means there are things that computers inherently cannot do

We can prove this using a counting argument. We will show there is no onto function from the set of all Turing Machines to the set of all languages over {0,1}. (Works for any Σ) Hence there are languages that have no decider.

Then we will prove something stronger:
There are semi-decidable (r.e.) languages that are NOT decidable



Let L be any set and 2^L be the power set of L

Theorem: There is no onto map from L to 2^L

Proof: Assume, for a contradiction, that there is an onto map $f : L \to 2^L$

Let $S = \{ x \in L \mid x \notin f(x) \}$

If $S = f(y)$ then $y \in S$ if and only if $y \notin S$

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Can give a more constructive argument!

Theorem: There is no onto function from the positive integers to the real numbers in (0, 1)

Proof: Suppose f is any function mapping the positive integers to the real numbers in (0, 1):

1 ⟶ 0.28347279…
2 ⟶ 0.88388384…
3 ⟶ 0.77635284…
4 ⟶ 0.11111111…
5 ⟶ 0.12345678…
:   :

$$[\text{ n-th digit of r }] = \begin{cases} 1 & \text{if [n-th digit of f(n)]} \neq 1 \\ 2 & \text{otherwise} \end{cases}$$

$f(n) \neq r$ for all n (Here, r = 11121...) So f is not onto

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THE MORAL:
No matter what L is,
2^L *always* has more elements than L

Not all languages over {0,1} are decidable, in fact: not all languages over {0,1} are semi-decidable

{decidable languages over {0,1}}

{semi-decidable languages over {0,1}}

{Turing Machines}   **{Languages over {0,1}}**

{Strings of 0s and 1s}   **{Sets of strings of 0s and 1s}**

Set L   **Set of all subsets of L: 2^L**

Let $Z^+ = \{1,2,3,4\ldots\}$. There exists a bijection between Z^+ and $Z^+ \times Z^+$ (or Q^+)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5) …
(2,1)	(2,2)	(2,3)	(2,4)	(2,5) …
(3,1)	(3,2)	(3,3)	(3,4)	(3,5) …
(4,1)	(4,2)	(4,3)	(4,4)	(4,5) …
(5,1)	(5,2)	(5,3)	(5,4)	(5,5) …

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THE ACCEPTANCE PROBLEM

A_{TM} = { (M, w) | M is a TM that accepts string w }

Theorem: A_{TM} is semi-decidable (r.e.) but NOT decidable

A_{TM} is r.e. :

Define a TM U as follows:

On input (M, w), U runs M on w. If M ever accepts, accept. If M ever rejects, reject.

NB. When we write "input **(M, w)"** we really mean "input code for (code for **M, w)"**

THE ACCEPTANCE PROBLEM

A_{TM} = { (M, w) | M is a TM that accepts string w }

Theorem: A_{TM} is semi-decidable (r.e.) but NOT decidable

A_{TM} is r.e. :

Define a TM U as follows: **U is a *universal TM***

On input (M, w), U runs M on w. If M ever accepts, accept. If M ever rejects, reject.

Therefore,
U accepts (M,w) $\Leftrightarrow$ M accepts w $\Leftrightarrow$ (M,w) $\in A_{TM}$
Therefore, U *recognizes* A_{TM}

A_{TM} = { (M,w) | M is a TM that accepts string w }

A_{TM} is undecidable: (proof by contradiction)

Assume machine H decides A_{TM}

$$H(\,(M,w)\,) = \begin{cases} \text{Accept} & \text{if M accepts w} \\ \text{Reject} & \text{if M does not accept w} \end{cases}$$

Construct a new TM D as follows: on input M, run H on (M,M) and output the opposite of H

$$D(\,M\,) = \begin{cases} \text{Reject} & \text{if M accepts M} \\ \text{Accept} & \text{if M does not accept M} \end{cases}$$

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$$D(\,\boxed{D}\,) = \begin{cases} \text{Reject} & \text{if D accepts } \boxed{D} \\ \text{Accept} & \text{if D does not accept } \boxed{D} \end{cases}$$

OUTPUT OF H

	M_1	M_2	M_3	M_4 …	
M_1	accept	accept	accept	reject	accept
M_2	reject	accept	reject	reject	reject
M_3	accept	reject	reject	accept	accept
M_4	accept	reject	reject	reject	accept
:					

OUTPUT OF H

	M_1	M_2	M_3	M_4 …	D
M_1	accept	accept	accept	reject	accept
M_2	reject	accept	reject	reject	reject
M_3	accept	reject	reject	accept	accept
M_4	accept	reject	reject	reject	accept
:					
D	reject	reject	accept	accept	**?**

Theorem: A_{TM} is r.e. but NOT decidable

Theorem: $\neg A_{TM}$ is not even r.e.!

A_{TM} = { (M,w) | M is a TM that accepts string w }

A_{TM} is undecidable: A constructive proof:

Let machine H semi-decides A_{TM} (Such ∃ , why?)

H((M,w)) = { **Accept** if M accepts w; **Reject or No output** if M does not accept w }

Construct a new TM D as follows: on input M, run H on (M,M) and output

D(M) = {
Reject if H (**M, M**) Accepts
Accept if H (**M M**) Rejects
No output if H (**M, M** has No output

A_{TM} = { (M,w) | M is a TM that accepts string w }

A_{TM} is undecidable: A constructive proof:

Let machine H semi-decides A_{TM} (Such ∃ , why?)

H((M,w)) = { **Accept** if M accepts w; **Reject or No output** if M does not accept w }

Construct a new TM D as follows: on input M, run H on (M,M) and output

D(D) = {
Reject if H (**D, D**) Accepts
Accept if H (**D, D**) Rejects
No output if H (**D, D** has No output

H((D,D)) = No output **No Contradictions !**

We have shown:

Given any machine H for semi-deciding A_{TM}, we can *effectively construct* a TM D such that (D,D) $\notin A_{TM}$ but H fails to tell us that.

That is, H *fails* to be a decider on instance (D,D).

In other words,

Given any "good" candidate for deciding the *Acceptance Problem*, we can effectively construct an instance where the candidate fails.

THE classical HALTING PROBLEM

$HALT_{TM}$ = { (M,w) | M is a TM that halts on string w }

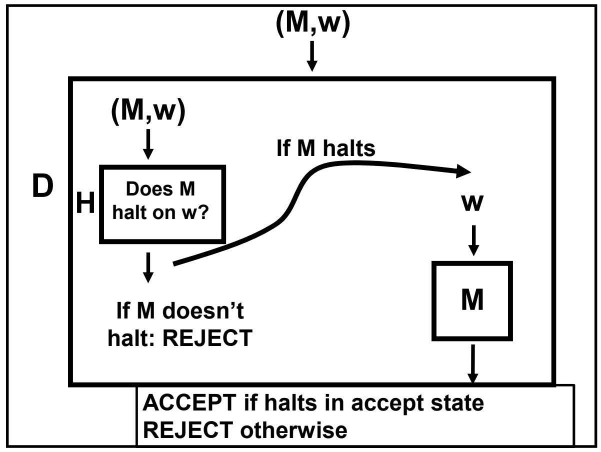
Theorem: $HALT_{TM}$ is undecidable

Proof: Assume, for a contradiction, that TM H decides $HALT_{TM}$

We use H to construct a TM D that decides A_{TM}

On input (M,w), D runs H on (M,w):

- **If H rejects then reject**
- **If H accepts, run M on w until it halts:**
  - **Accept if M accepts ie halts in an accept state**
  - **Otherwise reject**



In many cases, one can show that a language L is undecidable by showing that if it is decidable, then so is A_{TM}

We reduce deciding A_{TM} to deciding the language in question

$$A_{TM} \leq L$$

We just showed: $A_{TM} \leq Halt_{TM}$
Is $Halt_{TM} \leq A_{TM}$?

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Read chapter 4 of the book for next time