15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

REVIEW for MIDTERM 1

THURSDAY Feb 6

Midterm 1 will cover everything we have seen so far

The PROBLEMS will be from Sipser, Chapters 1, 2, 3

It will be Closed-Book, Closed-Everything

- **1. Deterministic Finite Automata and Regular Languages**
- **2. Non-Deterministic Finite Automata**
- **3. Pumping Lemma for Regular Languages; Regular Expressions**
- **4. Minimizing DFAs**
- **5. PDAs, CFGs; Pumping Lemma for CFLs**
- **6. Equivalence of PDAs and CFGs**
- **7. Chomsky Normal Form**
- **8. Turing Machines**

Machines		**Syntactic Rules**
DFAs		
$\updownarrow$		
NFAs	$\longleftrightarrow$	Regular Expressions
PDAs	$\longleftrightarrow$	Context-Free Grammars

THE REGULAR OPERATIONS

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Negation: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

REGULAR EXPRESSIONS

σ **is a regexp representing** $\{\sigma\}$

ε **is a regexp representing** $\{\varepsilon\}$

$\varnothing$ **is a regexp representing** $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) **represents** $L_1 \cdot L_2$

$(R_1 \cup R_2)$ **represents** $L_1 \cup L_2$

$(R_1)^*$ **represents** L_1^*

How can we test if two regular expressions are the same?

Length n	R_1		R_2
	↓		↓
O(n) states	N_1		N_2
	↓		↓
O(2^n) states	M_1		M_2
	↓		↓
	$M_{1\ MIN}$	?=	$M_{2\ MIN}$

Another way???

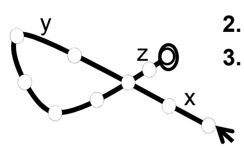
THEOREMS
and
CONSTRUCTIONS

THE PUMPING LEMMA
(for Regular Languages)

Let L be a regular language with |L| = ∞

Then there is an integer P such that if $w \in L$ and $|w| \geq P$ then can write w = xyz, where:

1. **$|y| > 0$**
2. **$|xy| \leq P$**
3. **$xy^iz \in L$ for any $i \geq 0$**

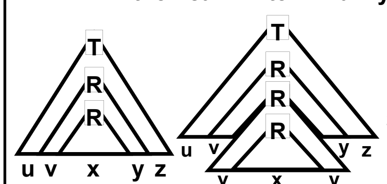


THE PUMPING LEMMA
(for Context Free Grammars)

Let L be a context-free language with |L| = ∞

Then there is an integer P such that if $w \in L$ and $|w| \geq P$

then can write w = uvxyz, where:



1. **$|vy| > 0$**
2. **$|vxy| \leq P$**
3. **$uv^ixy^iz \in L$, for any $i \geq 0$**

CONVERTING NFAs **TO DFAs**

Input: NFA N = (Q, Σ, δ, Q_0, F)

Output: DFA M = (Q′, Σ, δ′, q_0′, F′)

$Q' = 2^Q$

$\delta' : Q' \times \Sigma \rightarrow Q'$

$\delta'(R,\sigma) = \bigcup_{r \in R} \varepsilon(\,\delta(r,\sigma)\,)$ *

$q_0' = \varepsilon(Q_0)$

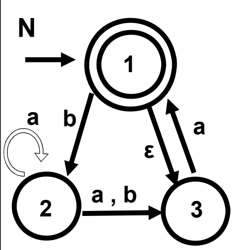
$F' = \{ R \in Q' \mid f \in R \text{ for some } f \in F \}$

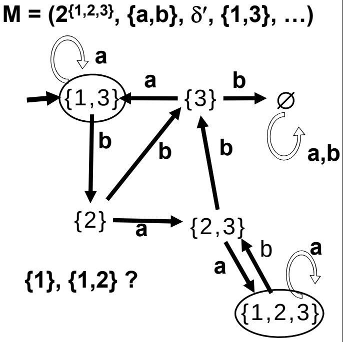
* For **R ⊆ Q**, the **ε-closure of R, ε(R) = {q** that can be reached from some r ∈ **R** by traveling along zero or more **ε** arrows}

Given: NFA N = ({1,2,3}, {a,b}, δ , {1}, {1})

Construct: Equivalent DFA M

M = ($2^{\{1,2,3\}}$, {a,b}, δ′, {1,3}, …)





ε({1}) = {1,3}

{1}, {1,2} ?

EQUIVALENCE

L can be represented by a regexp
⇔
L is a regular language

✓ **L can be represented by a regexp ⇒ L is a regular language**

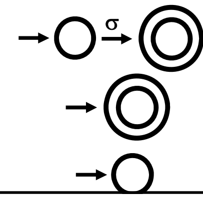
Induction on the length of R:

Base Cases (R has length 1):

R = σ

R = ε

R = ∅



Inductive Step:

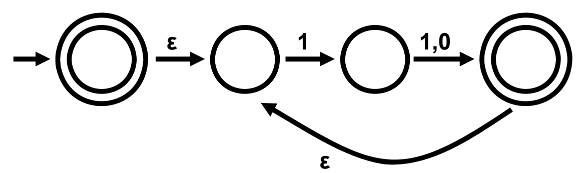
Assume R has length k > 1,
and that every regexp of length < k
represents a regular language

Three possibilities for what R can be:

$R = R_1 \cup R_2$ (Closure under Union)
$R = R_1 R_2$ (Closure under Concat.)
$R = (R_1)^*$ (Closure under Star)

Therefore: L can be represented by a regexp ⇒ L is regular

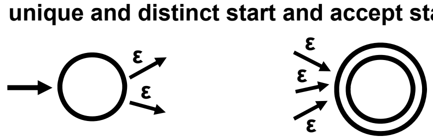
Transform (1(0 ∪ 1))* to an NFA

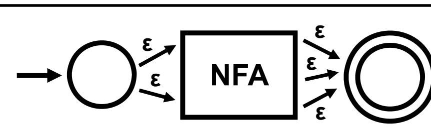


✓ **L is a regular language ⇒ L can be represented by a regexp**

Proof idea: Transform an NFA for L into a regular expression by removing states and re-labeling the arrows with regular expressions

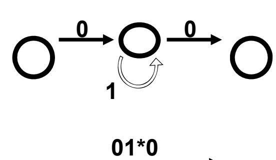
Add unique and distinct start and accept states

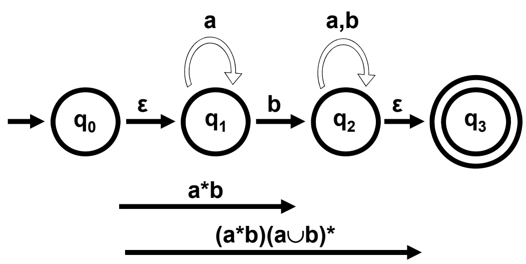




While machine has more than 2 states:

Pick an internal state, rip it out and re-label the arrows with regexps, to account for the missing state





$R(q_0,q_3) = (a^*b)(a \cup b)^*$

THEOREM

For every regular language L, there exists a UNIQUE (up to re-labeling of the states) minimal DFA M such that L = L(M)

EXTENDING δ

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, extend δ to $\hat{\delta} : Q \times \Sigma^* \to Q$ as follows:

$\hat{\delta}(q, \varepsilon) = q$

$\hat{\delta}(q, \sigma) = \delta(q, \sigma)$

$\hat{\delta}(q, w_1 \ldots w_{k+1}) = \delta(\hat{\delta}(q, w_1 \ldots w_k), w_{k+1})$

Note: $\delta(q_0, w) \in F \Leftrightarrow$ **M accepts w**

String $w \in \Sigma^*$ distinguishes states q_1 and q_2 iff exactly ONE of $\hat{\delta}(q_1, w)$, $\hat{\delta}(q_2, w)$ is a final state

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Definition:

- $p \sim q$ **iff p is indistinguishable from q**
- $p \nsim q$ **iff p is distinguishable from q**

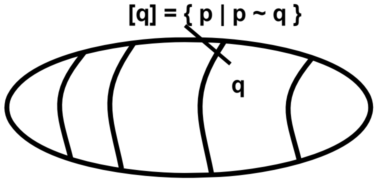
Proposition: ~ is an equivalence relation

- $p \sim p$ **(reflexive)**
- $p \sim q \Rightarrow q \sim p$ **(symmetric)**
- $p \sim q$ **and** $q \sim r \Rightarrow p \sim r$ **(transitive)**

Proposition: ~ is an equivalence relation

so ~ partitions the set of states of M into disjoint equivalence classes

$[q] = \{ p \mid p \sim q \}$



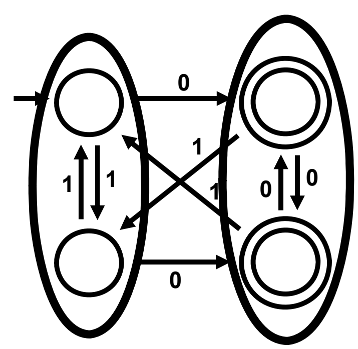
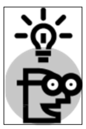


TABLE-FILLING ALGORITHM

Input: DFA M = (Q, Σ, δ, q_0, F)

Output: (1) D_M = { (p,q) | p,q ∈ Q and p $\not\sim$ q }

(2) E_M = { [q] | q ∈ Q }



IDEA:

- **We know how to find those pairs of states that ε distinguishes…**
- **Use this and recursion to find those pairs distinguishable with *longer* strings**
- **Pairs left over will be indistinguishable**

TABLE-FILLING ALGORITHM

Input: DFA M = (Q, Σ, δ, q_0, F)

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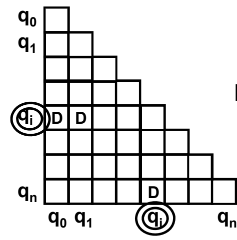


Base Case: p accepts and q rejects ⇒ p $\not\sim$ q

Recursion: if there is σ ∈ Σ and states p′, q′ satisfying

$$\begin{array}{l} \delta(p, \sigma) = p' \\ \quad\quad \not\sim \quad \Rightarrow p \not\sim q \\ \delta(q, \sigma) = q' \end{array}$$

Repeat until no more new D's

Algorithm MINIMIZE

Input: DFA M

Output: DFA M_{MIN}

(1) Remove all inaccessible states from M

(2) Apply Table-Filling algorithm to get E_M = { [q] | q is an accessible state of M }

$M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\ MIN}, F_{MIN})$

$Q_{MIN} = E_M$, $q_{0\ MIN} = [q_0]$, $F_{MIN} = \{ [q] \mid q \in F \}$

$\delta_{MIN}([q],\sigma) = [\ \delta(q,\sigma)\ ]$

Claim: $M_{MIN} \equiv M$

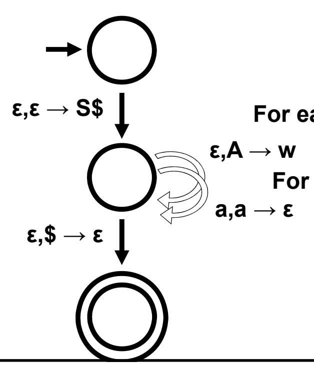
A Language L is generated by a CFG

⇔

L is recognized by a PDA

Suppose L is generated by a CFG G = (V, Σ, R, S)

Construct P = (Q, Σ, Γ, δ, q, F) that recognizes L

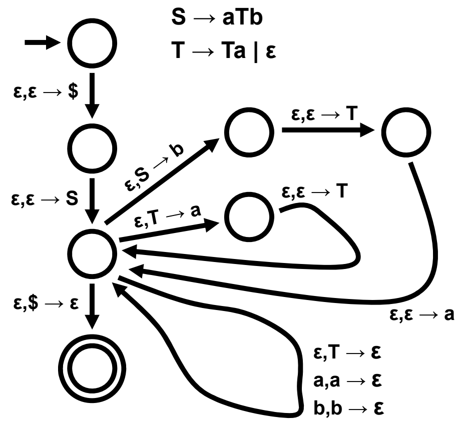


For each rule 'A → w' ∈ R:

For each terminal a ∈ Σ:

S → aTb

T → Ta | ε



A Language L is generated by a CFG

⇐

L is recognized by a PDA

Given PDA P = (Q, Σ, Γ, δ, q, F)

Construct a CFG G = (V, Σ, R, S) such that L(G) = L(P)

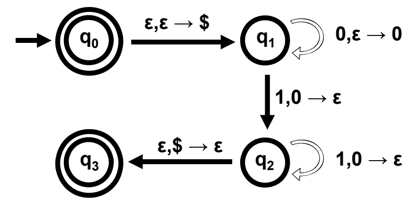
First, simplify P to have the following form:

(1) It has a single accept state, q_{accept}

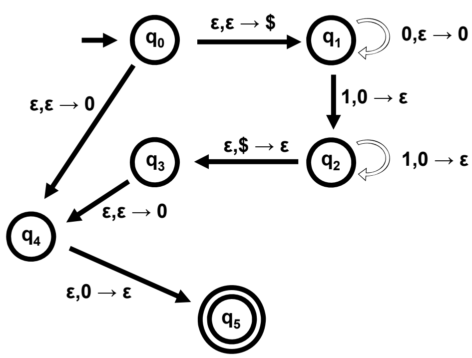
(2) It empties the stack before accepting

(3) Each transition either pushes a symbol or pops a symbol, but not both at the same time

SIMPLIFY



SIMPLIFY



Idea For Our Grammar G:
For every pair of states p and q in PDA P,

G will have a variable A_{pq} which generates all strings x that can take:

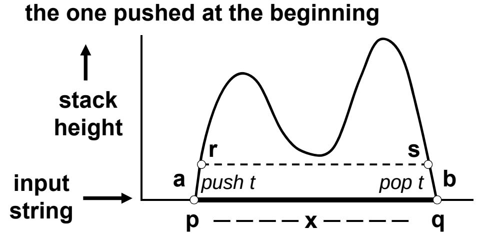
P from p with an empty stack
to q with an empty stack

$$V = \{A_{pq} \mid p,q \in Q\}$$

$$S = A_{q_0 q_{acc}}$$

x = ayb takes p with empty stack to q with empty stack

1. The symbol t popped at the end is exactly the one pushed at the beginning

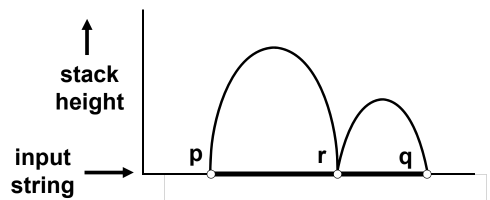


δ(p, a, ε) → (r, t)

δ(s, b, t) → (q, ε)

$$A_{pq} \to aA_{rs}b$$

2. The symbol popped at the end is not the one pushed at the beginning



$$A_{pq} \to A_{pr}A_{rq}$$

Formally:

$V = \{A_{pq} \mid p, q \in Q\}$

$S = A_{q_0 q_{acc}}$

For every $p, q, r, s \in Q$, $t \in \Gamma$ and $a, b \in \Sigma_\varepsilon$

If $(r, t) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, t)$

Then add the rule $A_{pq} \to aA_{rs}b$

For every $p, q, r \in Q$,

add the rule $A_{pq} \to A_{pr}A_{rq}$

For every $p \in Q$,

add the rule $A_{pp} \to \varepsilon$

THE CHOMSKY NORMAL FORM

A context-free grammar is in Chomsky normal form if every rule is of the form:

$A \to BC$	B, C are variables (not the start var)
$A \to a$	a is a terminal
$S \to \varepsilon$	S is the start variable

$S \to 0S1$
$S \to TT$
$T \to \varepsilon$

$S_0 \to TU \mid \varepsilon$
$T \to 0$
$U \to SV \mid 1$
$S \to TU$
$V \to 1$

Theorem: If G is in CNF, $w \in L(G)$ and $|w| > 0$, then any derivation of w in G has length $2|w| - 1$

Theorem: Any context-free language can be generated by a context-free grammar in Chomsky normal form

"Can transform any CFG into Chomsky normal form"

Theorem: Any CFL can be generated by a CFG in Chomsky normal form

Algorithm:

1. Add a new start variable ($S_0 \to S$)
2. Eliminate all $A \to \varepsilon$ rules:
   For each occurrence of A on the RHS of a rule, add a new rule that removes that occurrence *(unless this new rule was previously removed)*
3. Eliminate all $A \to B$ rules:
   For each rule with B on LHS of a rule, add a new rule that puts A on the LHS instead *(unless this new rule was previously removed)*
4. Convert $A \to u_1u_2 \ldots u_k$ to $A \to u_1A_1$, $A_1 \to u_2A_2$, ...
   If u_i is a terminal, replace u_i with U_i and add $U_i \to u_i$

Convert the following into Chomsky normal form:

$A \to BAB \mid B \mid \varepsilon$
$B \to 00 \mid \varepsilon$

$S_0 \to A$
$A \to BAB \mid B \mid \varepsilon$
$B \to 00 \mid \varepsilon$

$\Rightarrow$

$S_0 \to A \mid \varepsilon$
$A \to BAB \mid B \mid BB \mid AB \mid BA$
$B \to 00$

$\Downarrow$

$S_0 \to BAB \mid 00 \mid BB \mid AB \mid BA \mid \varepsilon$
$A \to BAB \mid 00 \mid BB \mid AB \mid BA$
$B \to 00$

$\Downarrow$

$S_0 \to BC \mid DD \mid BB \mid AB \mid BA \mid \varepsilon$, $C \to AB$,
$A \to BC \mid DD \mid BB \mid AB \mid BA$, $B \to DD$, $D \to 0$

FORMAL DEFINITIONS

deterministic DFA
A ^ finite automaton ^ is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states (finite)

Σ is the alphabet (finite)

$\delta : Q \times \Sigma \to Q$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

Let $w_1, \ldots, w_n \in \Sigma$ and $w = w_1 \ldots w_n \in \Sigma^*$
Then **M accepts w** if there are $r_0, r_1, \ldots, r_n \in Q$, s.t.
1. $r_0 = q_0$
2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for i = 0, ..., n-1, and
3. $r_n \in F$

A *non-deterministic* finite automaton (NFA) is a 5-tuple $N = (Q, \Sigma, \delta, Q_0, F)$

Q is the set of states

Σ is the alphabet

$\delta : Q \times \Sigma_\varepsilon \to 2^Q$ is the transition function

$Q_0 \subseteq Q$ is the set of start states

$F \subseteq Q$ is the set of accept states

2^Q is the set of all possible subsets of Q
$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

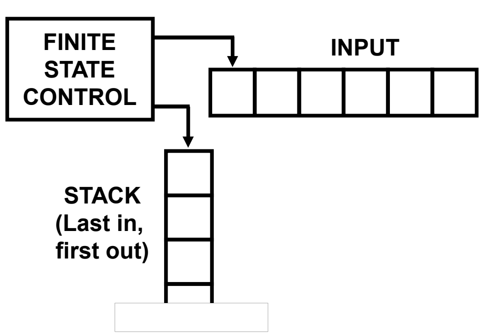
Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (ε = empty string)

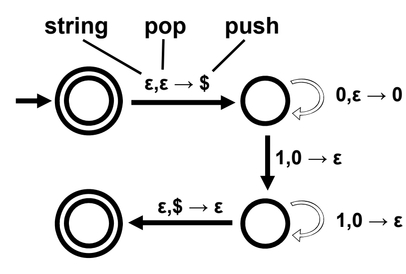
Then N accepts w if there are $r_0, r_1, \ldots, r_n \in Q$ such that
1. **$r_0 \in Q_0$**
2. **$r_{i+1} \in \delta(r_i, w_{i+1})$ for i = 0, ..., n-1, and**
3. **$r_n \in F$**

L(N) = the language recognized by N
= set of all strings machine N accepts

A language L is recognized by an NFA N if L = L(N).

PUSHDOWN AUTOMATA (PDA)





Definition: A (non-deterministic) PDA is a tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, where:

- **Q is a finite set of states**
- **Σ is the input alphabet**
- **Γ is the stack alphabet**
- $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow 2^{Q \times \Gamma_\varepsilon}$
- $q_0 \in Q$ **is the start state**
- $F \subseteq Q$ **is the set of accept states**

2^Q **is the set of subsets of Q and** $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

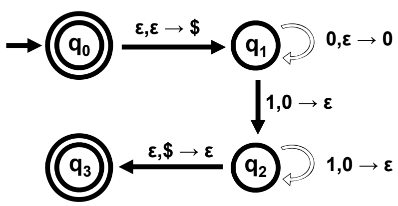
Let $w \in \Sigma^*$ and suppose w can be written as $w_1 \ldots w_n$ where $w_i \in \Sigma_\varepsilon$ (recall $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$)

Then P accepts w if there are

$r_0, r_1, \ldots, r_n \in Q$ and

$s_0, s_1, \ldots, s_n \in \Gamma^*$ (sequence of stacks) **such that**

1. **$r_0 = q_0$ and $s_0 = \varepsilon$** (**P** starts in q_0 with empty stack)
2. **For i = 0, ..., n-1:**
   $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$
   (**P** moves correctly according to state, stack and symbol read)
3. **$r_n \in F$** (**P** is in an accept state at the end of its input)



$Q = \{q_0, q_1, q_2, q_3\}$ $\quad \Sigma = \{0,1\}$ $\quad \Gamma = \{\$,0,1\}$

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow 2^{Q \times \Gamma_\varepsilon}$

$\delta(q_1,1,0) = \{ (q_2,\varepsilon) \}$ $\quad \delta(q_2,1,1) = \varnothing$

$\delta(q_2,\varepsilon,\$) = \{ (q_3,\varepsilon) \}$

CONTEXT-FREE GRAMMARS

A context-free grammar (CFG) is a tuple $G = (V, \Sigma, R, S)$, where:

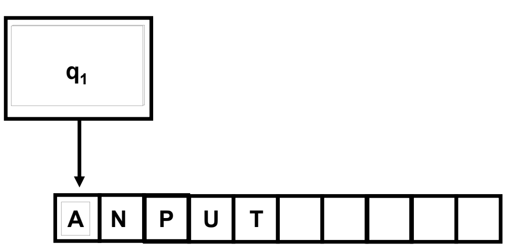
- **V is a finite set of variables**
- **Σ is a finite set of terminals (disjoint from V)**
- **R is set of production rules of the form $A \rightarrow W$, where $A \in V$ and $W \in (V \cup \Sigma)^*$**
- **$S \in V$ is the start variable**

$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$ Strings Generated by G

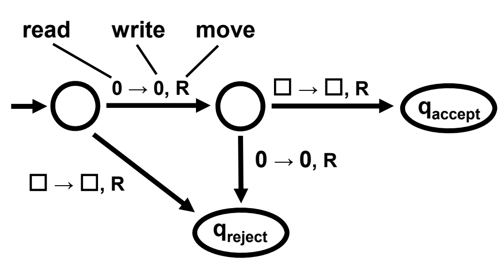
$G = \{ \{S\}, \{0,1\}, R, S \}$ $\quad R = \{ S \rightarrow 0S1, S \rightarrow \varepsilon \}$

$L(G) = \{ 0^n1^n \mid n \geq 0 \}$ Strings Generated by G

TURING MACHINE



INFINITE TAPE



Definition: A Turing Machine is a 7-tuple
$T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

Q is a finite set of states

Σ is the input alphabet, where $\square \notin \Sigma$

Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$

$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L,R\}$

$q_0 \in Q$ is the start state

$q_{accept} \in Q$ is the accept state

$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$

A Turing Machine M accepts input w if there is a sequence of configurations $C_1, \ldots, C_k$ such that

1. C_1 is a *start* configuration of M on input w, ie C_1 is q_0w
2. each C_i *yields* C_{i+1}, ie M can legally go from C_i to C_{i+1} in a single step

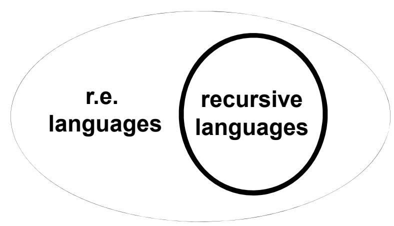
ua q_i bv	*yields*	u q_j acv	if	$\delta(q_i, b) = (q_j, c, L)$
ua qi bv	*yields*	uac q_j v	if	$\delta(q_i, b) = (q_j, c, R)$

A Turing Machine M accepts input w if there is a sequence of configurations $C_1, \ldots, C_k$ such that

1. C_1 is a *start* configuration of M on input w, ie C_1 is q_0w
2. each C_i *yields* C_{i+1}, ie M can legally go from C_i to C_{i+1} in a single step
3. C_k is an *accepting* configuration, ie the state of the configuration is q_{accept}

A language is called Turing-recognizable or recursively enumerable (r.e.) or semi-decidable if some TM recognizes it

A language is called decidable or recursive if some TM decides it



Theorem: If A and $\neg$A are r.e. then A is recursive

A_{DFA} = { (B, w) | B is a DFA that accepts string w }

Theorem: A_{DFA} is decidable

Proof Idea: Simulate B on w

A_{NFA} = { (B, w) | B is an NFA that accepts string w }

Theorem: A_{NFA} is decidable

A_{CFG} = { (G, w) | G is a CFG that generates string w }

Theorem: A_{CFG} is decidable

Proof Idea: Transform G into Chomsky Normal Form. Try all derivations of length up to 2|w|-1

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Happy studying!