15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

THE PUMPING LEMMA FOR REGULAR LANGUAGES and REGULAR EXPRESSIONS

TUESDAY Jan 21

WHICH OF THESE ARE REGULAR ?

B = $\{0^n1^n \mid n \geq 0\}$

C = { w | w has equal number of occurrences of 01 and 10 }

D = { w | w has equal number of 1s and 0s}

THE PUMPING LEMMA

Let L be a regular language with $|L| = \infty$

Then there is a positive integer P s.t.

if $w \in L$ and $|w| \geq P$
then can write w = xyz, where:

1. **$|y| > 0$ (y isn't ε)**
2. **$|xy| \leq P$**
3. **For *every* $i \geq 0$, $xy^iz \in L$**

Why is it called the pumping lemma? The word w gets PUMPED into something longer…

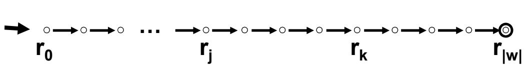
Proof: Let M be a DFA that recognizes L

Let P be the number of states in M

Assume $w \in L$ is such that $|w| \geq P$

We show: w = xyz

1. **$|y| > 0$**
2. **$|xy| \leq P$**
3. **$xy^iz \in L$ for *all* $i \geq 0$**



There must be j and k such that $j < k \leq P$, and $r_j = r_k$ (why?) (Note: k - j > 0)

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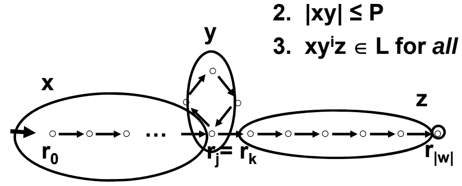


There must be j and k such that $j < k \leq P$, and $r_j = r_k$

USING THE PUMPING LEMMA

Let's prove that $B = \{0^n1^n \mid n \geq 0\}$ is not regular

Assume B is regular. Let $w = 0^P1^P$

If B is regular, can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, and for any $i \geq 0$, xy^iz is *also* in B

y must be all 0s: Why? **$|xy| \leq P$**

xyyz has more 0s than 1s

Contradiction!

USING THE PUMPING LEMMA

D = { w | w has equal number of 1s and 0s} is not regular

Assume D is regular. Let $w = 0^P1^P$ (w is in D!)

If D is regular, can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, where for any $i \geq 0$, xy^iz is *also* in D

y must be all 0s: Why? **$|xy| \leq P$**

xyyz has more 0s than 1s

Contradiction!

WHAT DOES C LOOK LIKE?

C = { w | w has equal number of occurrences of 01 and 10}

= { w | w = 1, w = 0, w = ε or w starts with a 0 and ends with a 0 or w starts with a 1 and ends with a 1 }

$$1 \cup 0 \cup \varepsilon \cup 0(0\cup 1)^*0 \cup 1(0\cup 1)^*1$$

REGULAR EXPRESSIONS
(expressions representing languages)

σ is a regexp representing $\{\sigma\}$

ε is a regexp representing $\{\varepsilon\}$

$\varnothing$ is a regexp representing $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 \cup R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

PRECEDENCE

$$* \quad \cdot \quad \cup$$

EXAMPLE

$$R_1^*R_2 \cup R_3 = ((R_1^*)R_2) \cup R_3$$

{ w | w has exactly a single 1 }

What language does ∅* represent?

{ w | w has length ≥ 3 and its 3rd symbol is 0 }

{ w | every odd position of w is a 1 }

EQUIVALENCE

L can be represented by a regexp ⇔ L is regular

1. **L can be represented by a regexp ⇒ L is regular**

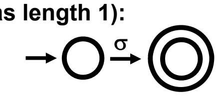
2. **L can be represented by a regexp ⇐ L is a regular language**

1. **Given regular expression R, we show there exists NFA N such that R represents L(N)**

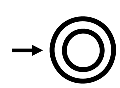
Induction on the *length* of R:

Base Cases (R has length 1):

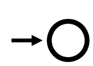
R = σ



R = ε



R = ∅



Inductive Step:

Assume R has length k > 1,
and that every regular expression of length < k represents a regular language

Three possibilities for R:

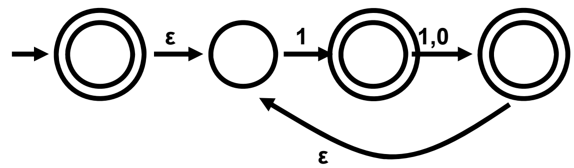
$R = R_1 \cup R_2$ (Union Theorem!)

$R = R_1 R_2$ (Concatenation)

$R = (R_1)^*$ (Star)

Therefore: L can be represented by a regexp
$\Rightarrow$ L is regular

Give an NFA that accepts the language represented by $(1(0 \cup 1))^*$

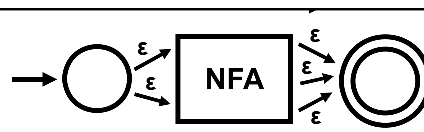


2. **L can be represented by a regexp**

$\Leftarrow$

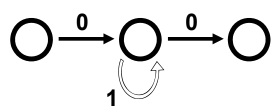
L is a regular language

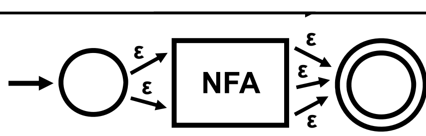
Proof idea: Transform an NFA for L into a regular expression by removing states and re-labeling arrows with regular expressions



Add unique and distinct start and accept states
While machine has more than 2 states:

Pick an internal state, rip it out and re-label the arrows with regexps, to account for the missing state





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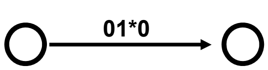
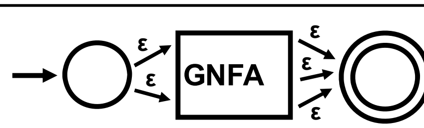
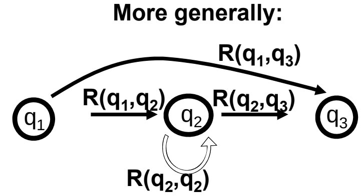
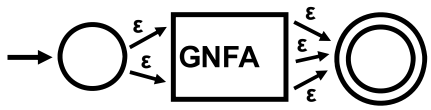




While machine has more than 2 states:

More generally:





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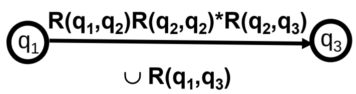
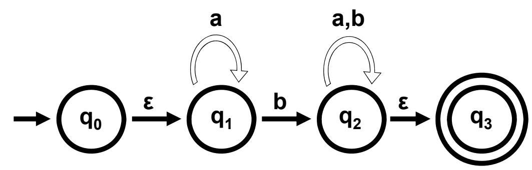




$R(q_0,q_3)$ =
represents L(N)

Formally: Add q_{start} and q_{accept} to create G (GNFA)

Run CONVERT(G): (Outputs a regexp)

If #states = 2

return the expression on the arrow going from q_{start} to q_{accept}

Formally: Add q_{start} and q_{accept} to create G (GNFA)

Run CONVERT(G): (Outputs a regexp)

If #states > 2

select $q_{rip} \in Q$ different from q_{start} and q_{accept}

$$\left.\begin{array}{l}\textbf{define } Q' = Q - \{q_{rip}\} \\ \textbf{define } R' \textbf{ as:}\end{array}\right\} \textbf{Defines: } G' \textbf{ (GNFA)}$$

$$R'(q_i,q_j) = R(q_i,q_{rip})R(q_{rip},q_{rip})^*R(q_{rip},q_j) \cup R(q_i,q_j)$$

(R′ = the regexps for edges in G′)

We note that G and G′ are *equivalent*

return CONVERT(G′)

Claim: CONVERT(G) is *equivalent* to G

Proof by induction on k (number of states in G)

Base Case:

✓ **k = 2**

Inductive Step:

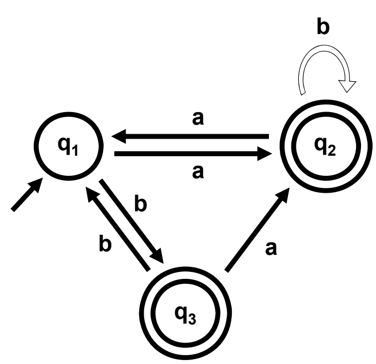
Assume claim is true for k-1 state GNFAs

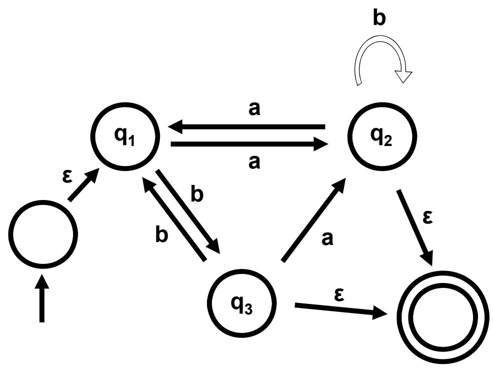
Recall that G and G′ are *equivalent*

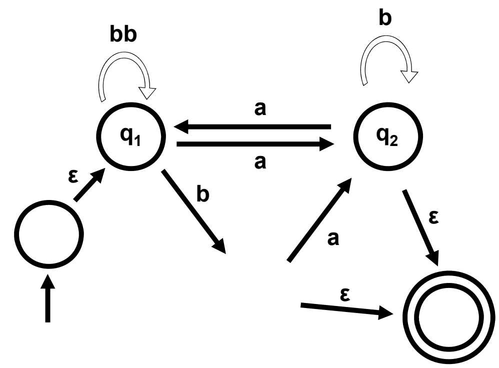
But, by the induction hypothesis, G′ is *equivalent* to CONVERT(G′)

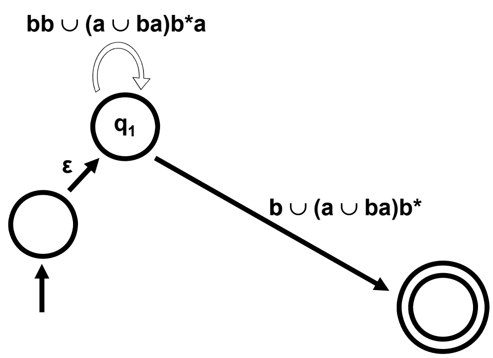
Thus: CONVERT(G′) *equivalent* to CONVERT(G)

QED



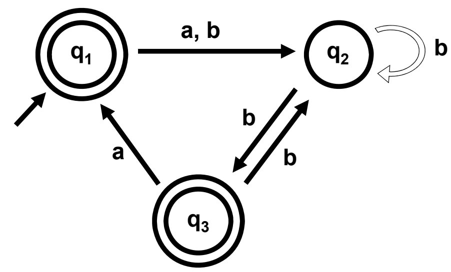


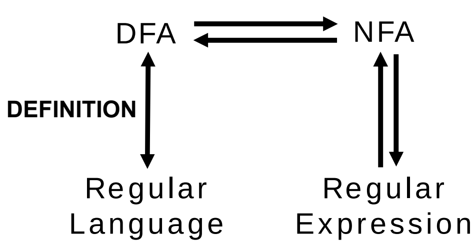




(bb ∪ (a ∪ ba)b*a)* (b ∪ (a ∪ ba)b*)

Convert the NFA to a regular expression





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Finish Chapter 1 of the book for next time