Spin Transformations of Discrete Surfaces

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(joint work with Ulrich Pinkall, Peter Schröder)

Immersions of a surface M can be expressed via maps $f : M \to \mathrm{Im}\mathbb{H}$ into the imaginary part of the quaternions. Two immersions f and $\tilde{f}$ are called *spin equivalent* [1] if there exists some function $\lambda : M \to \mathbb{H}$ such that

$$d\tilde{f} = \bar{\lambda} df \lambda, \tag{1}$$

and it is clear that spin equivalence implies conformal equivalence since the induced Riemannian metrics are related by a positive scaling: $|d\tilde{f}|^2 = |\lambda|^4 |df|^2$. In digital geometry processing, preservation of conformal structure provides a valuable tool for maintining signal integrity (e.g., aspect ratio of mesh elements). It is therefore natural to seek an analog of spin transformations in the discrete setting.

Kamberov, Pedit, and Pinkall observed that, as a condition on λ, equation (1) is equivalent to

$$0 = d(\bar{\lambda} df \lambda) = d\bar{\lambda} \wedge df \lambda - \bar{\lambda} df \wedge d\lambda = -2\mathrm{Im}(\bar{\lambda} df \wedge d\lambda)$$

whenever M is simply-connected [1]. In other words, $\bar{\lambda} df \wedge d\lambda$ must be equal to some real function ρ, which leads to the integrability condition $\bar{\lambda} df \wedge d\lambda = \rho |df|^2$. The function ρ represents the change in *mean curvature half-density* between an initial immersion f and its spin transform $\tilde{f}$, i.e., $\tilde{H}|d\tilde{f}| = H|df| + \rho|df|$.

For discretization we consider the alternate formulation

$$D\lambda = \rho\lambda \tag{2}$$

where D is the self-adjoint elliptic operator given by

$$D\lambda := -\frac{df \wedge d\lambda}{|df|^2},$$

which we call the *quaternionic Dirac operator*, since it is locally equivalent to the standard Dirac operator for a spin-1/2 particle in the plane [2]. We can now specify a surface (up to isometry and uniform scaling) by prescribing a change $\rho|df|$ in mean curvature half-density and solving the eigenvalue problem

$$(D - \rho)\lambda = \gamma\lambda \tag{3}$$

for the smallest eigenvalue γ. The resulting pair $(\lambda, \rho+\gamma)$ satisfies our integrability condition (2), with a small constant shift in the prescribed curvature change.

Consider any surface M composed of piecewise smooth faces σ with linear edges e_{ij}. If λ is also linear along edges, then we have

$$\int_\sigma D\lambda |df|^2 = \int_\sigma d(df\lambda) = \sum_{e_{ij}\in\partial\sigma} \int_{e_{ij}} df\lambda = \sum_{e_{ij}\in\partial\sigma} (f_j - f_i)\frac{\lambda_i + \lambda_j}{2},$$

which gives us one way to discretize D. In particular, when M is simplicial we end up with the sparse linear operator

$$\mathsf{D}_{ij} = -\frac{1}{2A_i} e_j$$

for each edge e_j of each 2-simplex σ_i. In a similar fashion, the operator ρ can be discretized as $\mathsf{R} = \mathsf{PB}$ where $\mathsf{B}_{ij} = 1/3$ for each vertex v_j of face σ_i and $\mathsf{P}_{ii} = \rho_i$ for some prescribed value ρ_i on each face. The resulting discrete operator $\mathsf{A} = \mathsf{D} - \mathsf{R}$ is rectangular, and so there are several options for formulating the eigenvalue problem (3). The system $\mathsf{B}^*\mathsf{A} = \mathsf{B}^*\mathsf{B}$ obtained by averaging values from faces back to vertices is problematic because this local averaging artificially places high-frequency modes in the null space of the system, distorting solutions. An alternative is to note that the generalized eigenvalue problem $D^2\lambda = \gamma D\lambda$ shares solutions with our original problem (3), and can be discretized as $\mathsf{D}^*\mathsf{D}\lambda = \gamma\mathsf{B}^*\mathsf{D}\lambda$. In practice the spectrum of eigenvalues computed via this discretization closely matches the spectrum of the smooth operator, and solutions appear to converge linearly with respect to the mean edge length of simplices [2].

One might also consider discretizations that capture the essential structure of the smooth theory. For instance, Springborn, Schröder, and Pinkall consider equivalence classes of discrete immersions where edge lengths are related by positive scale factors at vertices [3]. This notion of *discrete conformal equivalence* leads to a theory that closely mimics the smooth setting [4], but in terms of constructive algorithms is limited to immersions into the plane or sphere. Therefore, a natural question to ask is whether this same notion of conformal equivalence can be extended to all spin transformations.

References

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