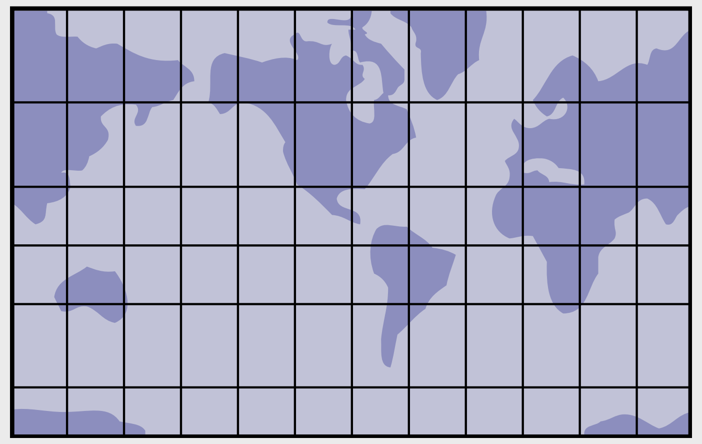
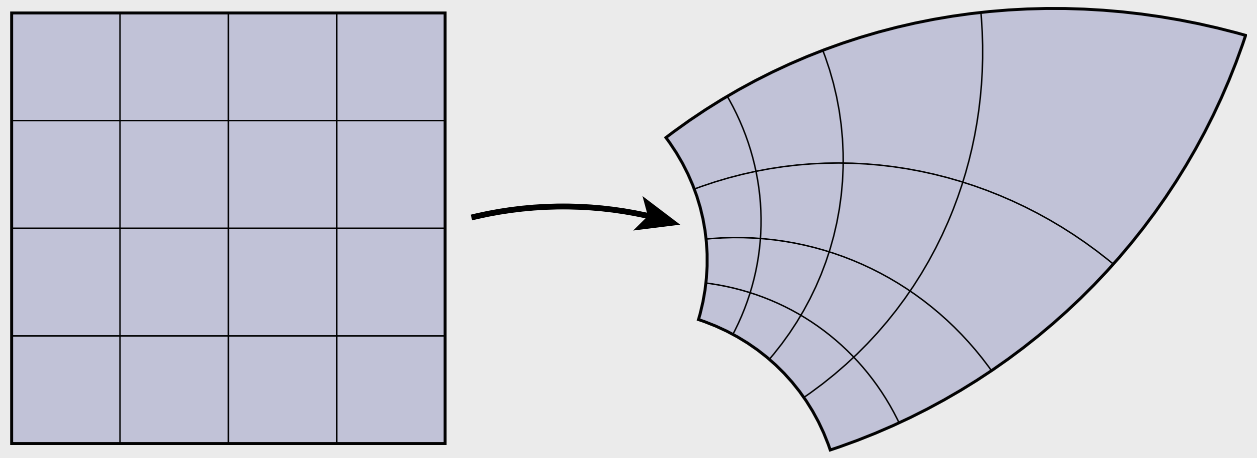
Conformal Geometry Processing

Keenan Crane • Symposium on Geometry Processing Grad School • Summer 2017

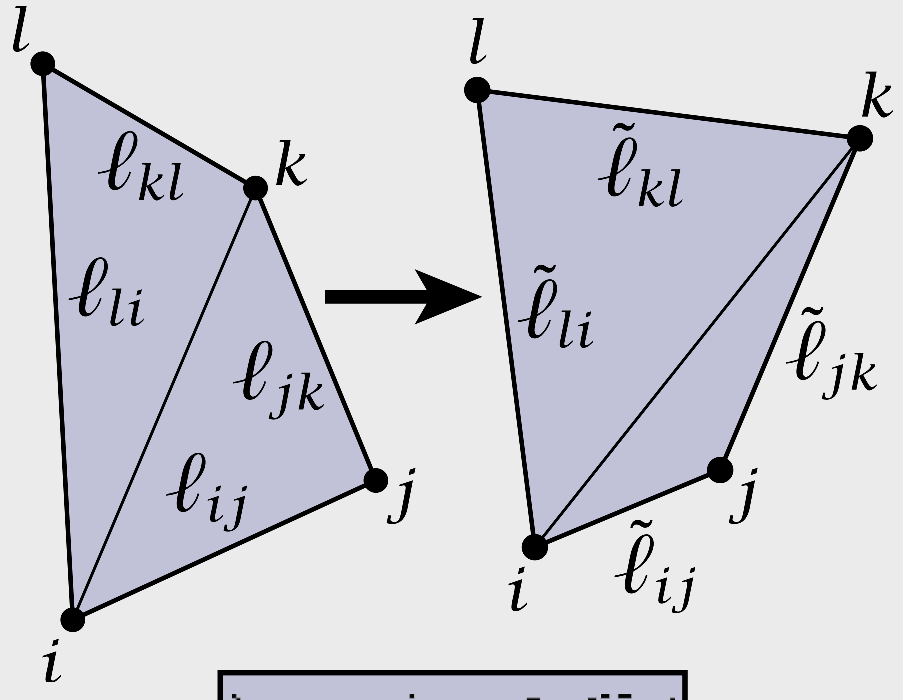
Outline



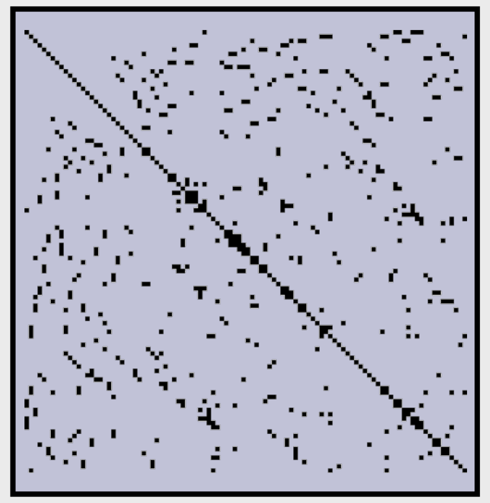
PART I: OVERVIEW



PART II: SMOOTH THEORY



PART III: DISCRETIZATION

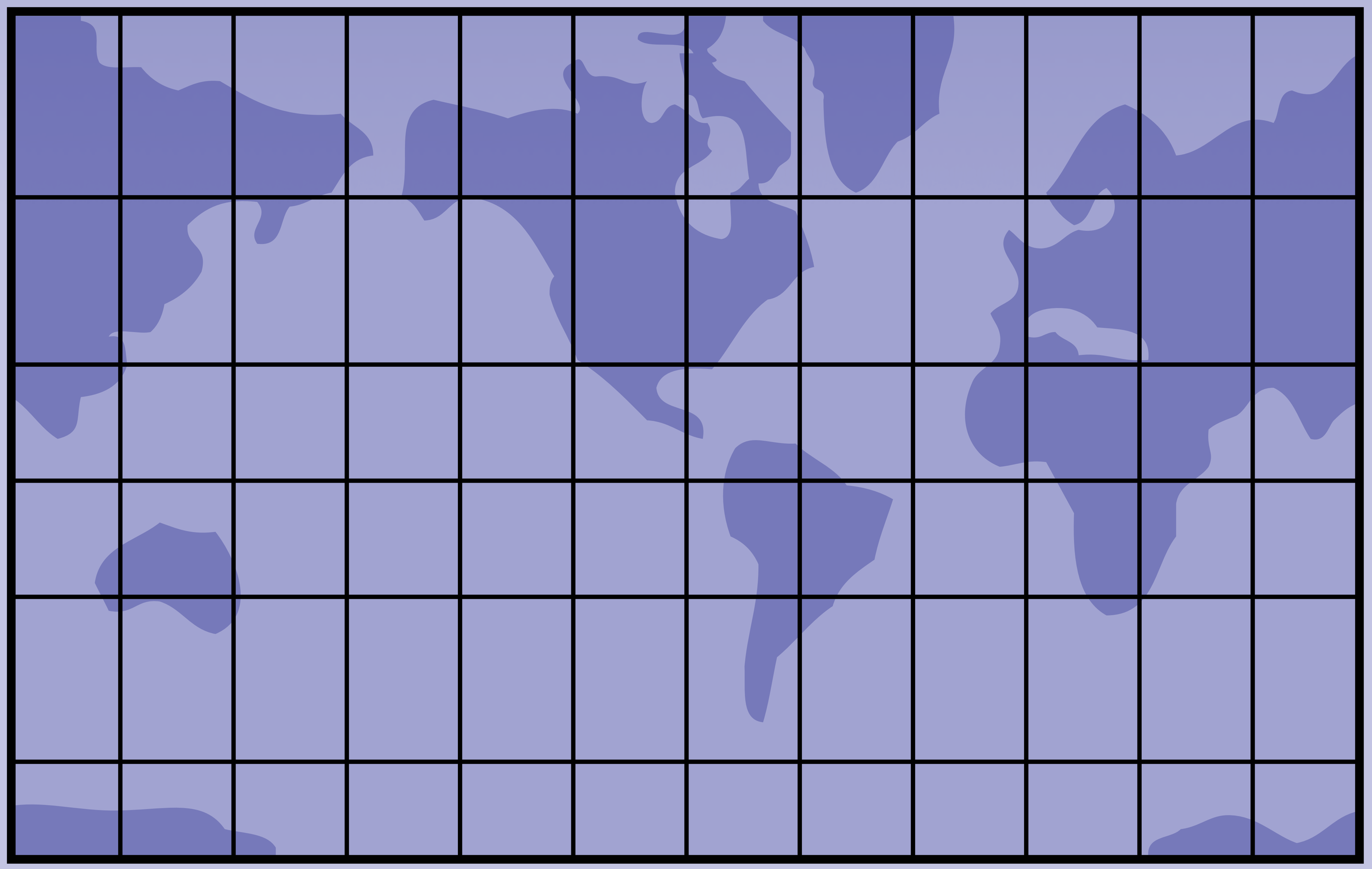


PART IV: ALGORITHMS

ACHTUNG!

These slides are a work in progress and ***there will be errors***. As always, *your brain* is the best tool for determining whether statements are actually true! Please do not hesitate to stop and ask for clarifications/corrections, or send email to `kmcrane@cs.cmu.edu`.

PART I: OVERVIEW



CONFORMAL GEOMETRY PROCESSING

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Motivation: Mapmaking Problem

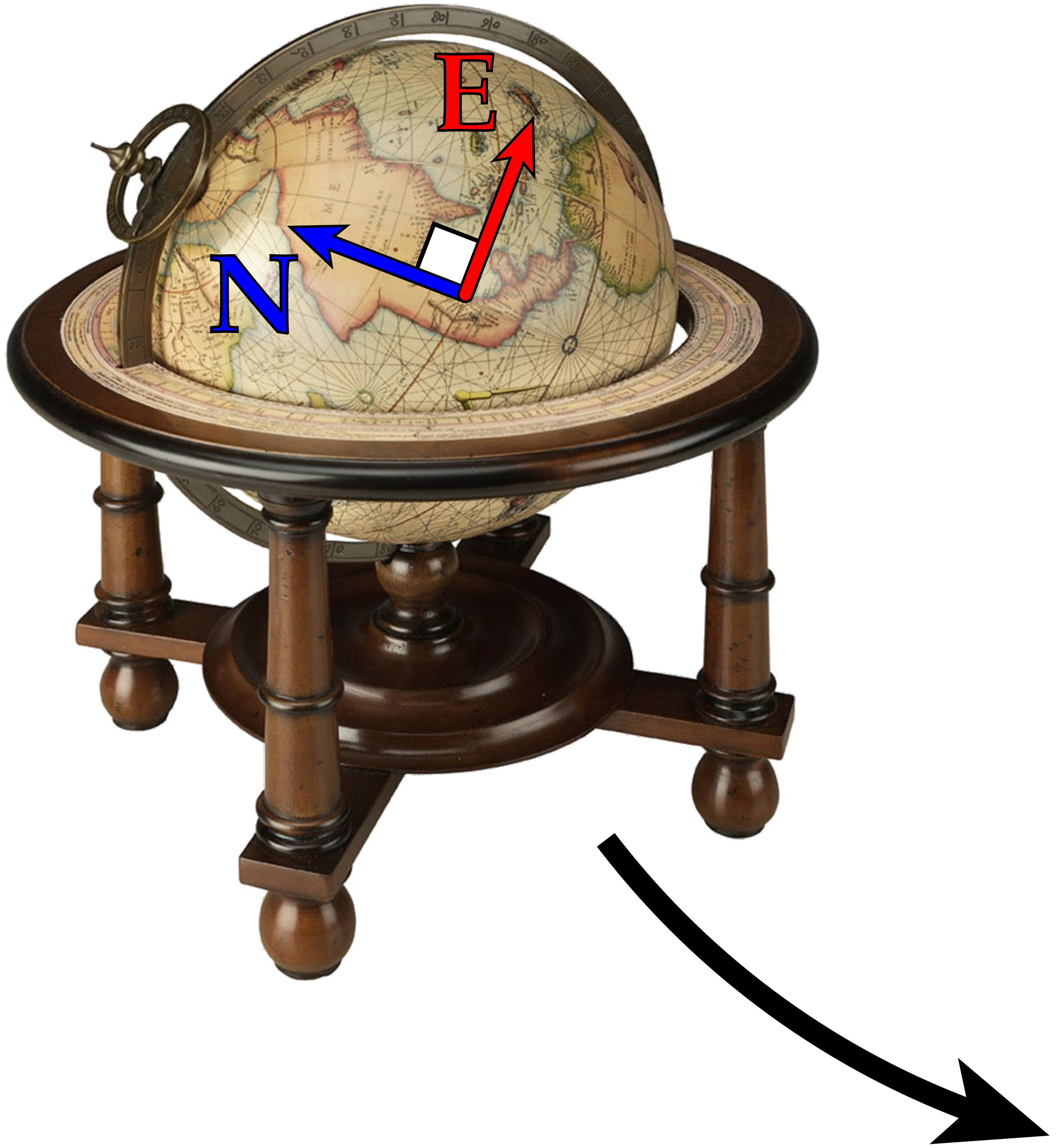
- How do you make a flat map of the round globe?
- Hard to do! Like trying to flatten an orange peel…



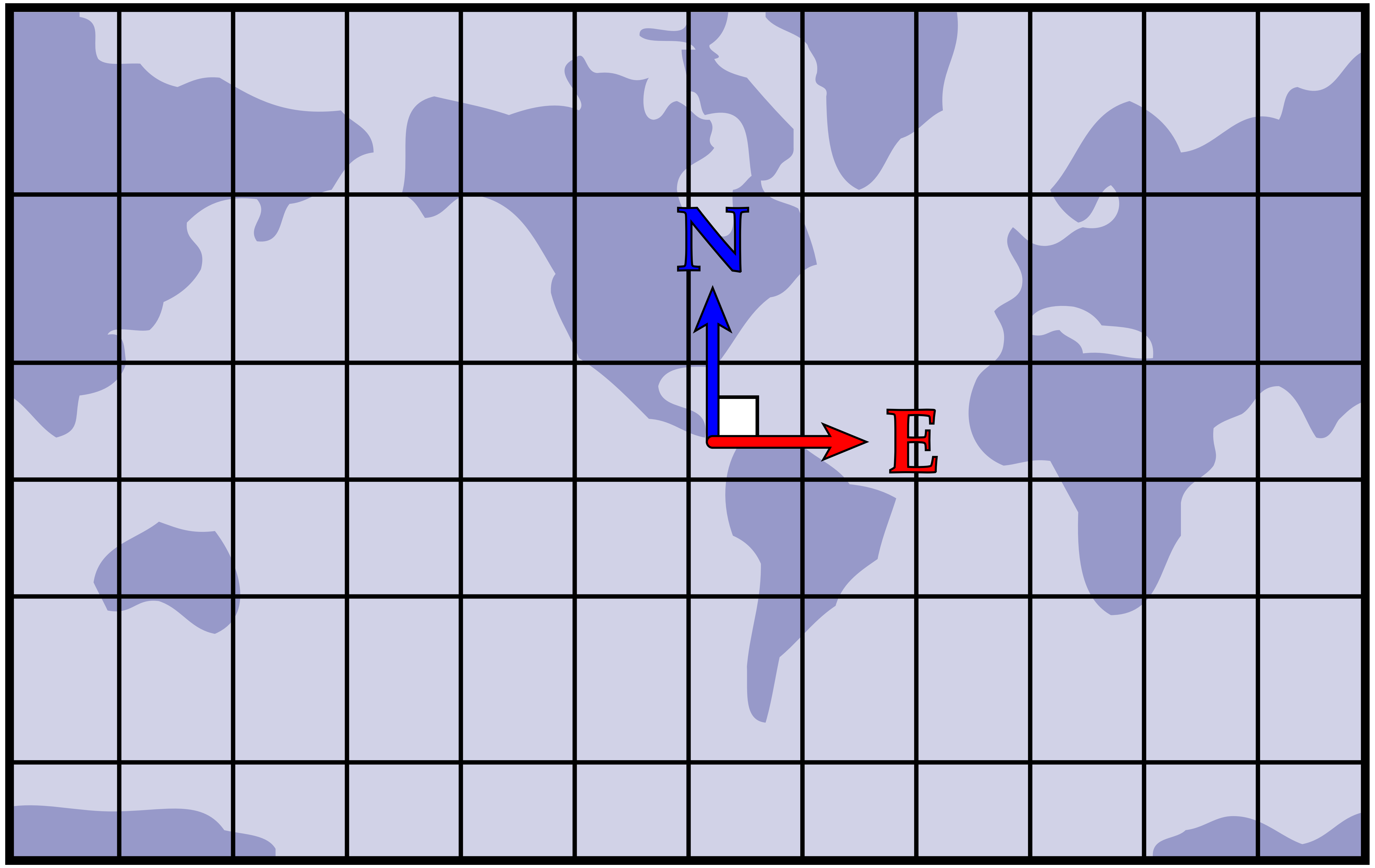
Impossible without some kind of distortion and/or cutting.

Conformal Mapmaking

- Amazing fact: can always make a map that exactly preserves **angles**.

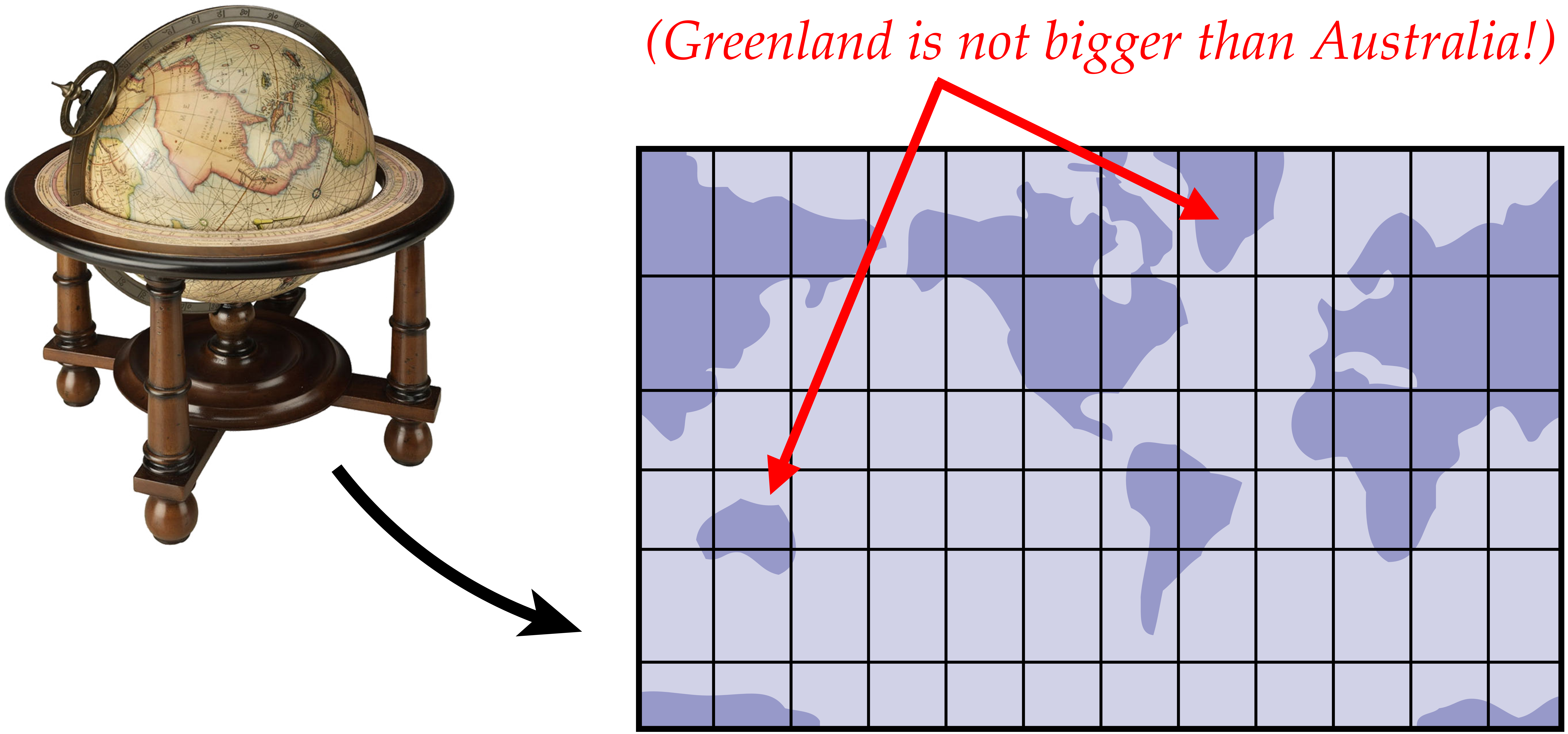


(Very useful for navigation!)



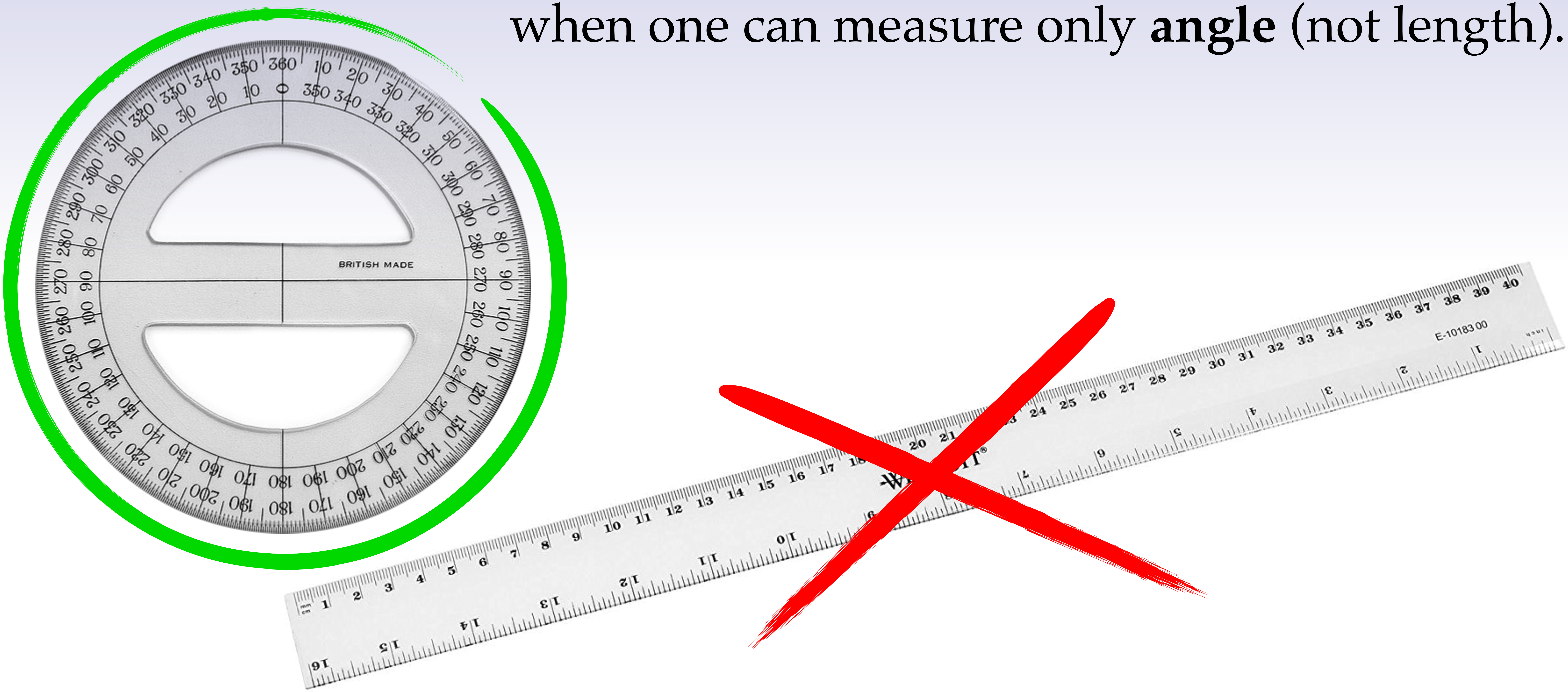
Conformal Mapmaking

- However, **areas** may be badly distorted…

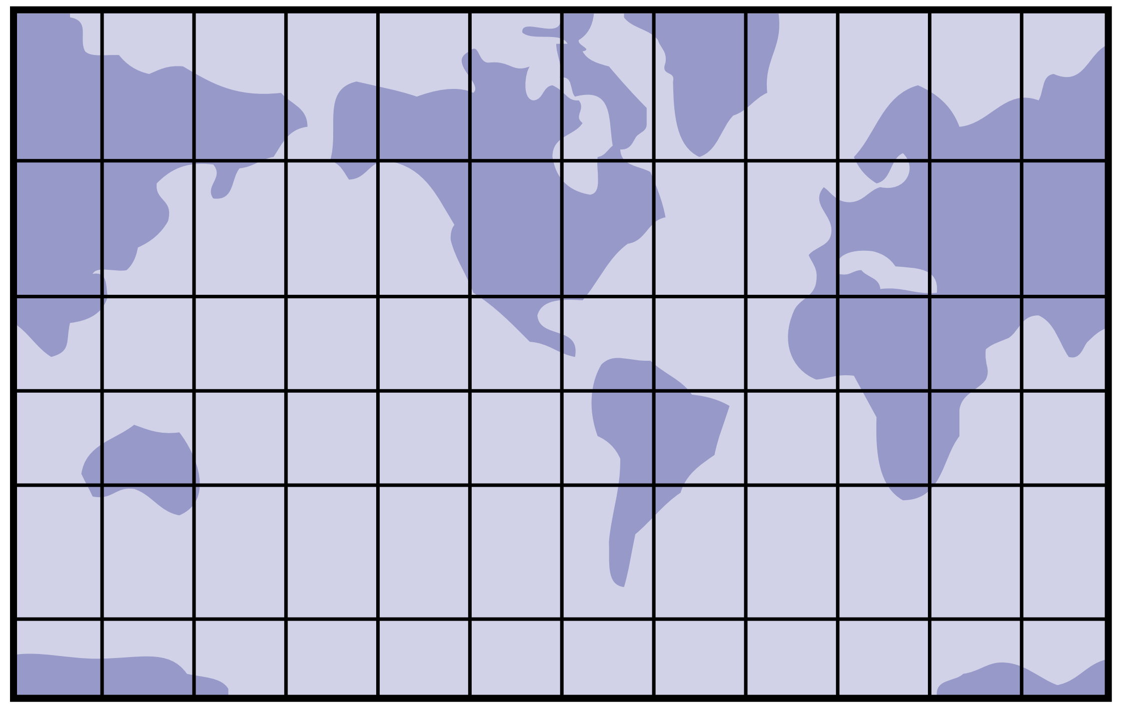


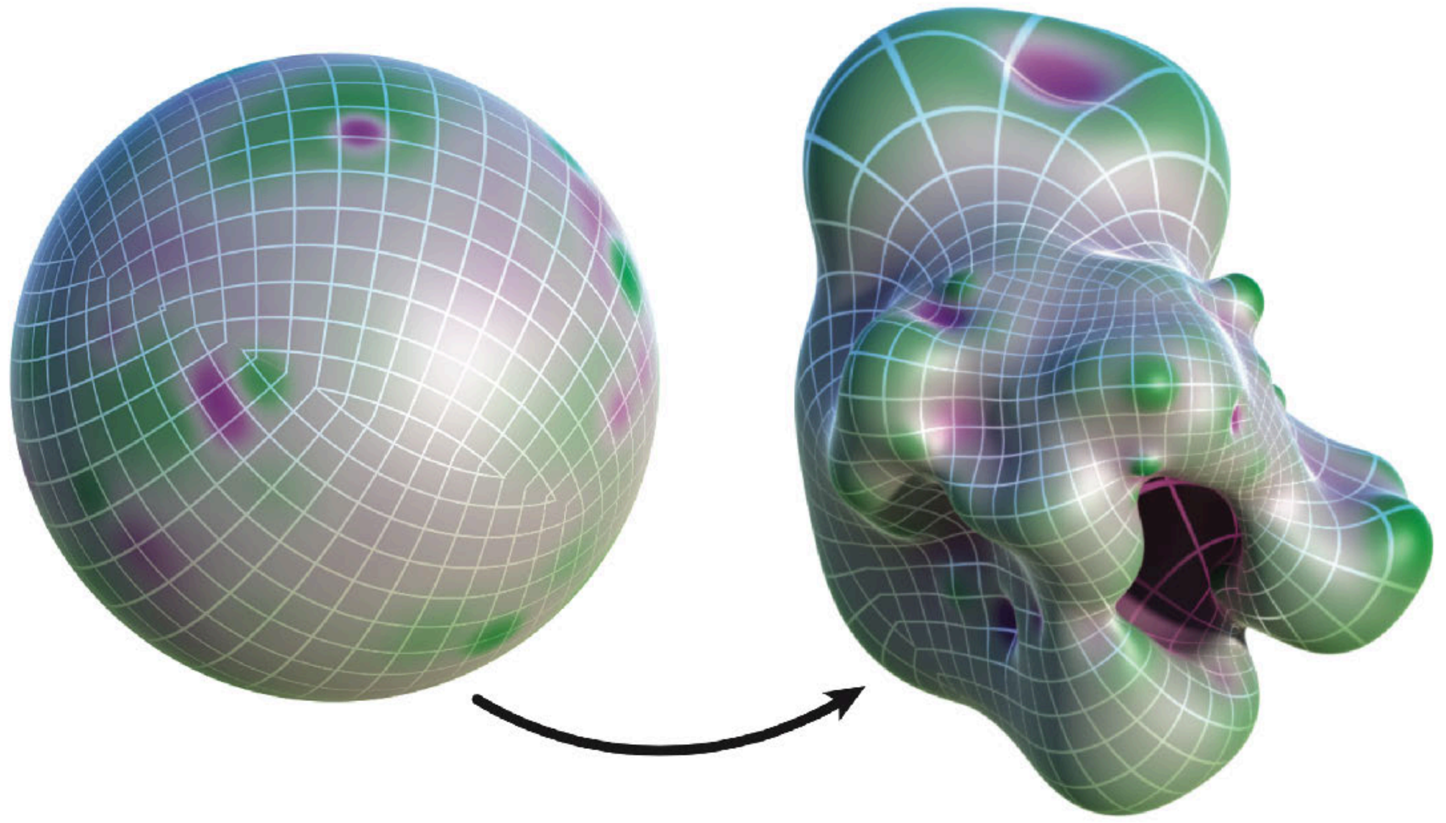
Conformal Geometry

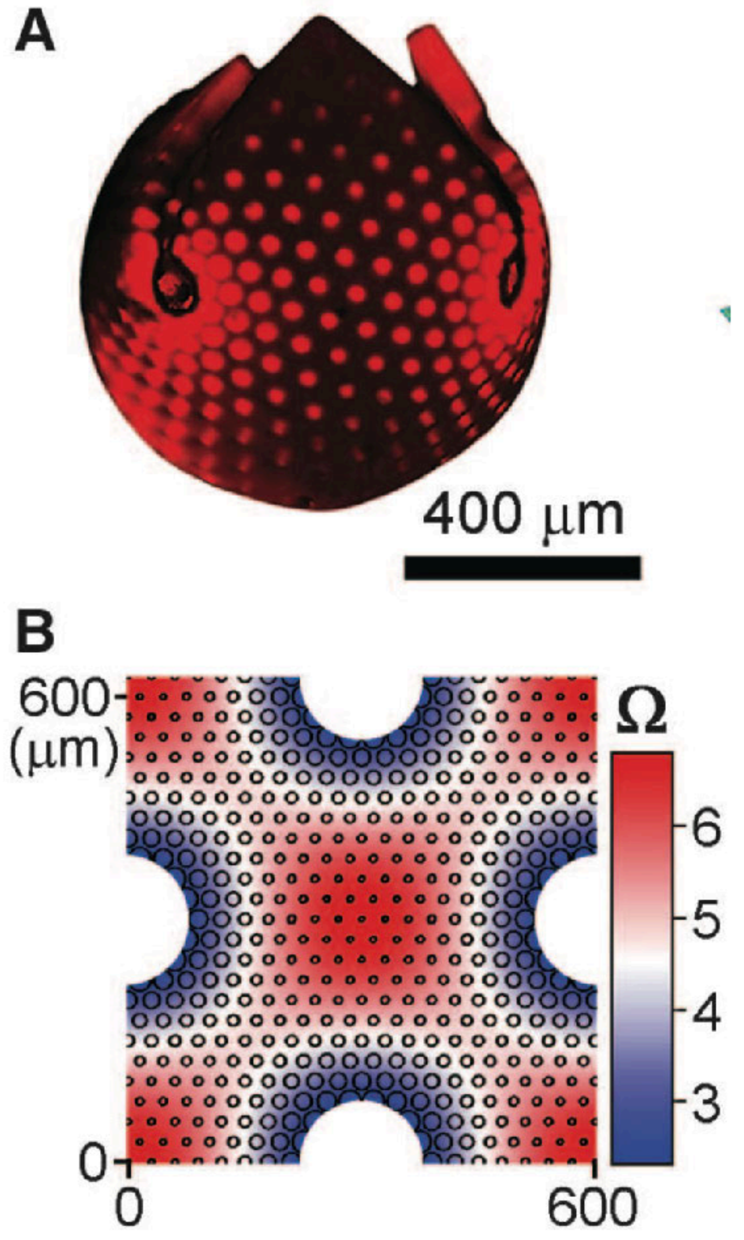
More broadly, *conformal geometry* is the study of shape when one can measure only **angle** (not length).

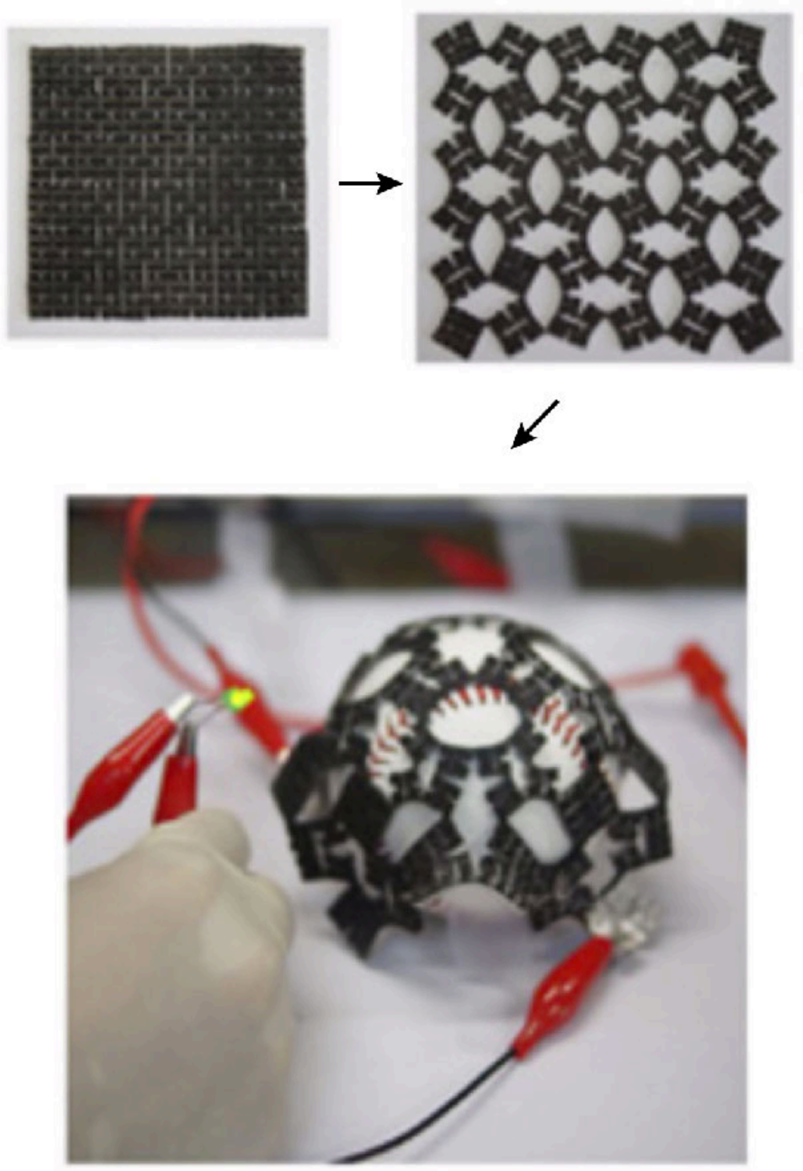


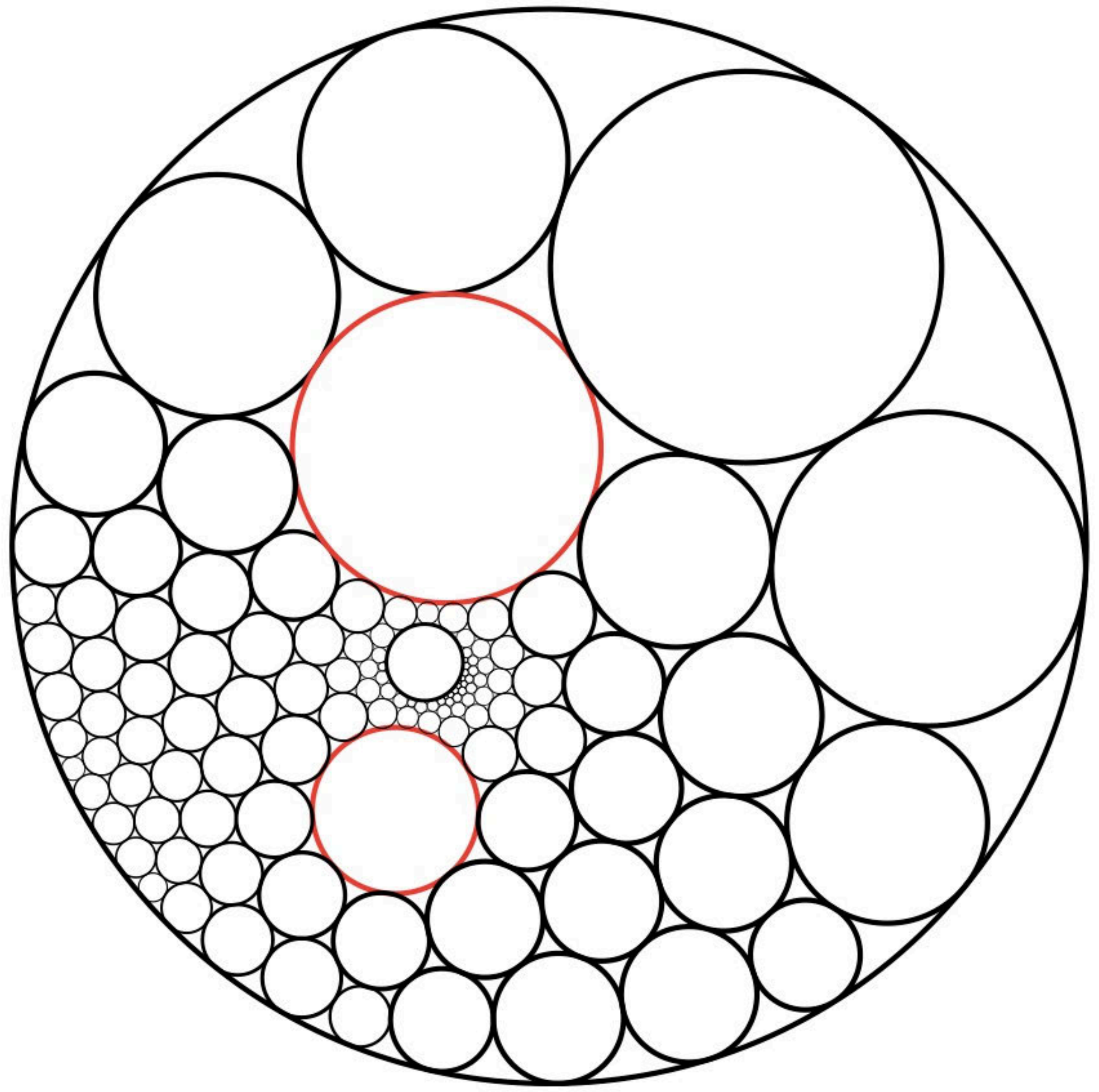
Conformal Geometry — Visualized

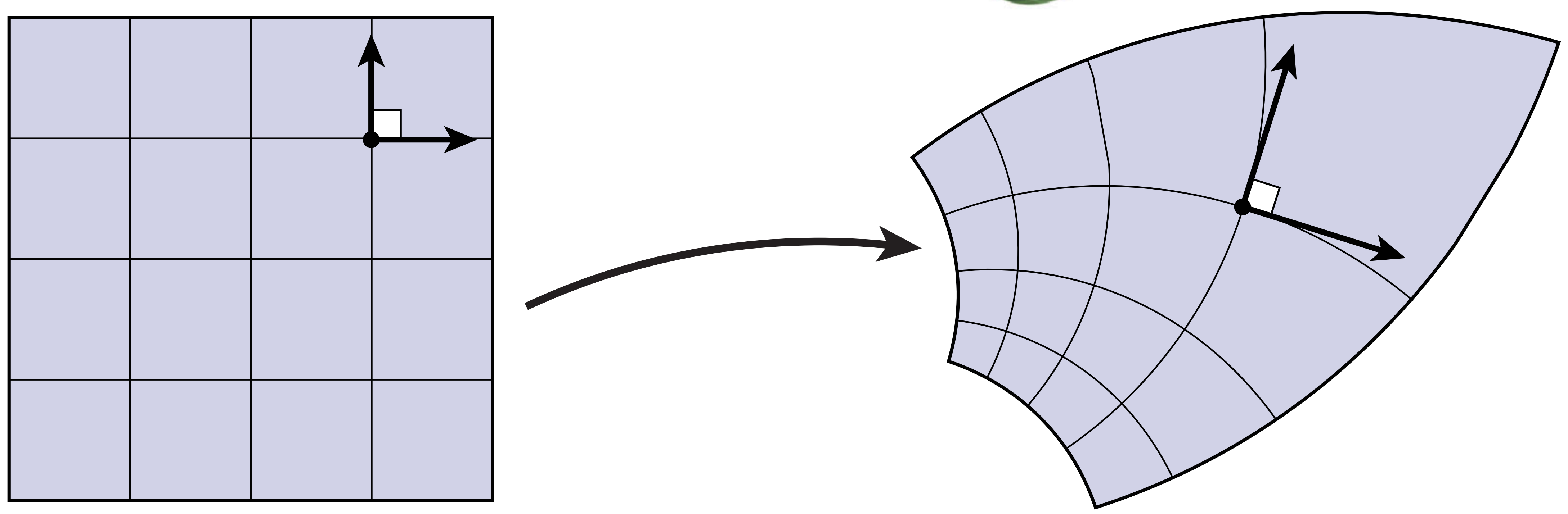


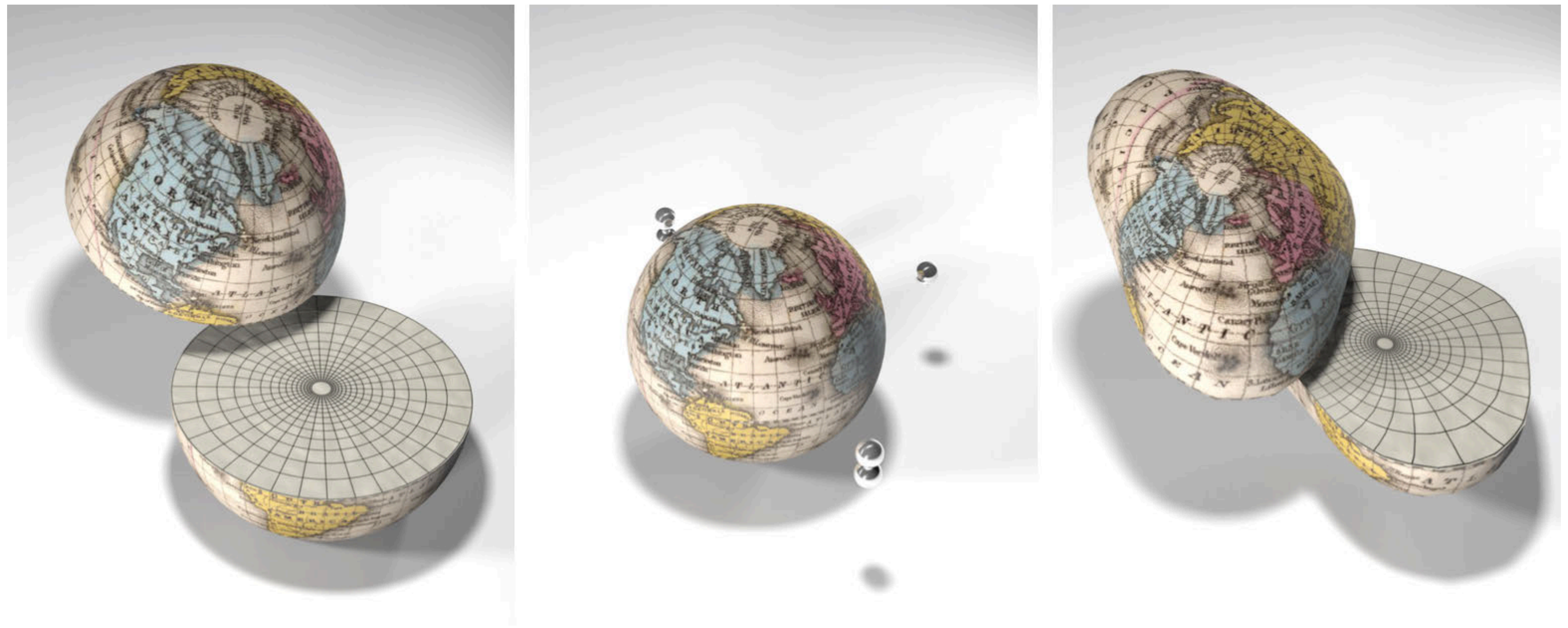






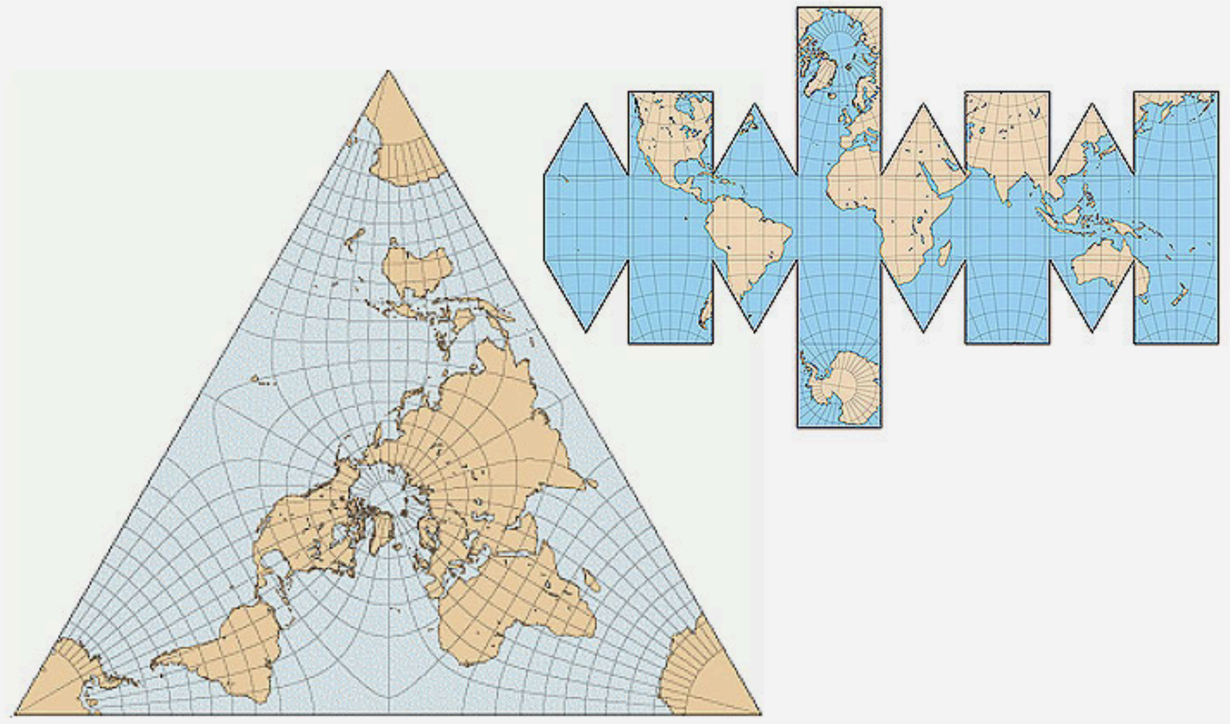




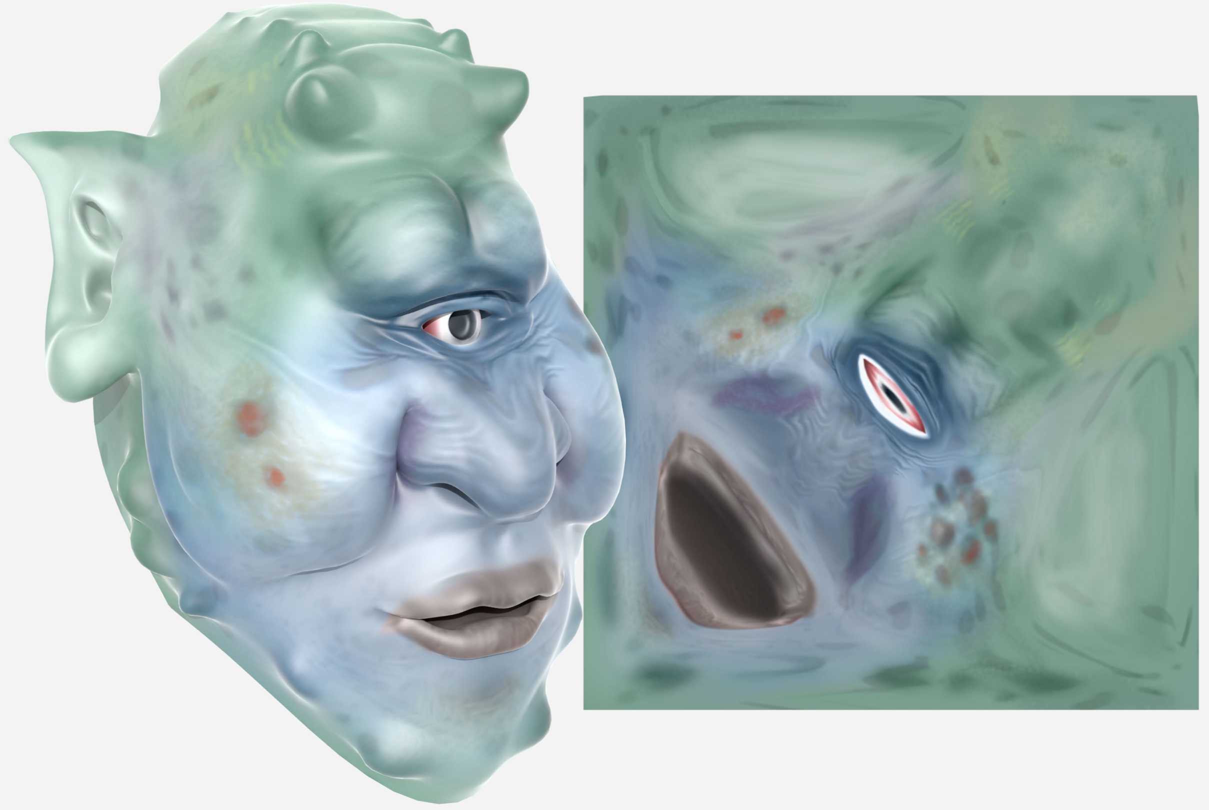


Applications of Conformal Geometry Processing

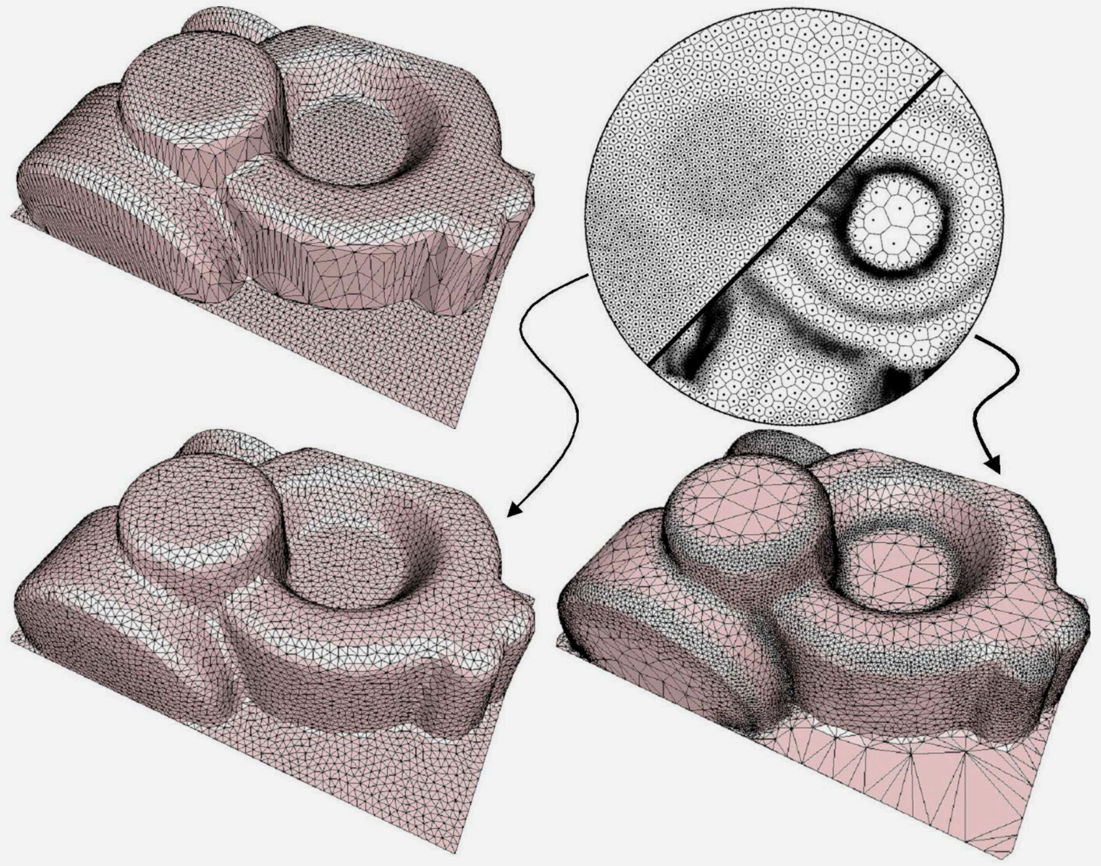
Basic building block for *many* applications…



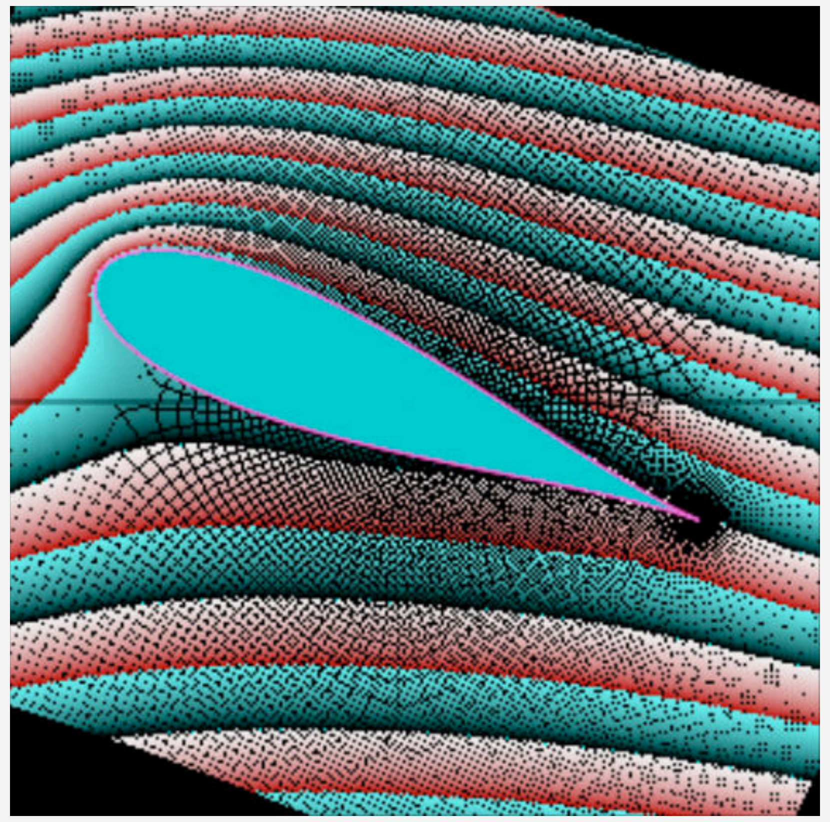
CARTOGRAPHY



TEXTURE MAPPING

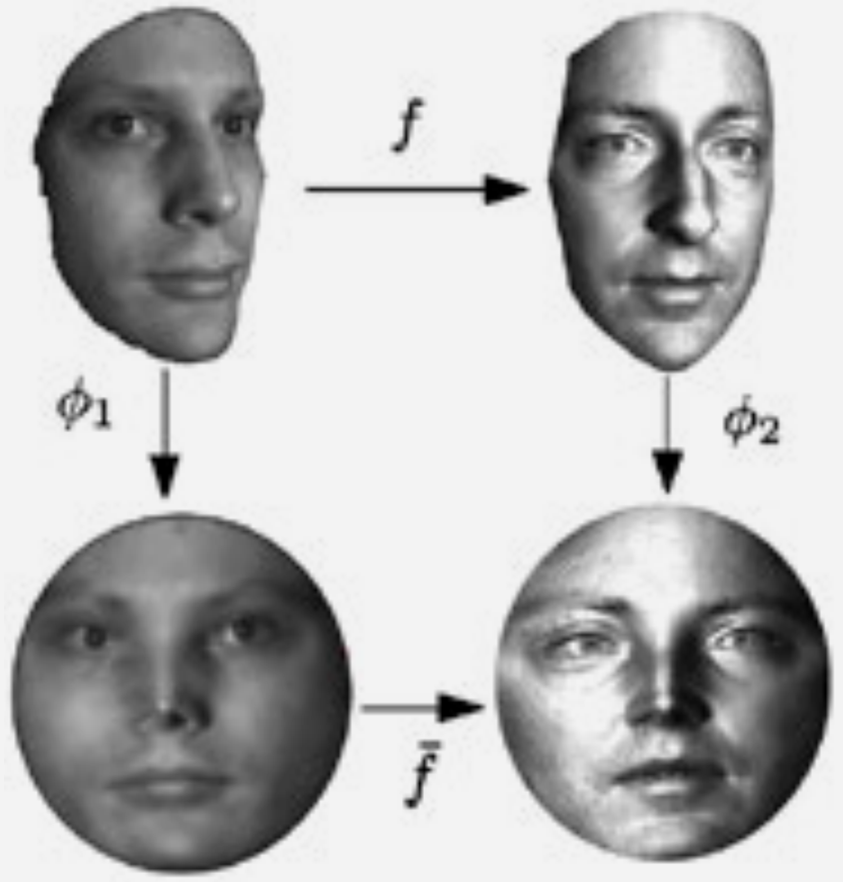


REMESHING

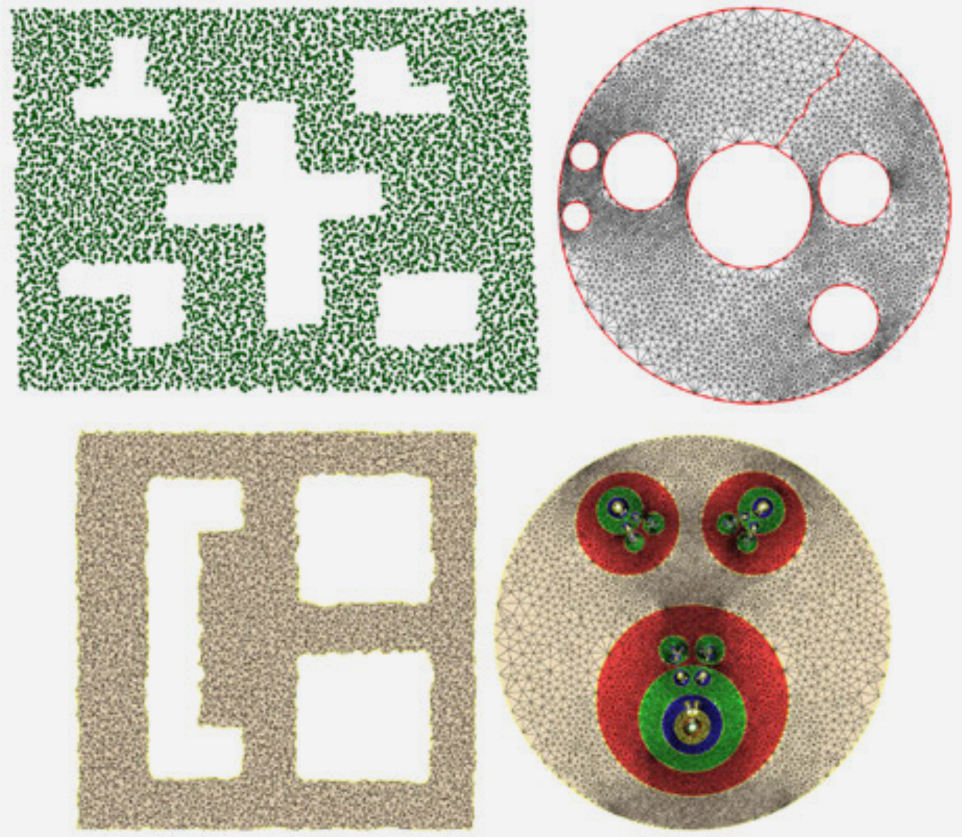


SIMULATION

3D FABRICATION



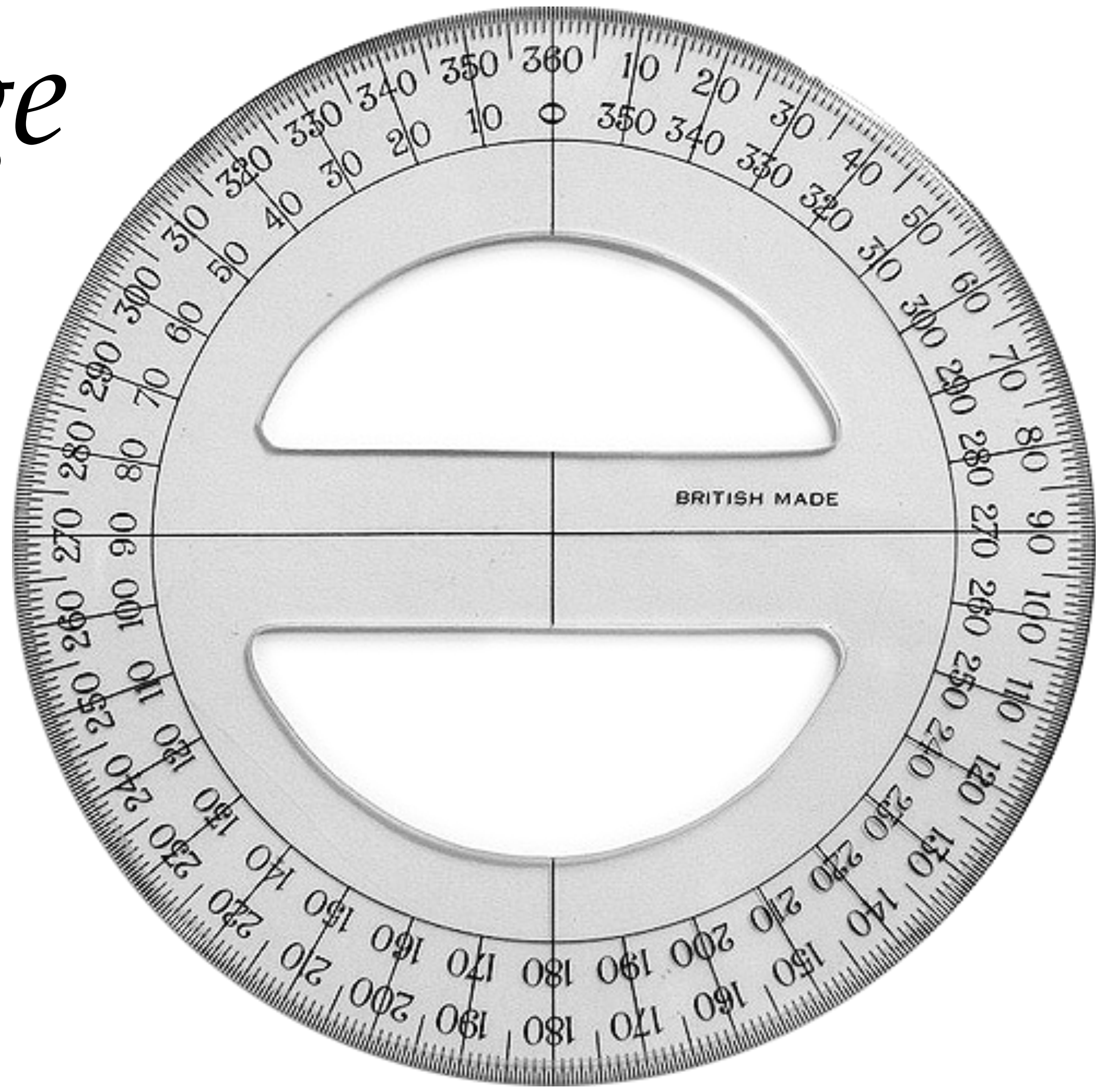
SHAPE ANALYSIS



SENSOR NETWORKS

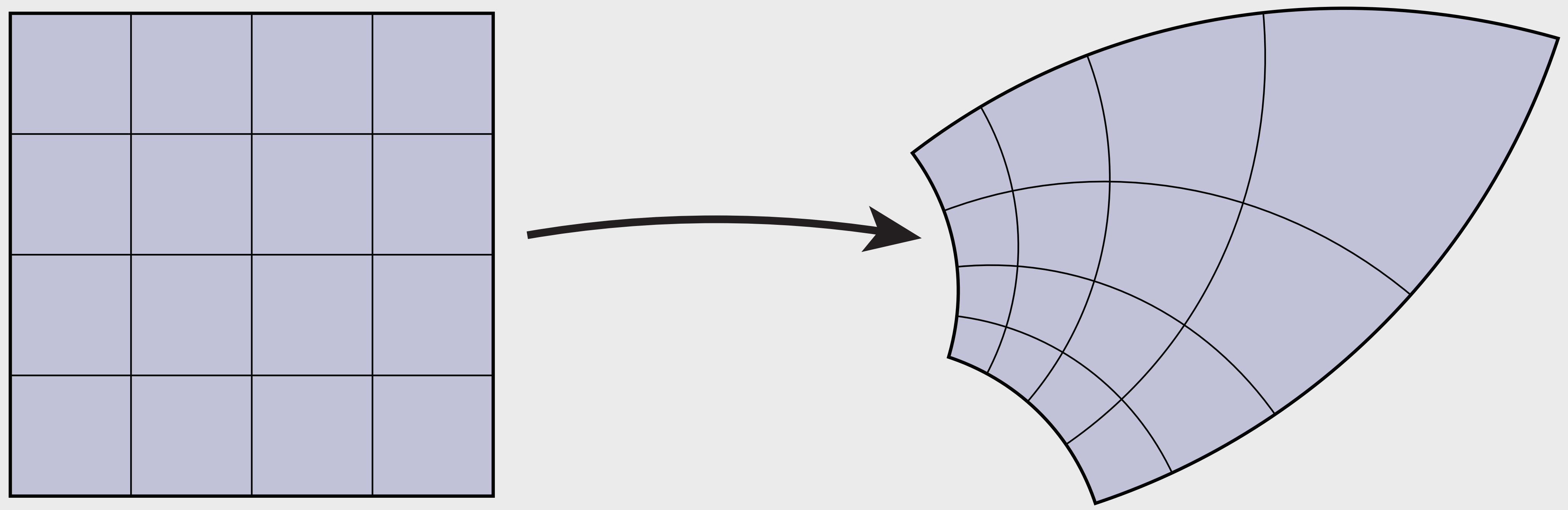
Why **Conformal**?

- Why so much interest in maps that preserve *angle*?
  - **QUALITY**: *Every conformal map is already "really nice"*
  - **SIMPLICITY**: *Makes "pen and paper" analysis easier*
  - **EFFICIENCY**: *Often yields computationally easy problems*
  - **GUARANTEES**: *Well understood, lots of theorems/knowledge*

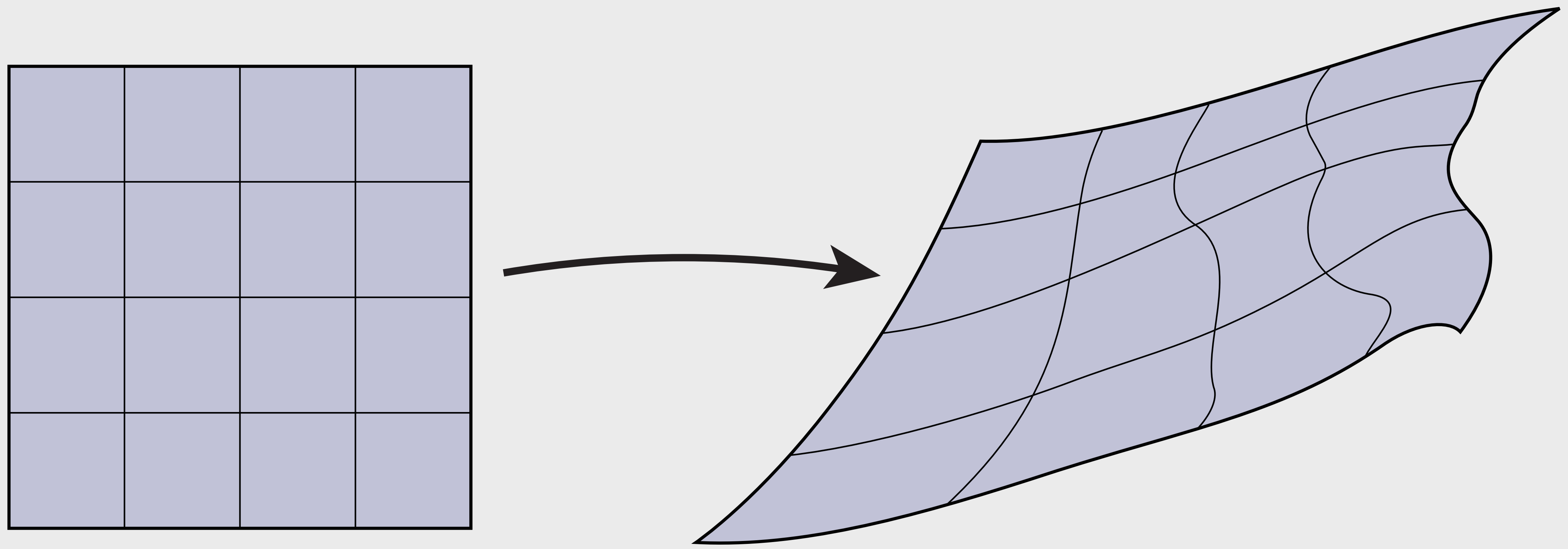


Conformal Maps are “Really Nice”

- Angle preservation already provides a lot of regularity
- E.g., every conformal map has infinitely many derivatives (C^∞)
- Scale distortion is smoothly distributed (harmonic)



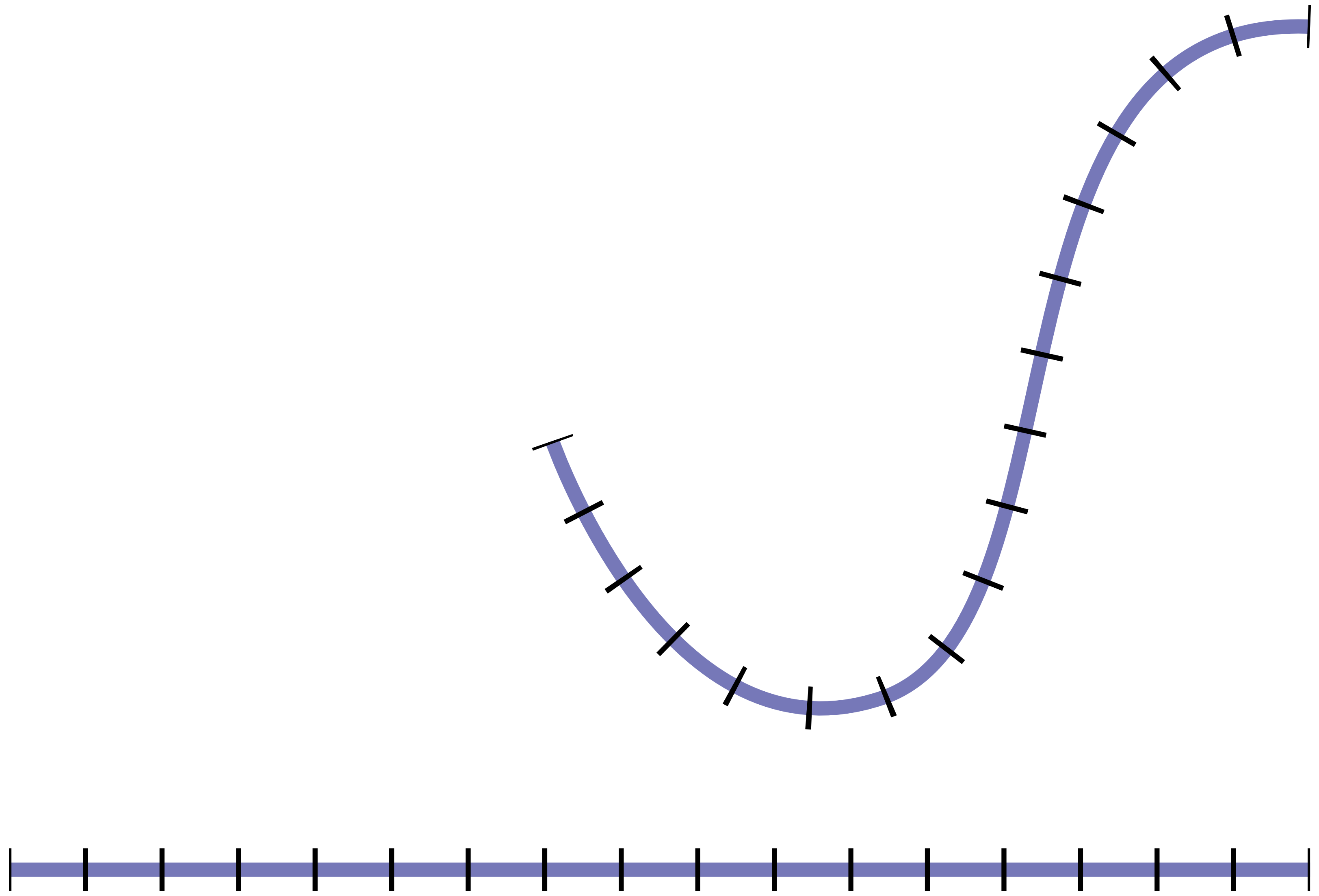
CONFORMAL

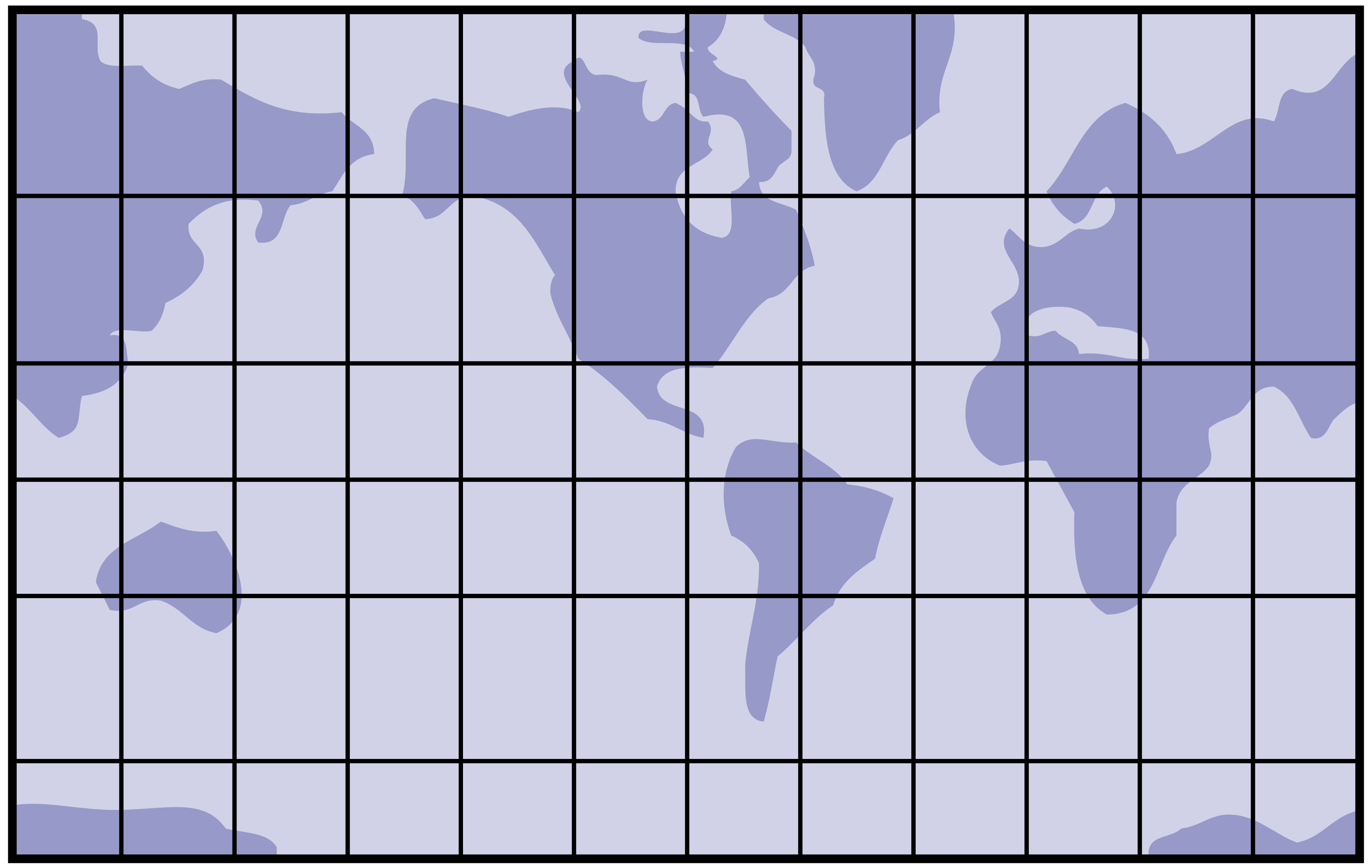


NOT CONFORMAL

Conformal Coordinates Make Life Easy

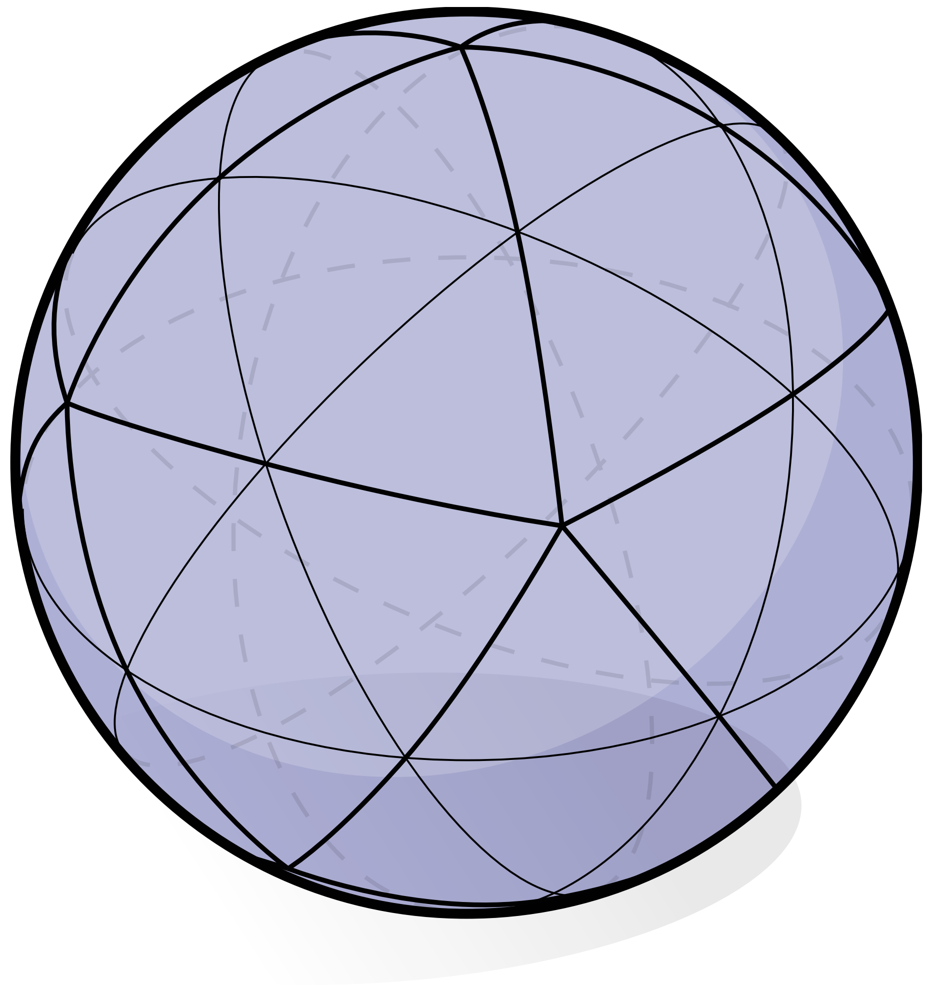
- Makes life easy “on pen and paper”
  - **Curves:** life greatly simplified by assuming *arc-length* parameterization
  - **Surfaces:** “arc-length” (isometric) not usually possible
    - conformal coordinates are “next best thing” (and always possible!)
    - only have to keep track of scale (rather than arbitrary Jacobian)



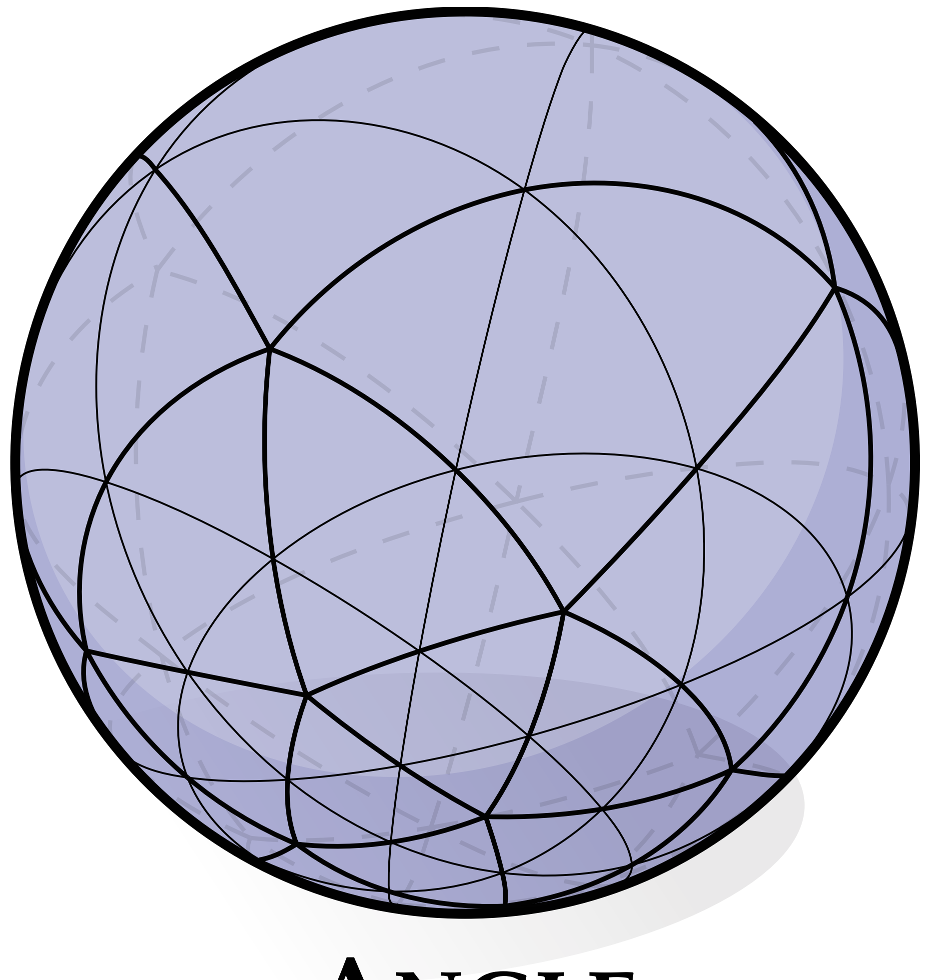


Aside: Isn't Area-Preservation "Just as Good?"

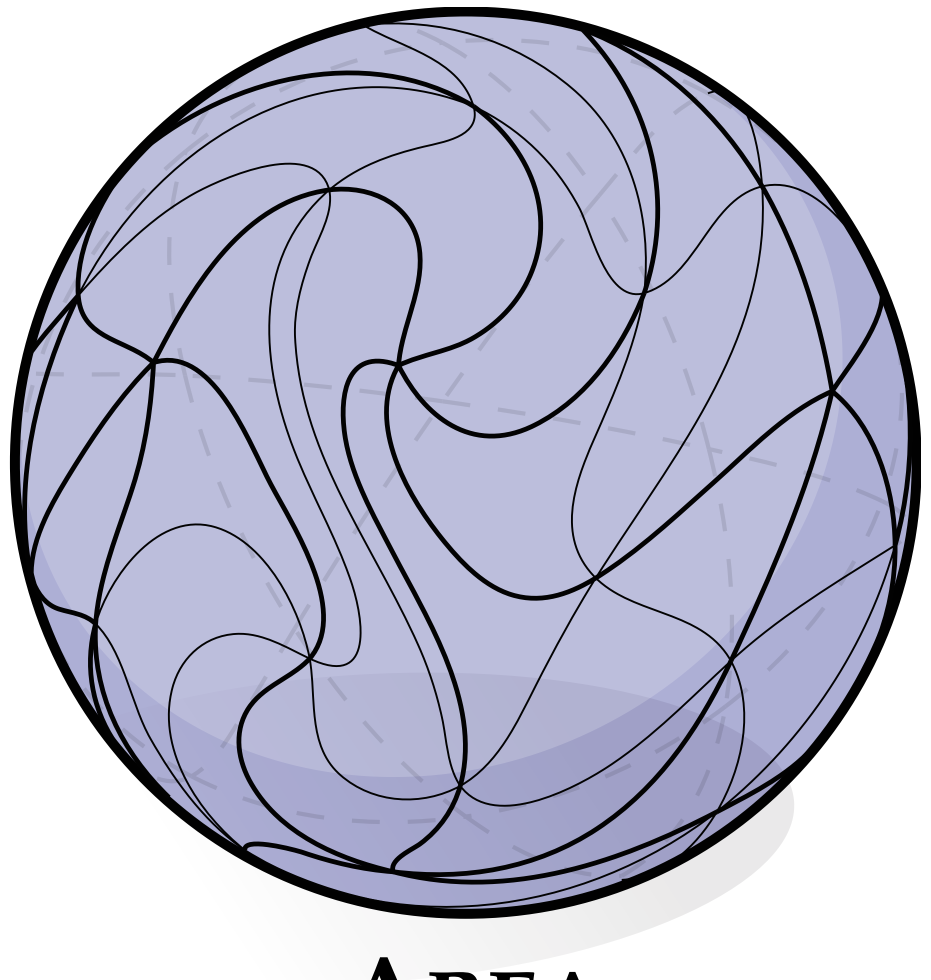
- **Q:** What's so special about *angle*? Why not preserve, say, *area* instead?
- **A:** Area-preservation alone can produce maps that are ***nasty!***
  - Don't even have to be smooth; *huge* space of possibilities.
- E.g., any motion of an incompressible fluid (e.g., swirling water):



ORIGINAL



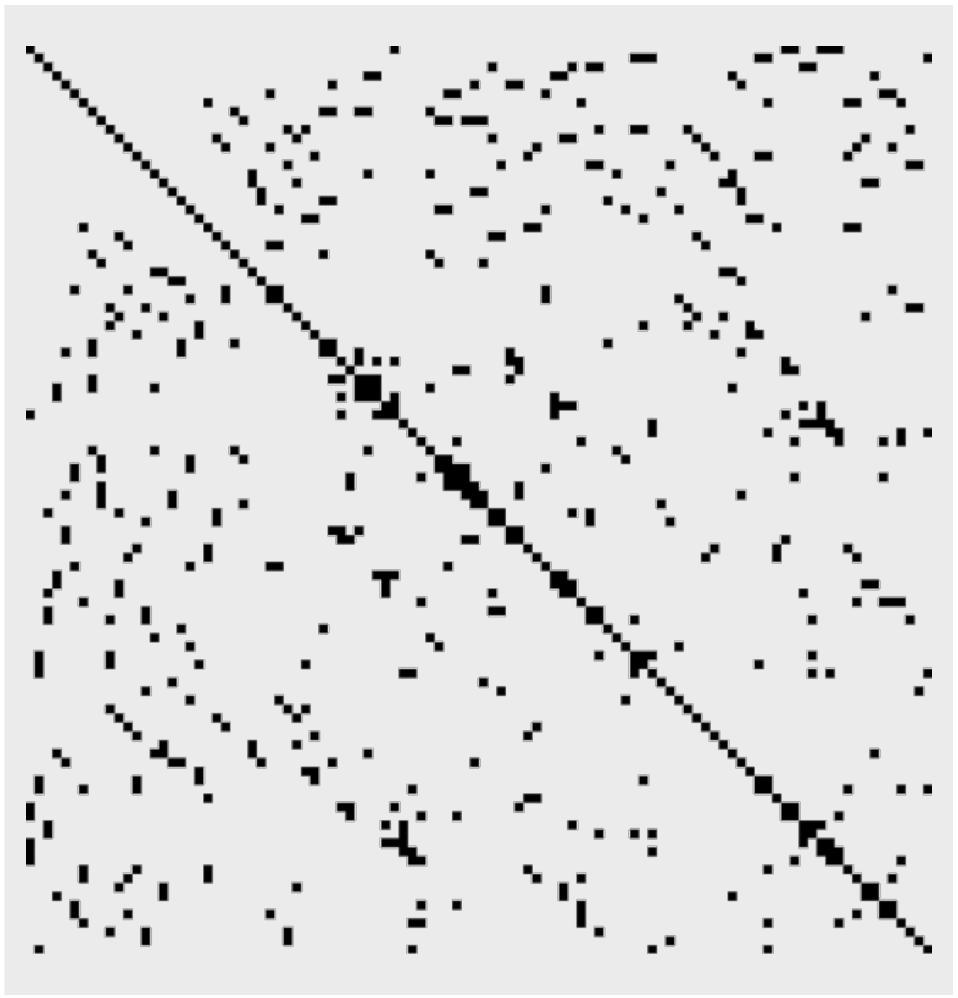
ANGLE PRESERVING

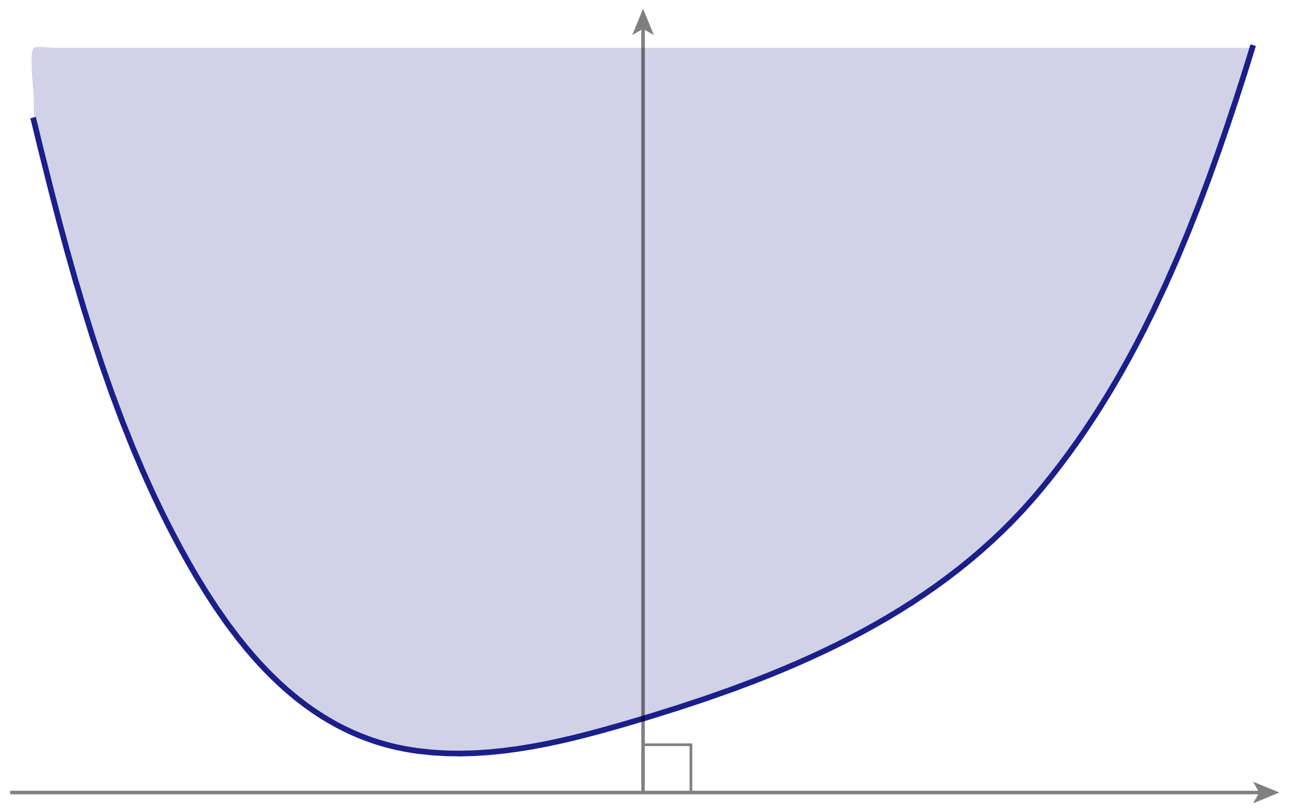


AREA PRESERVING

Computing Conformal Maps is Efficient

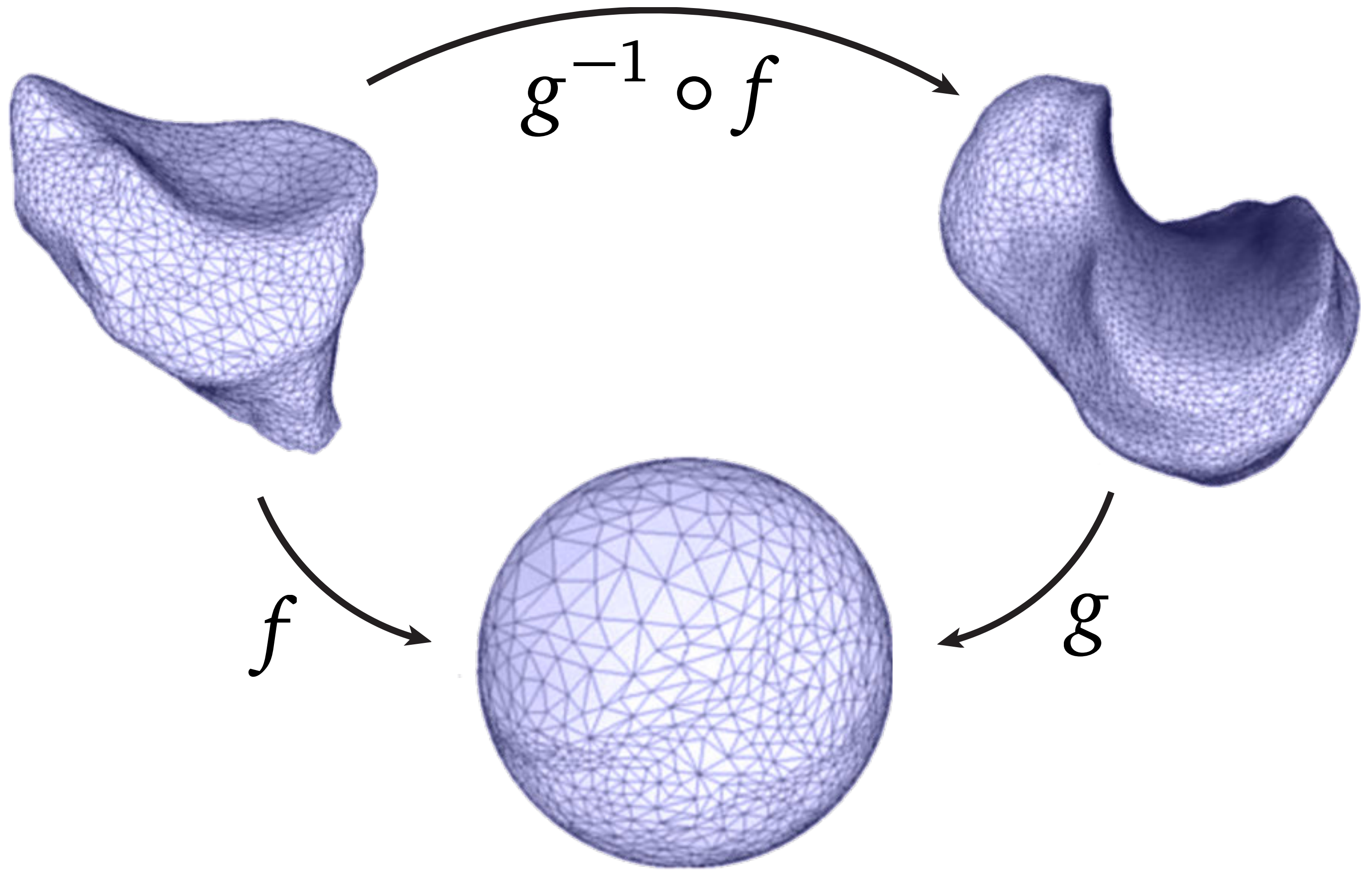
- Algorithms boil down to efficient, scalable computation
  - sparse linear systems / sparse eigenvalue problems
  - convex optimization problems
- Compare to more elaborate mapping problems
  - *bounded distortion*, *locally injective*, etc.
  - entail more difficult problems (e.g., *SOCP*)
- Much broader domain of applicability
  - real time vs. "just once"





Conformal Maps Help Provide Guarantees

- Established topic*
  - lots of existing theorems, analysis
  - connects to standard problems (e.g., Laplace)
  - makes it easier to provide guarantees (max principle, Delaunay, *etc.*)
- *Uniformization theorem* provides (nearly) canonical maps

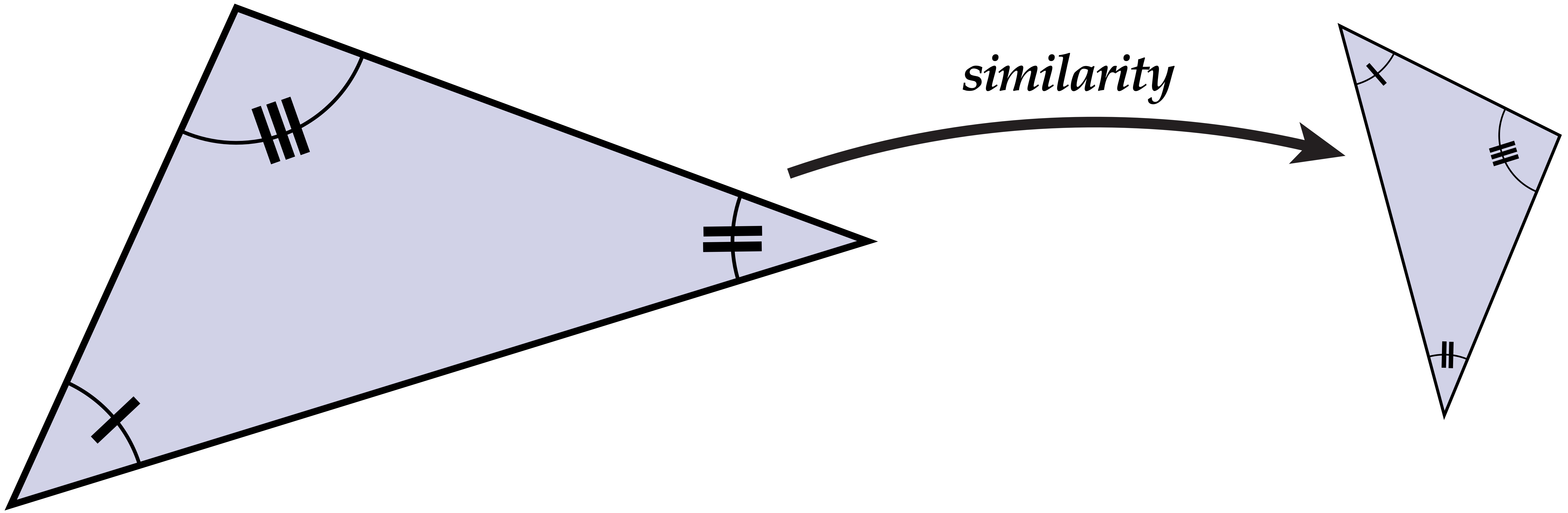


**Also makes it harder to do something truly new in conformal geometry processing…!*

Discrete Conformal Maps?

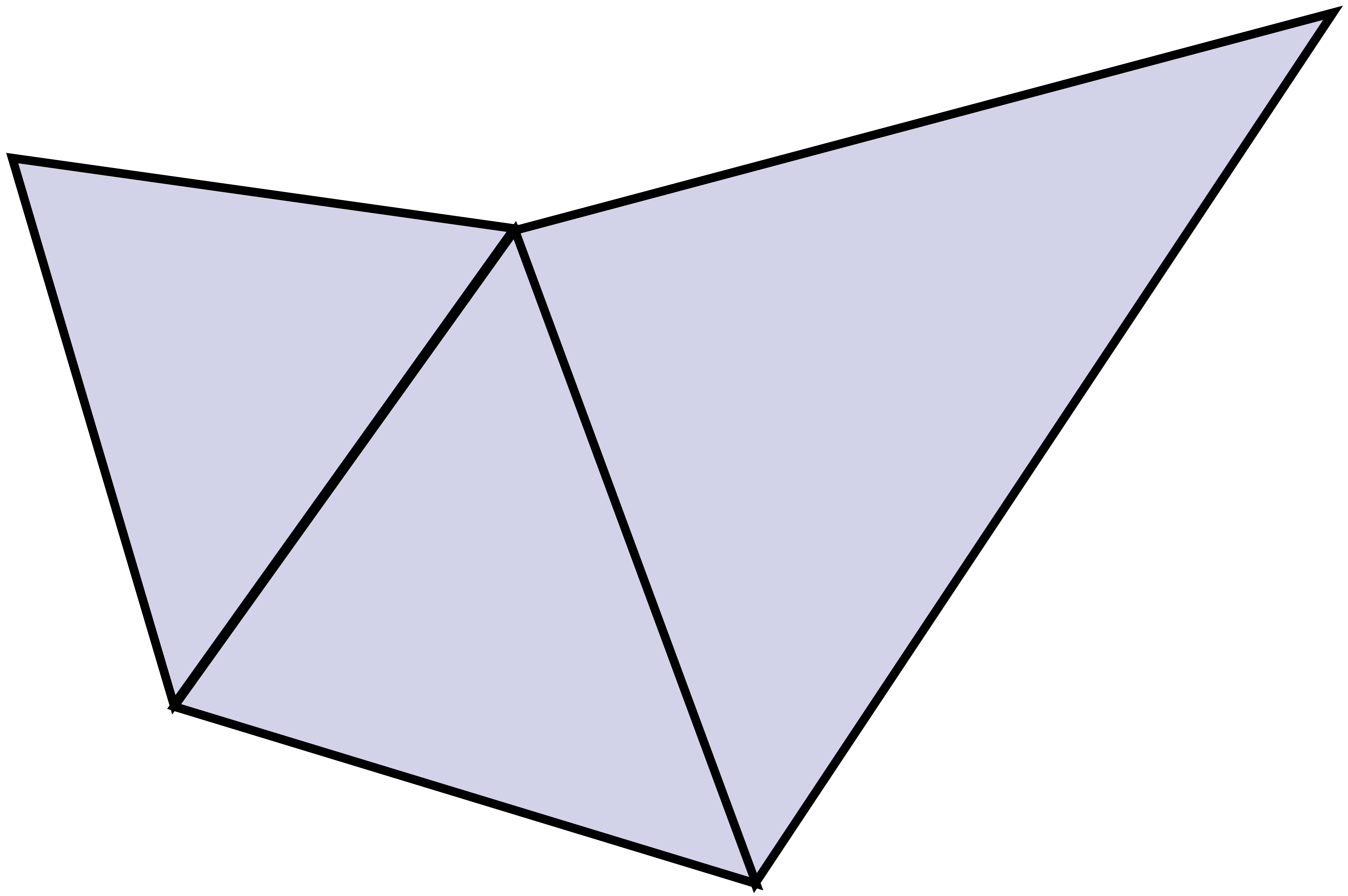
To compute conformal maps, we need some finite "discretization."

First attempt: preserve corner angles in a triangle mesh:



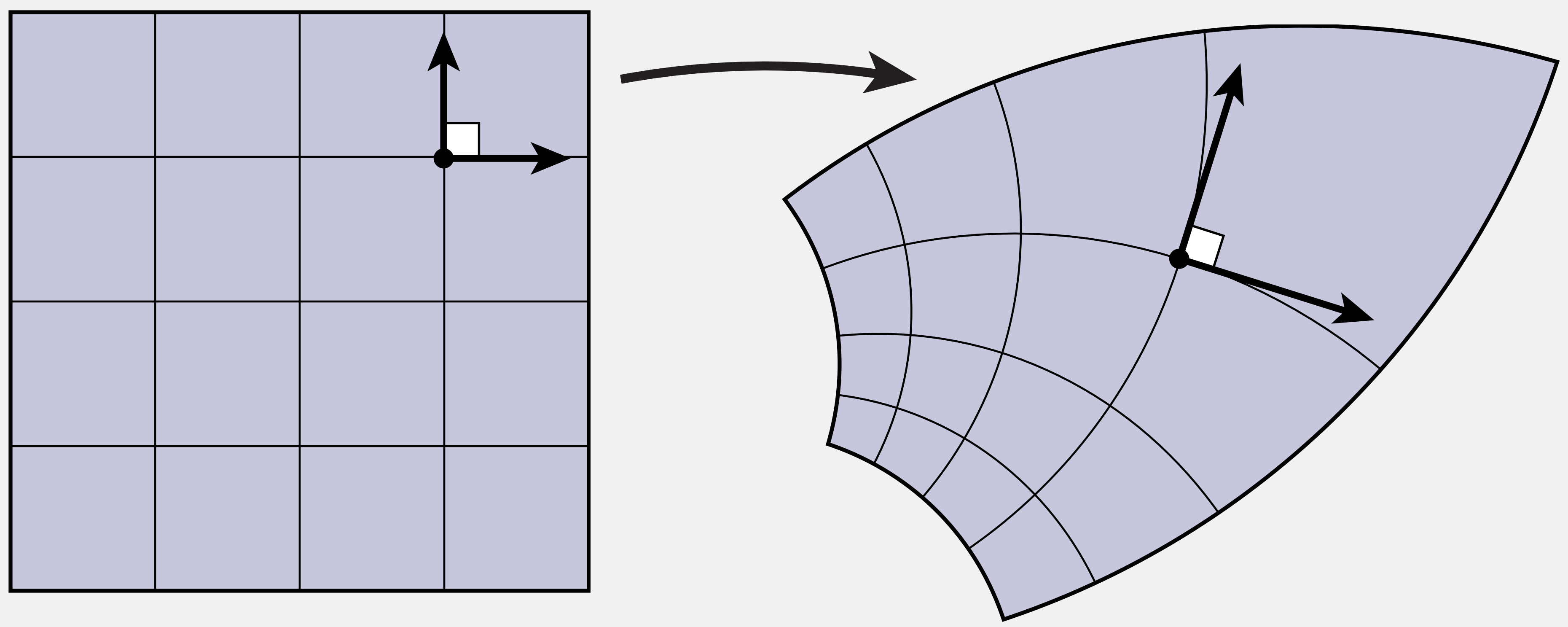
Rigidity of Angle Preservation

Problem: one triangle determines the entire map! (Too "rigid")

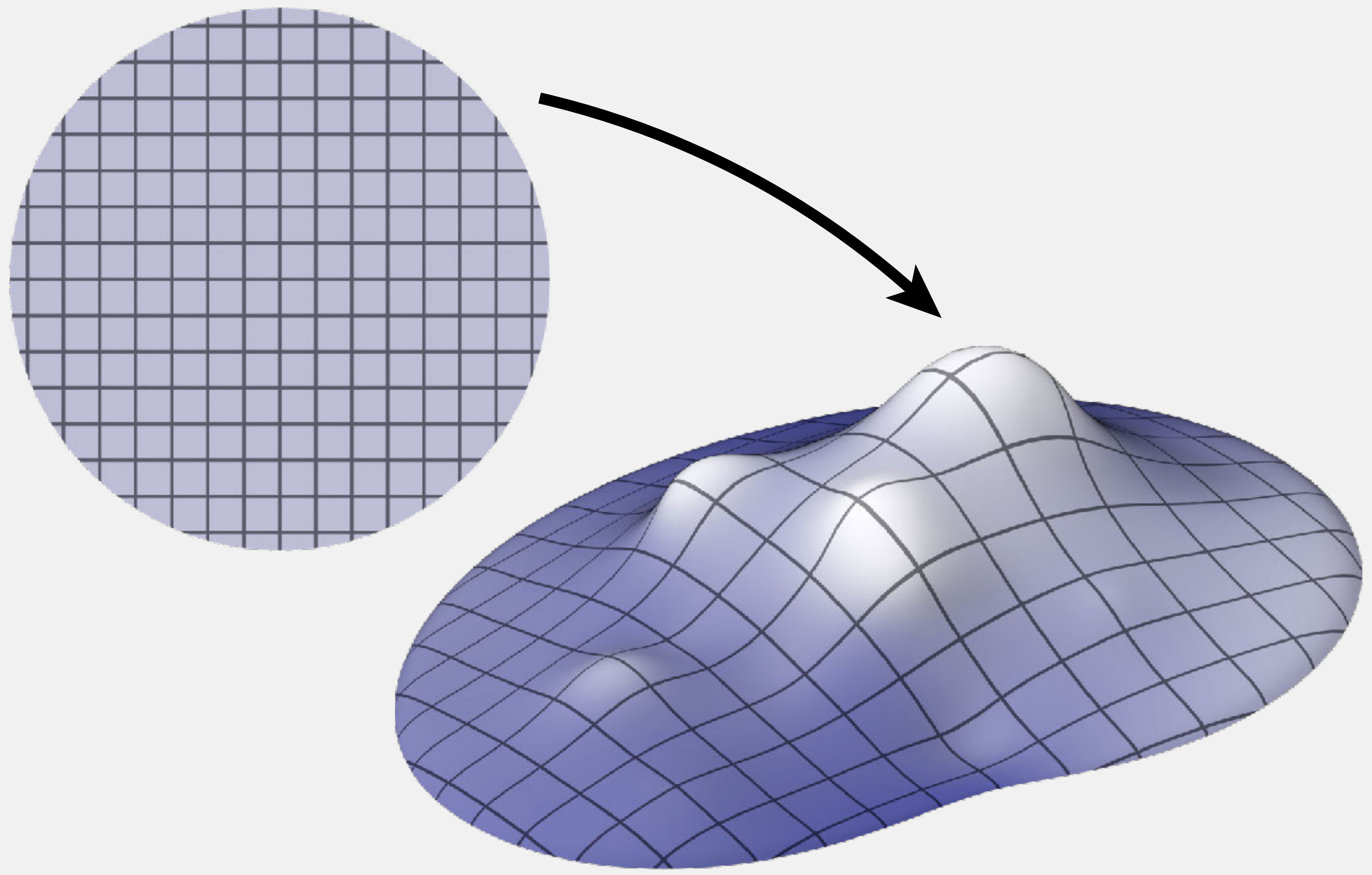


Need a different way of thinking…

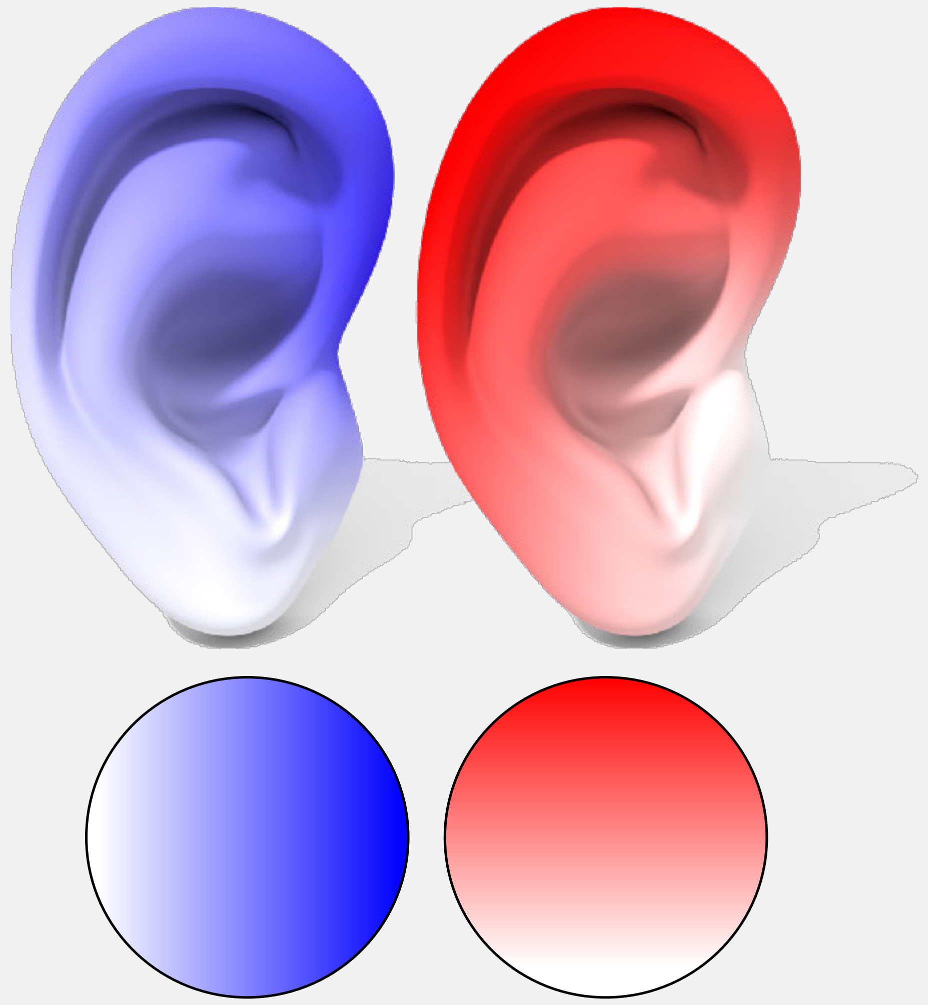
(Some) Characterizations of Conformal Maps



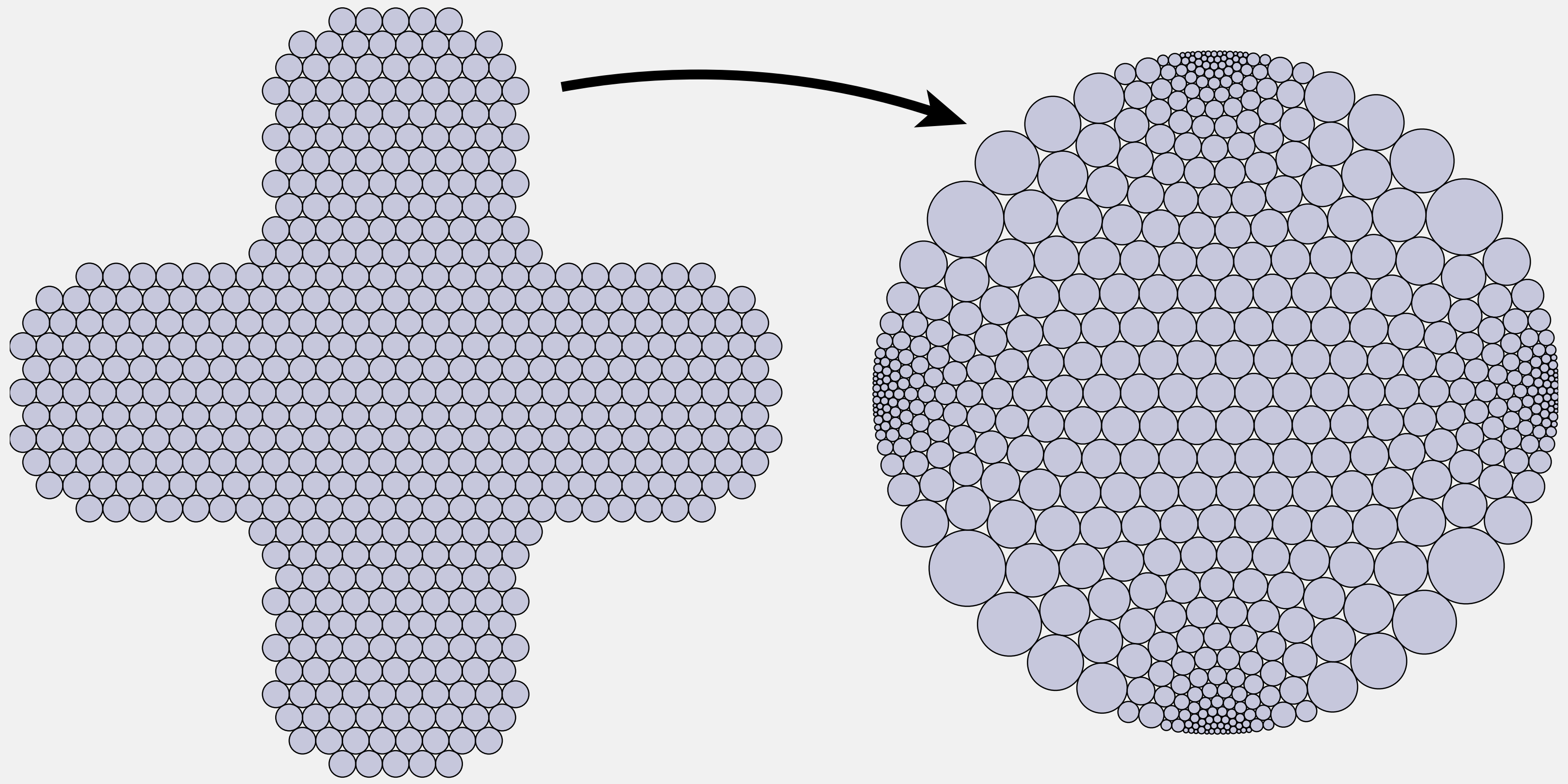
angle preservation



metric rescaling



conjugate harmonic functions



preservation of circles

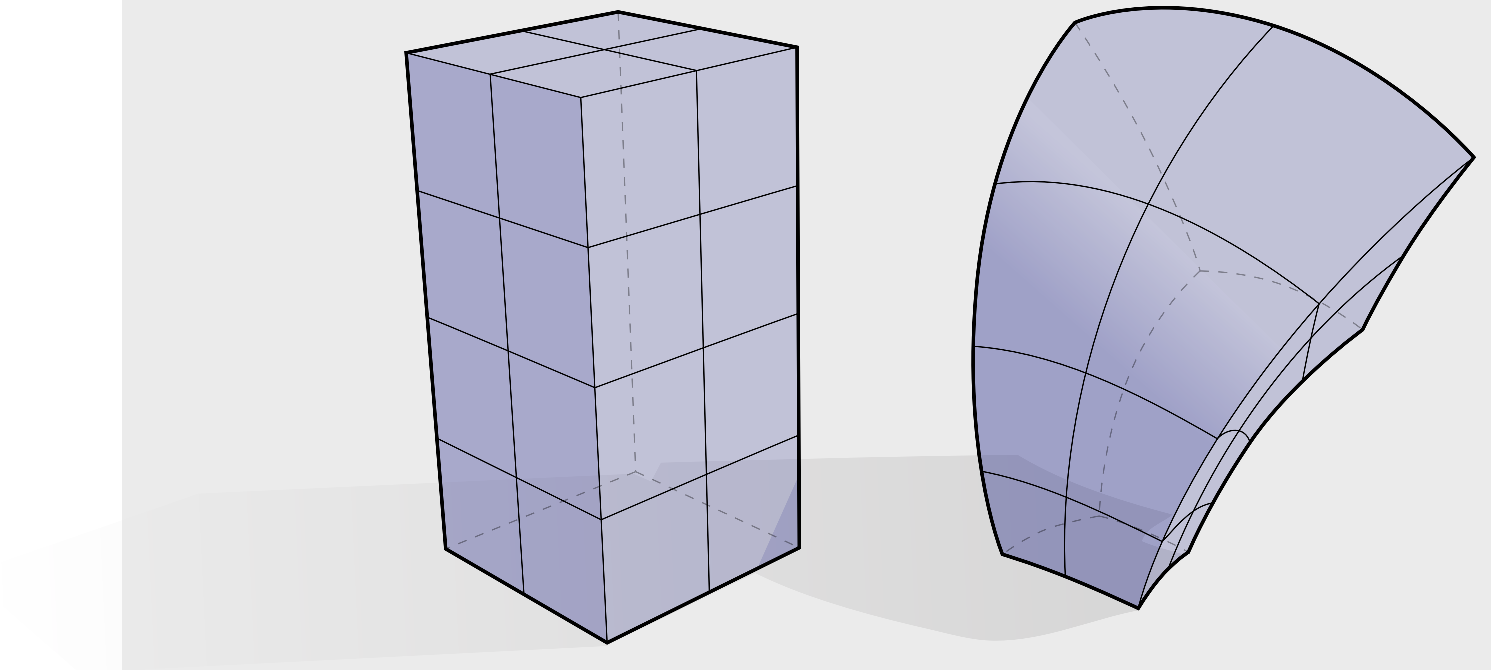
critical points of Dirichlet energy

(Some) Conformal Geometry Algorithms

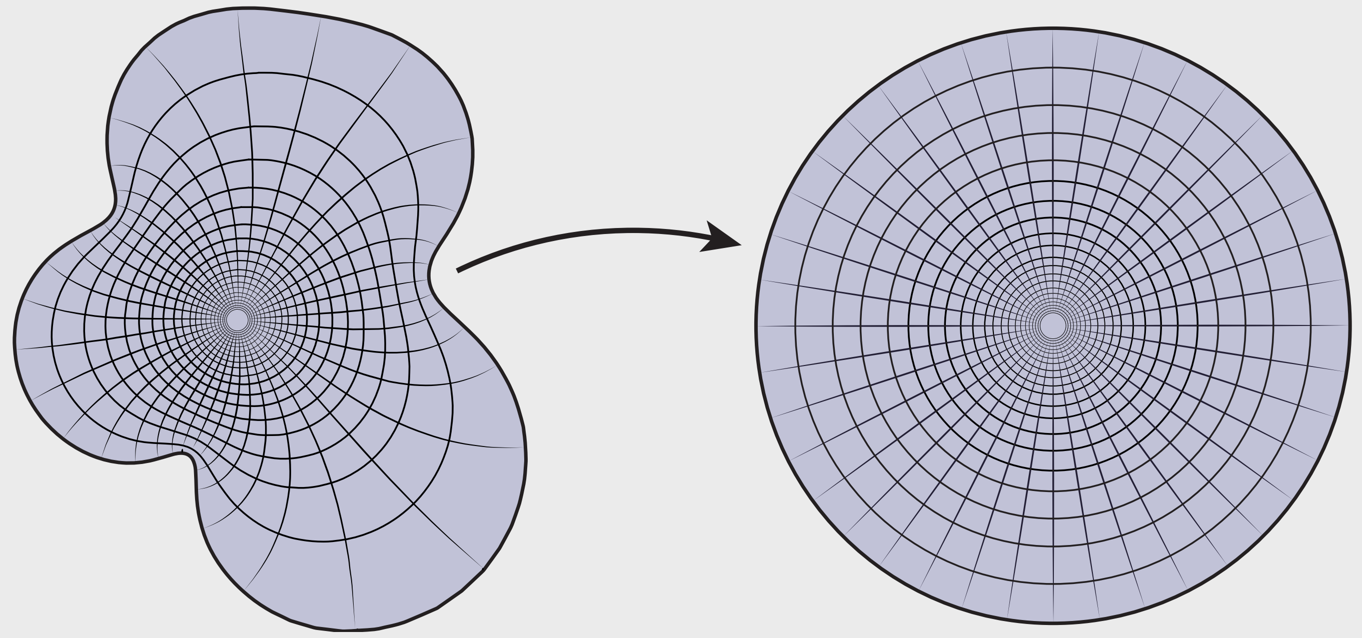
CHARACTERIZATION	ALGORITHMS
Cauchy-Riemann	*least square conformal maps (LSCM)*
Dirichlet energy	*discrete conformal parameterization (DCP)* *genus zero surface conformal mapping (GZ)*
angle preservation	*angle based flattening (ABF)*
circle preservation	*circle packing* *circle patterns (CP)*
metric rescaling	*conformal prescription with metric scaling (CPMS)* *conformal equivalence of triangle meshes (CETM)*
conjugate harmonic	*boundary first flattening (BFF)*

Some Key Ideas in Conformal Surface Geometry

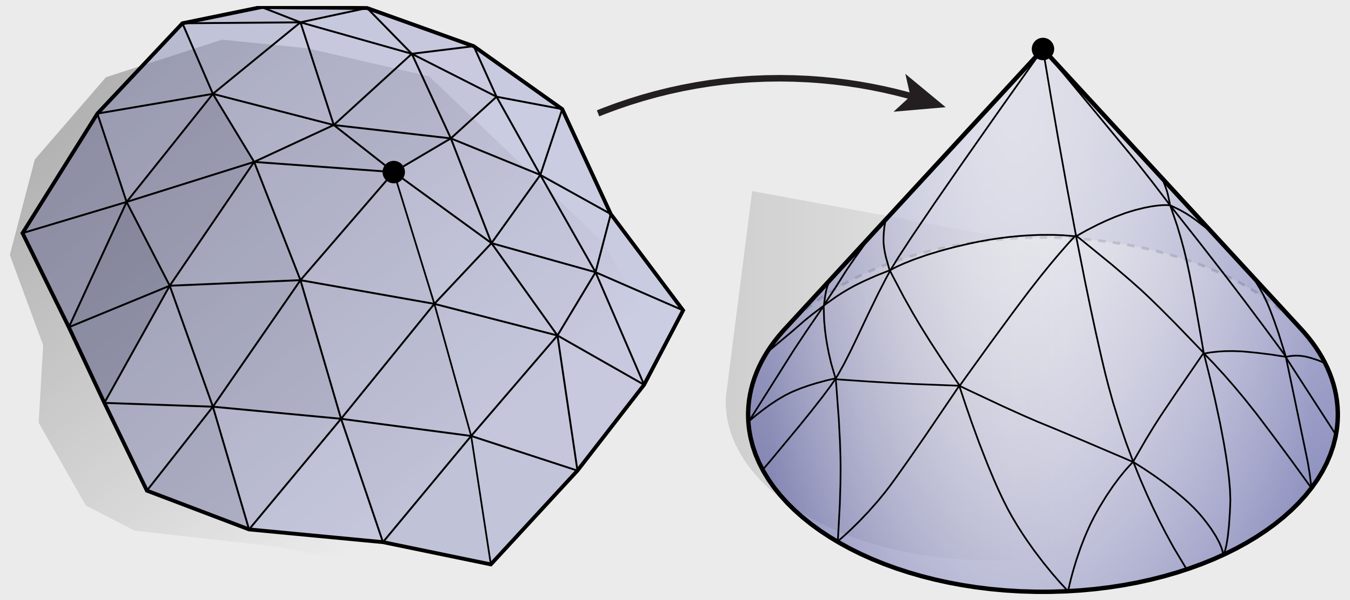
Möbius Transformations / Stereographic Projection



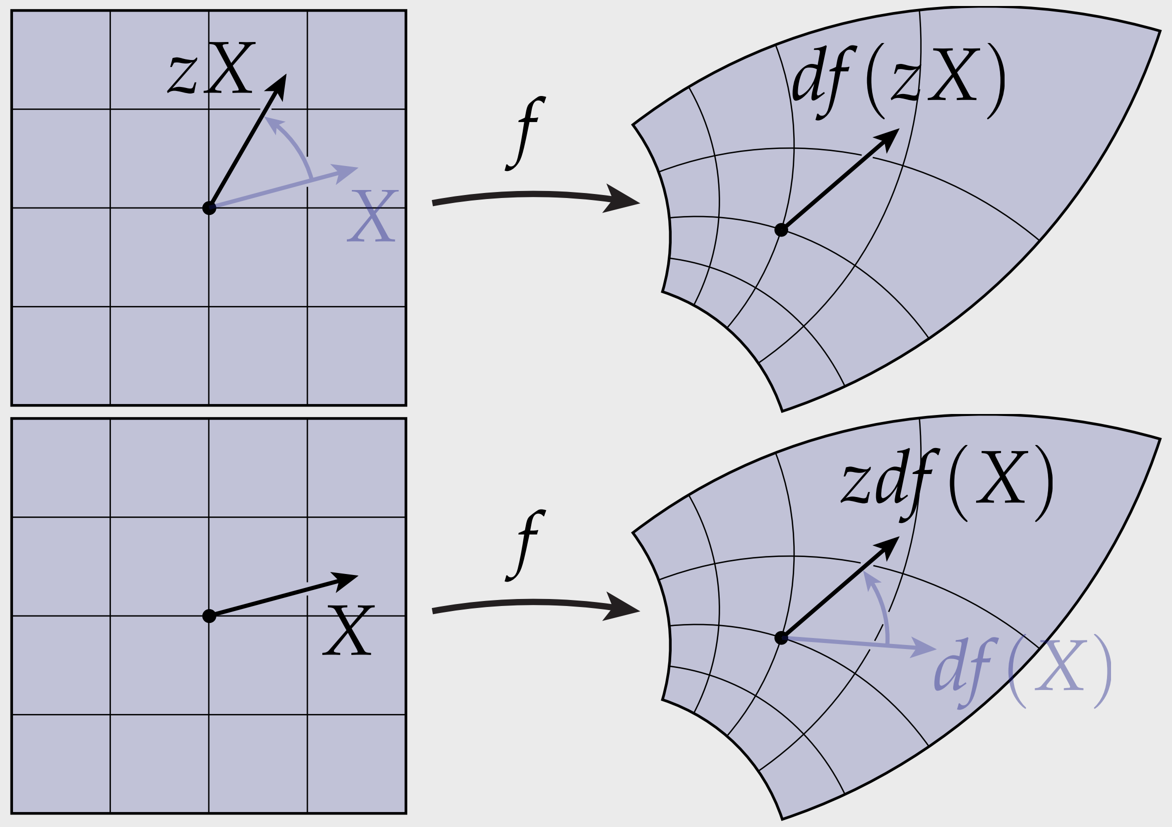
Riemann Mapping / Uniformization



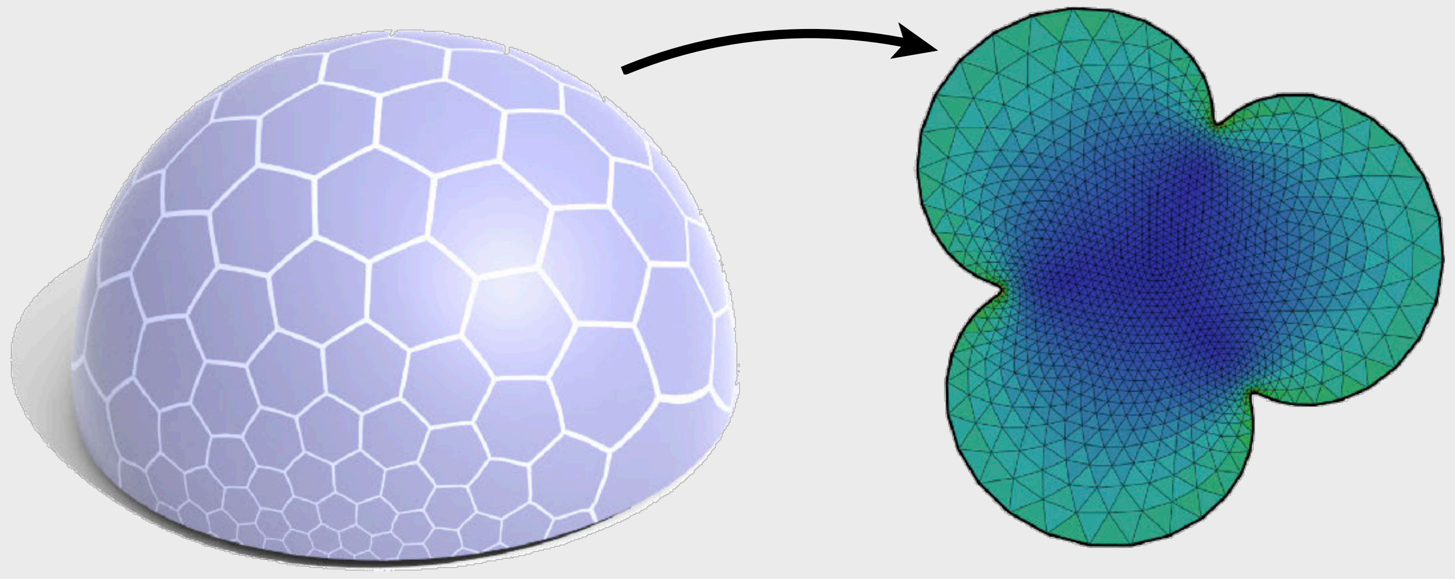
Cone Singularities



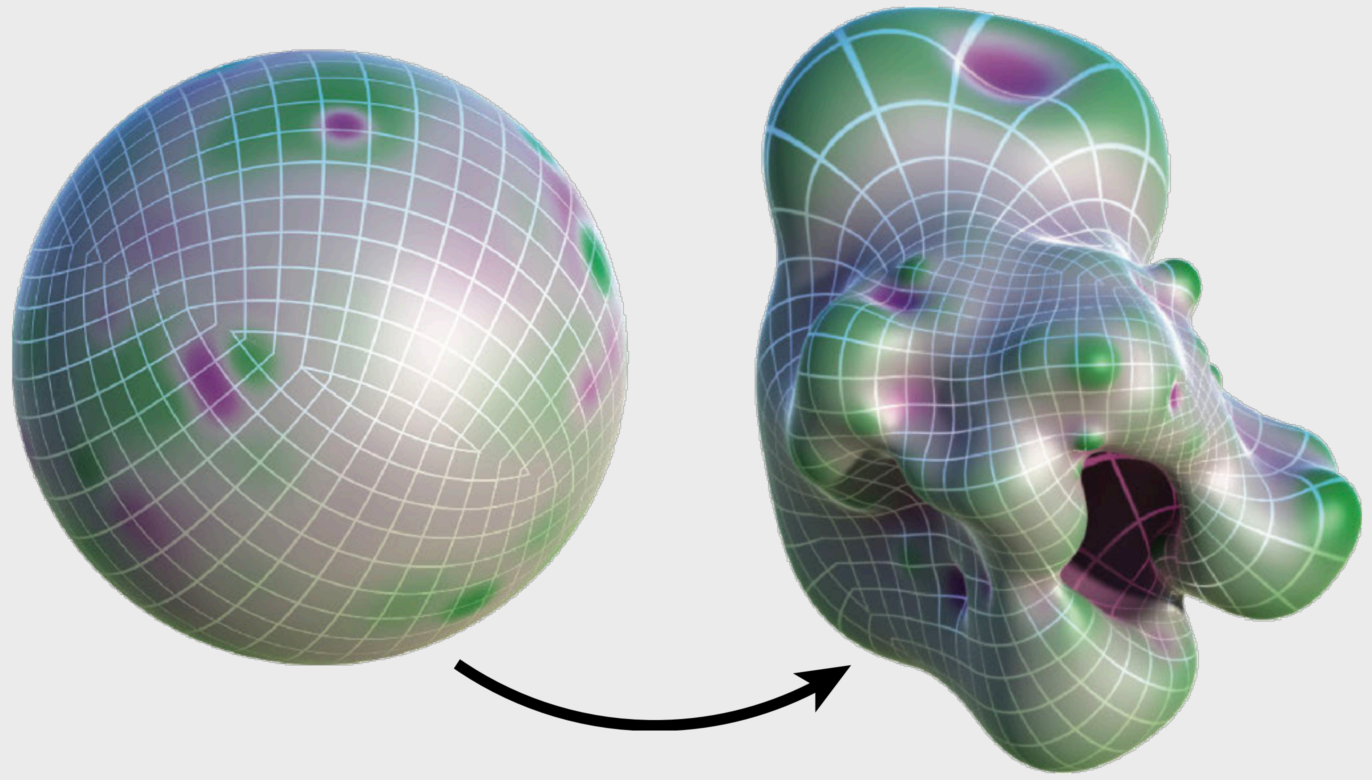
Cauchy-Riemann Equation



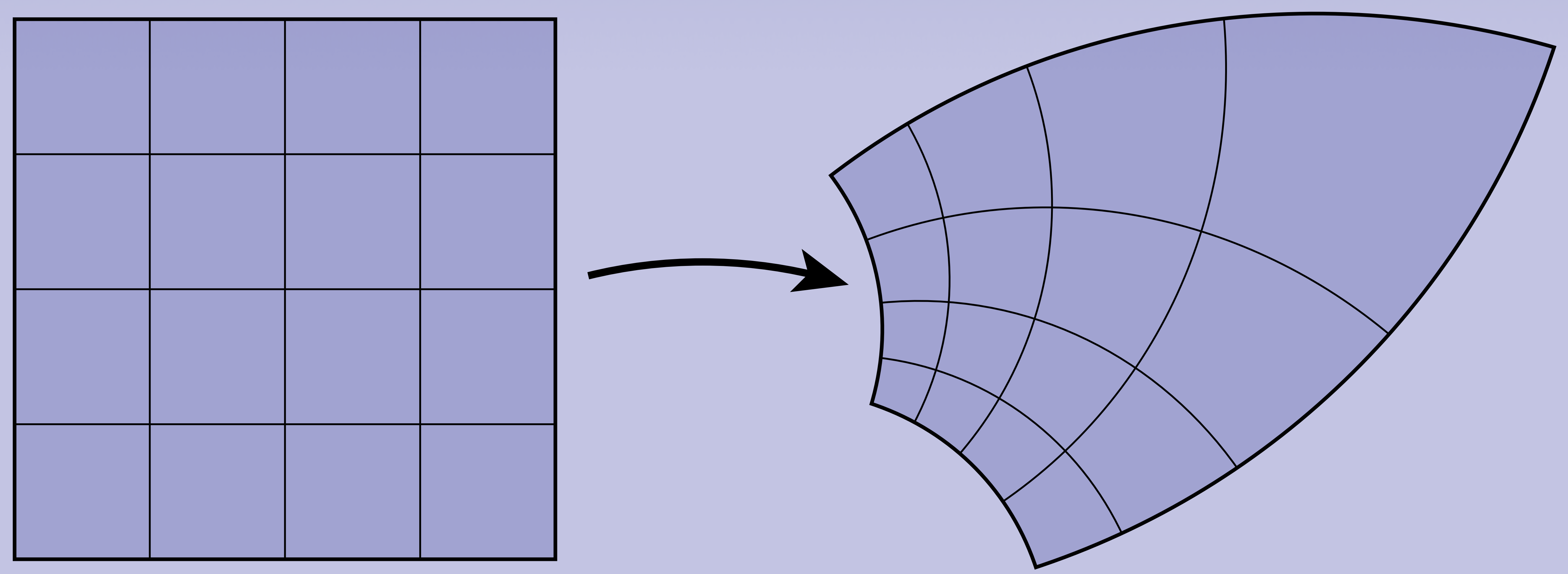
Ricci Flow / Cherrier Formula



Dirac Equation



Part II: Smooth Theory



Conformal Geometry Processing

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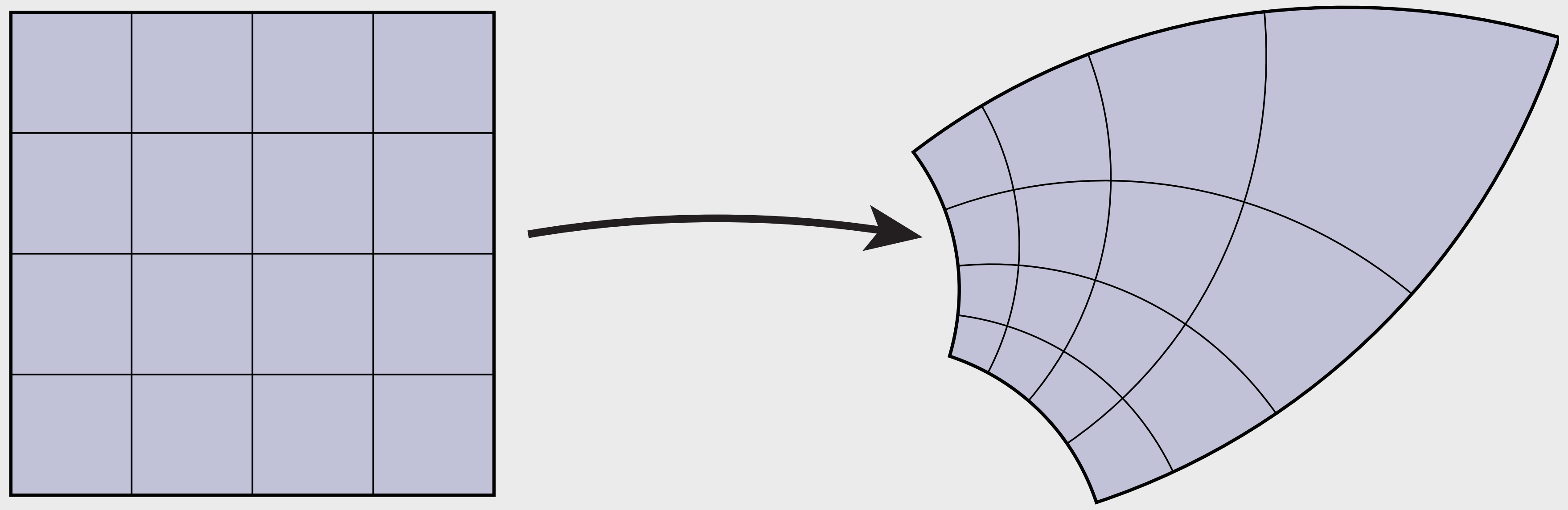
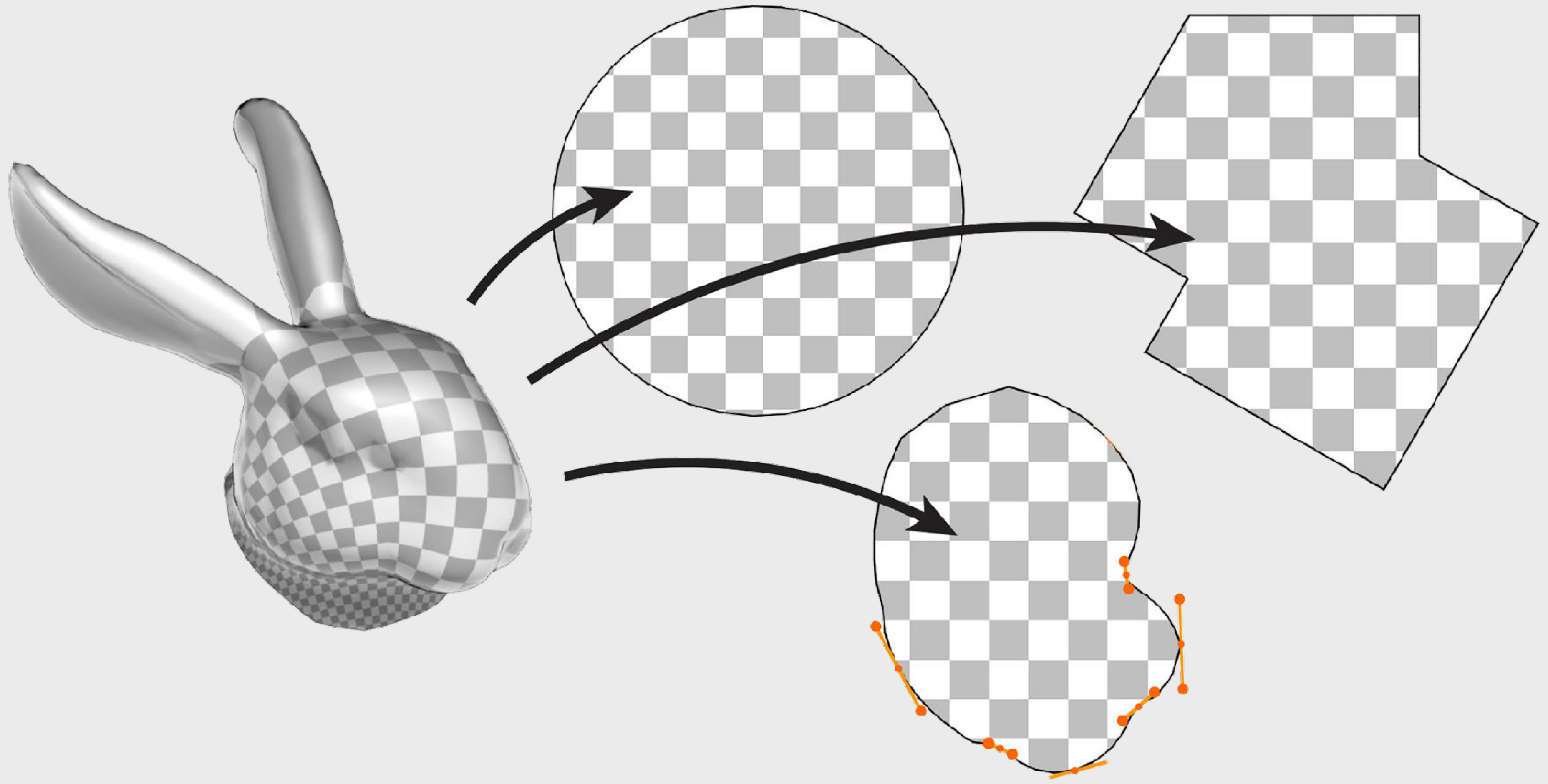
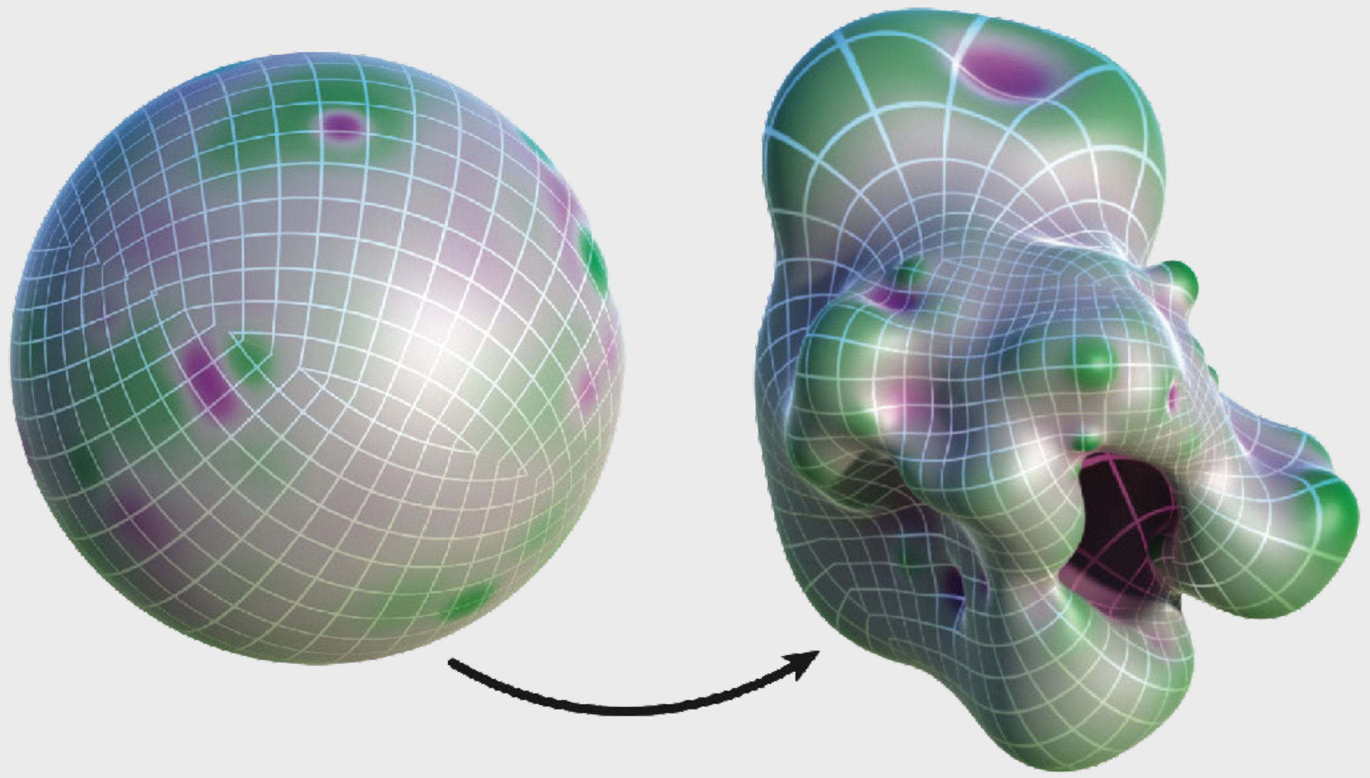
Conformal Maps of Surfaces



PLANE TO PLANE



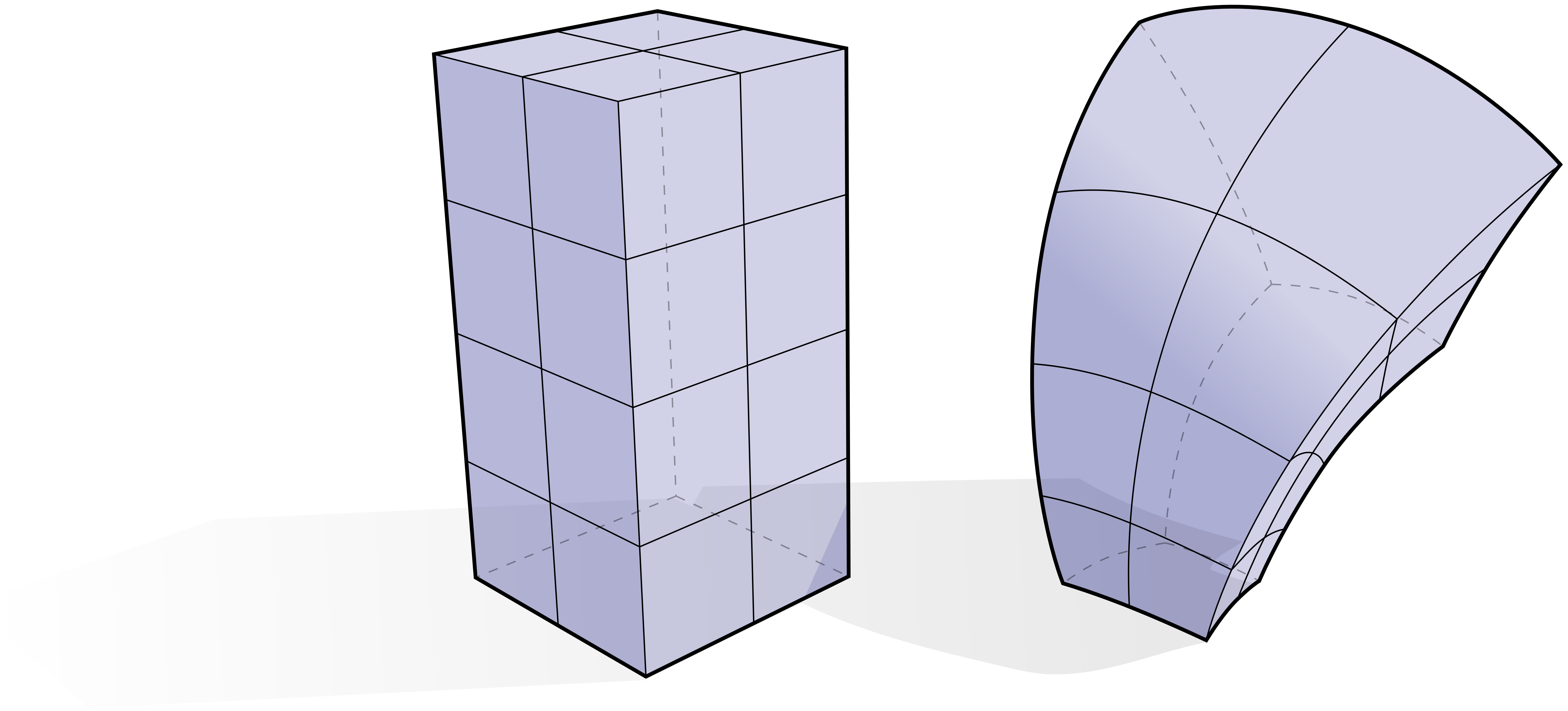
SURFACE TO PLANE



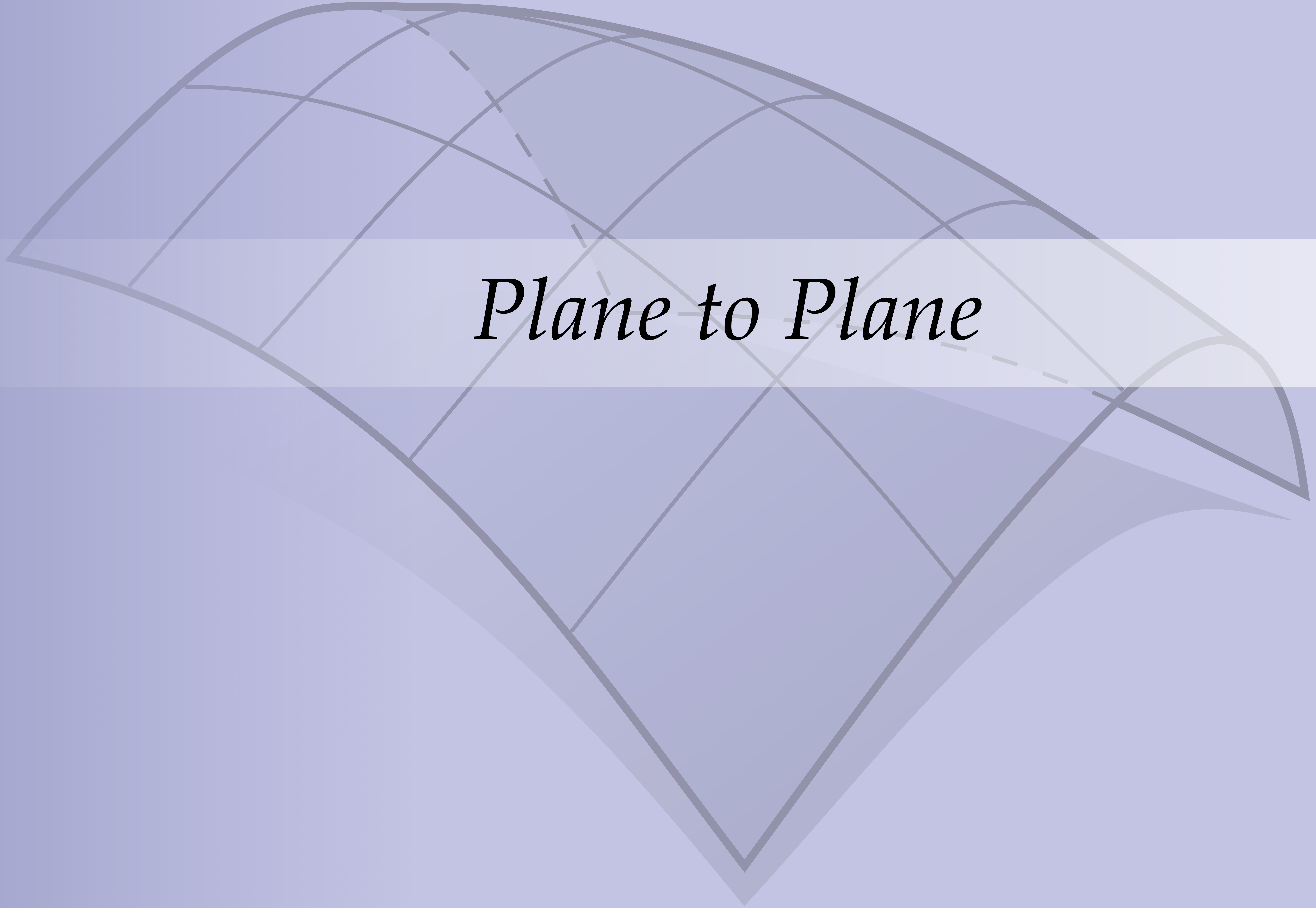
SURFACE TO SURFACE

Why Not Higher Dimensions?

Theorem (Liouville)**.** For $n \geq 3$, the only angle-preserving maps from $\mathbb{R}^n$ to itself (or from a region of $\mathbb{R}^n$ to $\mathbb{R}^n$) are Möbius transformations.



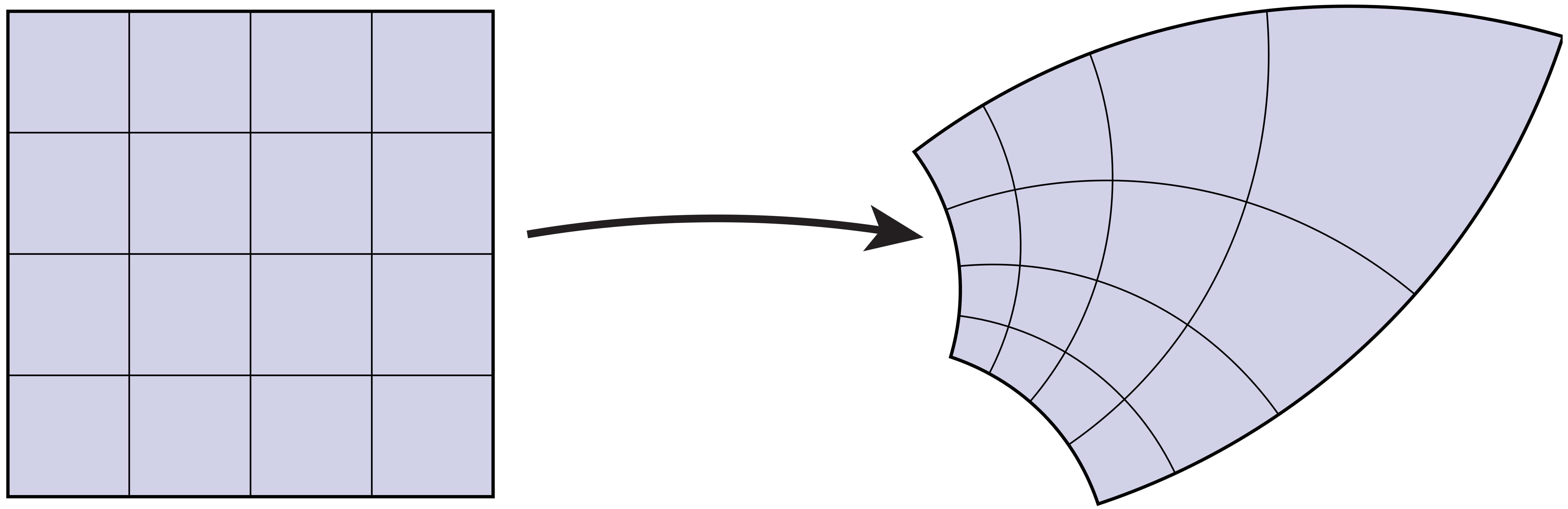
Key idea: conformal maps of volumes are *very* rigid.



Plane to Plane

Plane to Plane

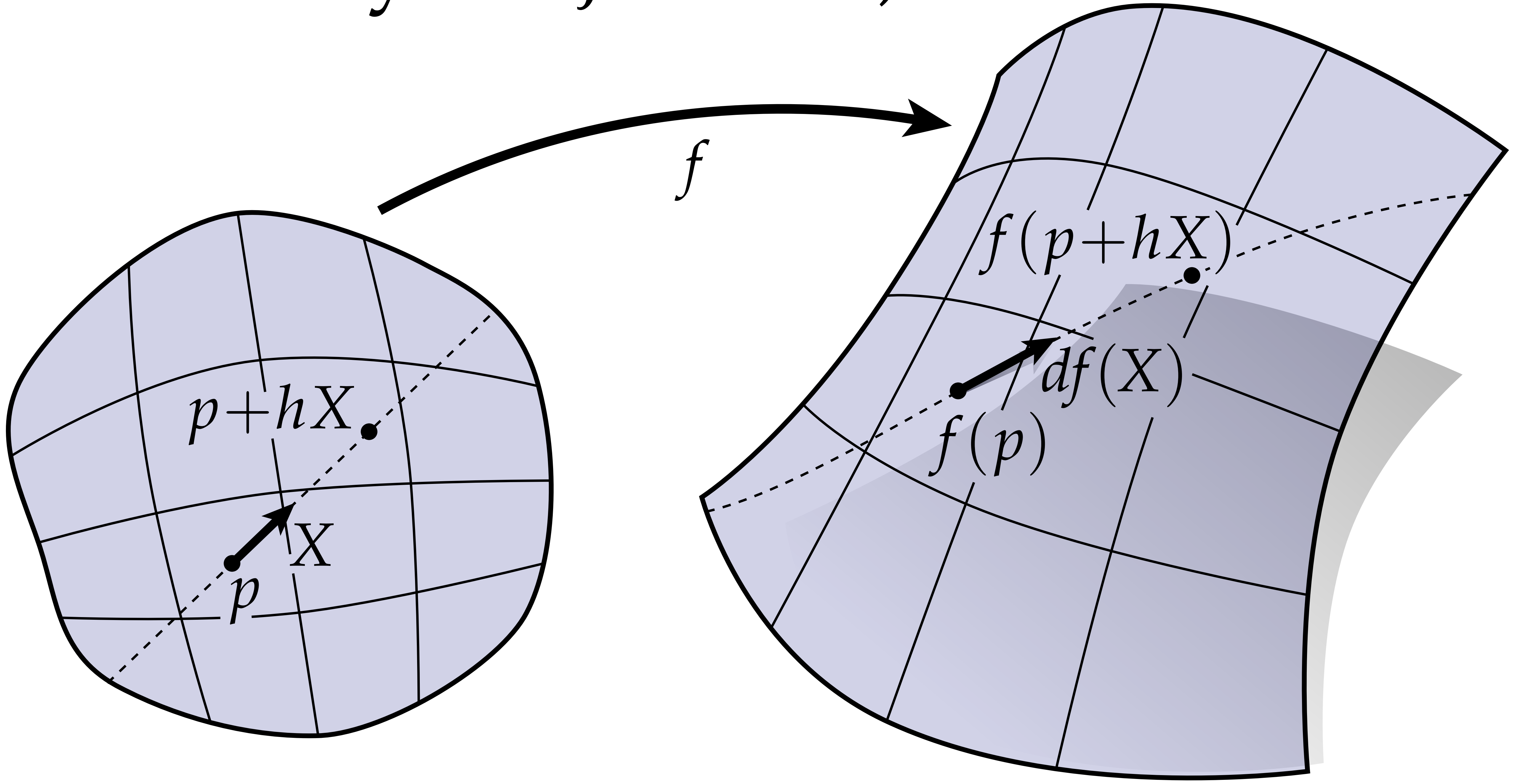
- Most basic case: conformal maps from region of 2D plane to 2D plane.
- Basic topic of complex analysis
- Fundamental equation: *Cauchy-Riemann*
- *Many* ideas we will omit (e.g., power series/analytic point of view)



Differential of a Map

- Basic idea we'll need to understand: *differential* of a map
- Describes how to *"push forward"* vectors under a differentiable map
- (In coordinates, differential is represented by the *Jacobian*)

$$df(X) = \lim_{h\to 0} \frac{f(p+hX) - f(p)}{h}$$

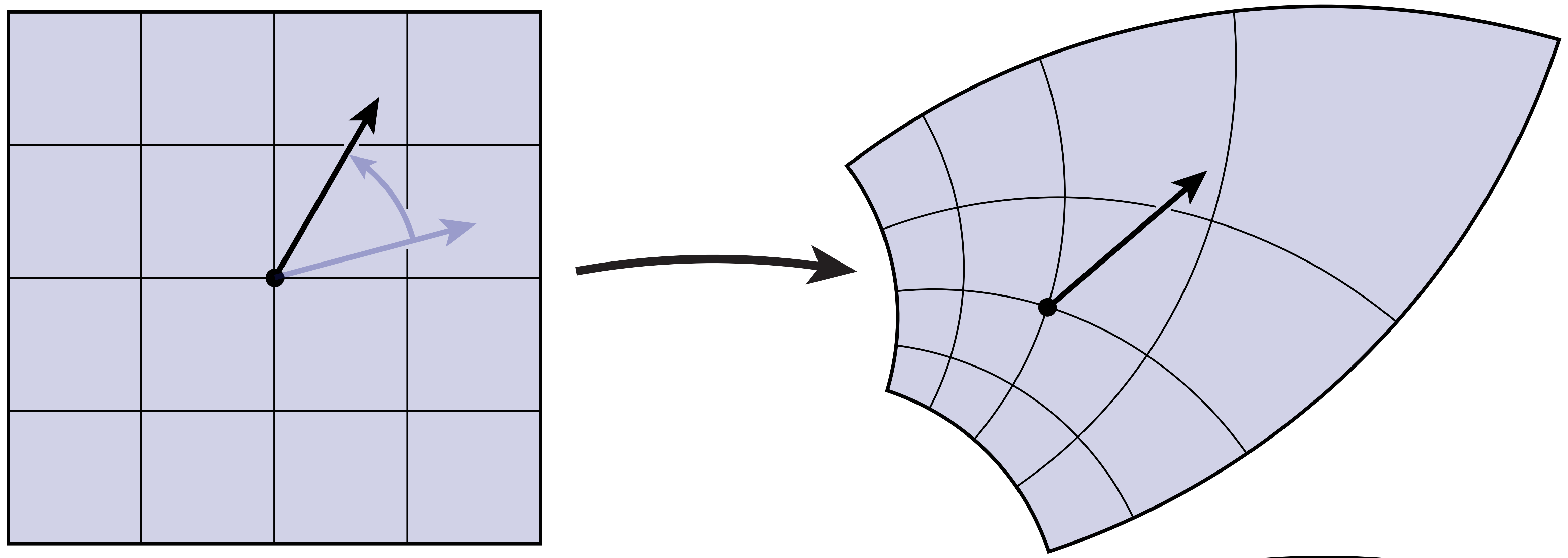


Intuition: "how do vectors get stretched out?"

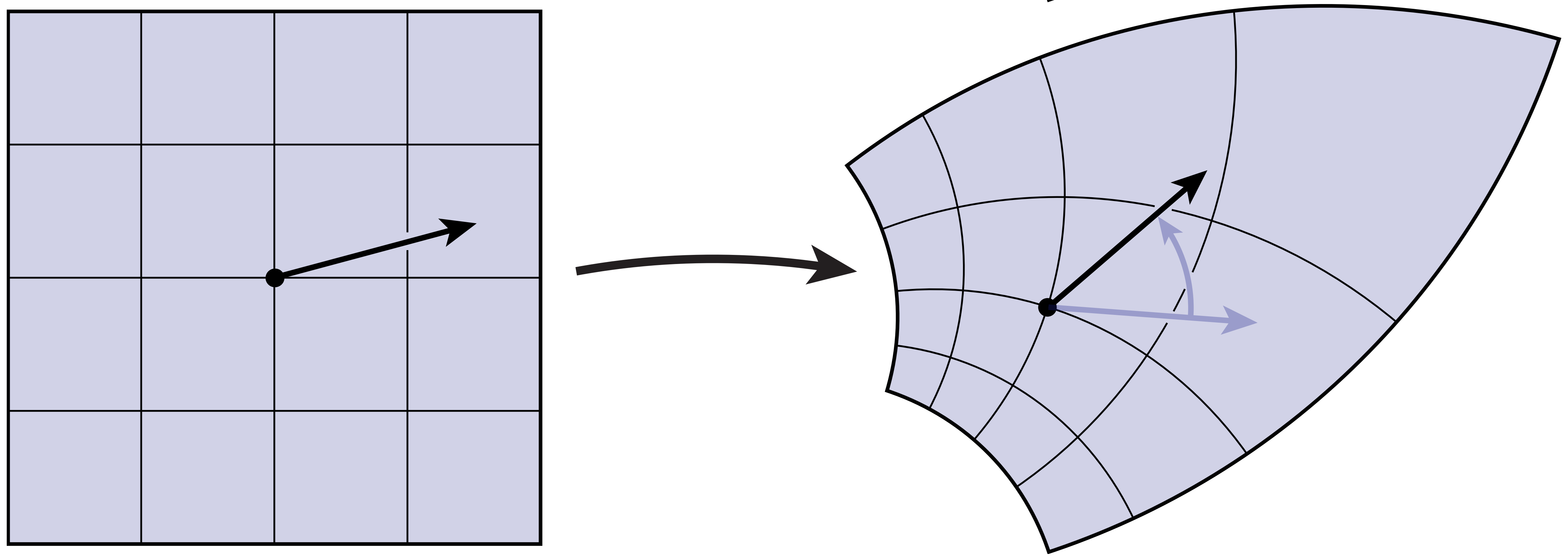
Conformal Map

- A map is conformal if two operations are equivalent:

1. rotate, then push forward vector



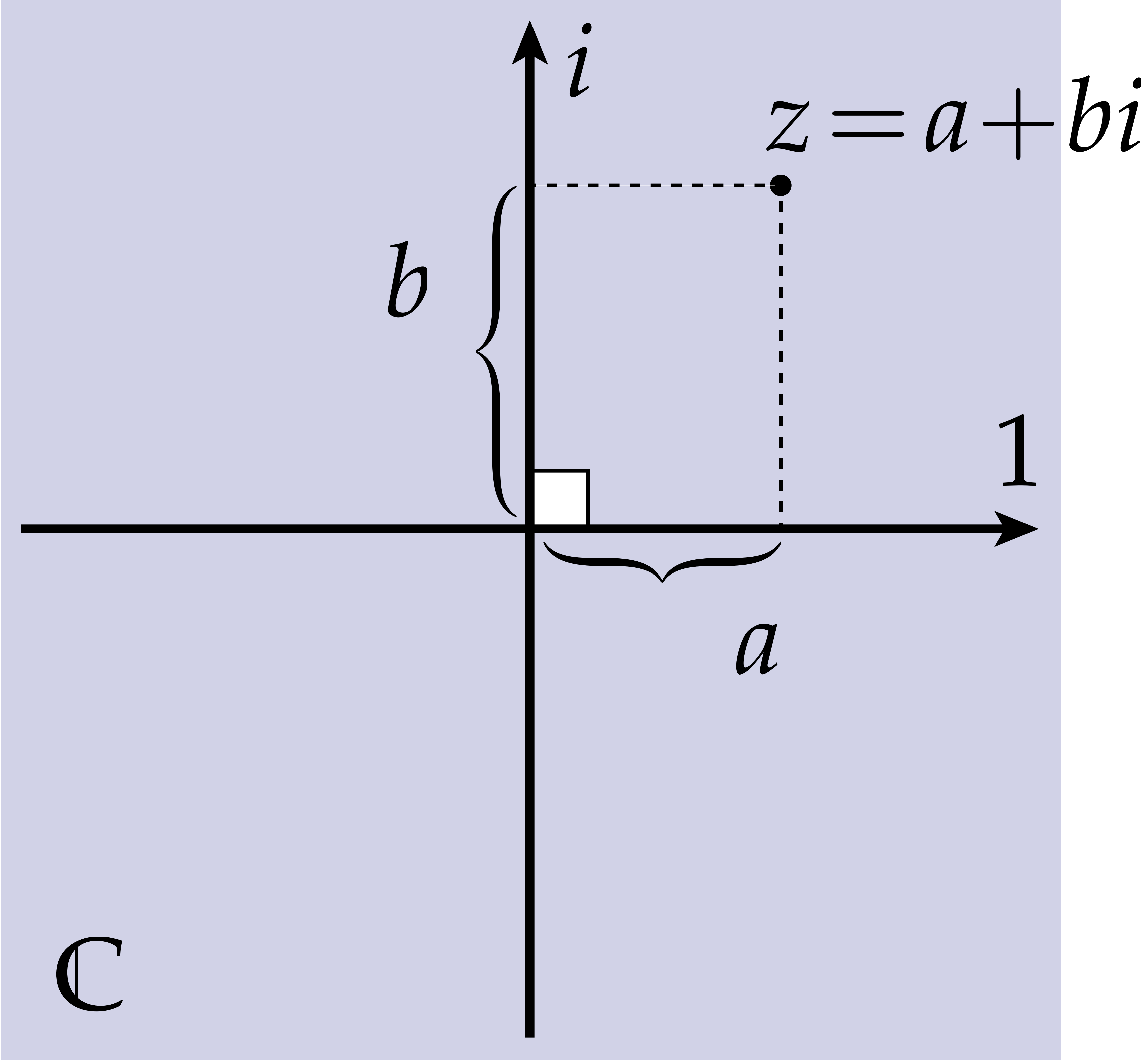
2. push forward vector, then rotate



(How can we write this condition more explicitly?)

Complex Numbers

- Not much different from the usual Euclidean plane
- Additional operations make it easy to express **scaling & rotation**
- Extremely natural for conformal geometry



- Two basis directions: 1 and i
- Points expressed as $z = a+bi$

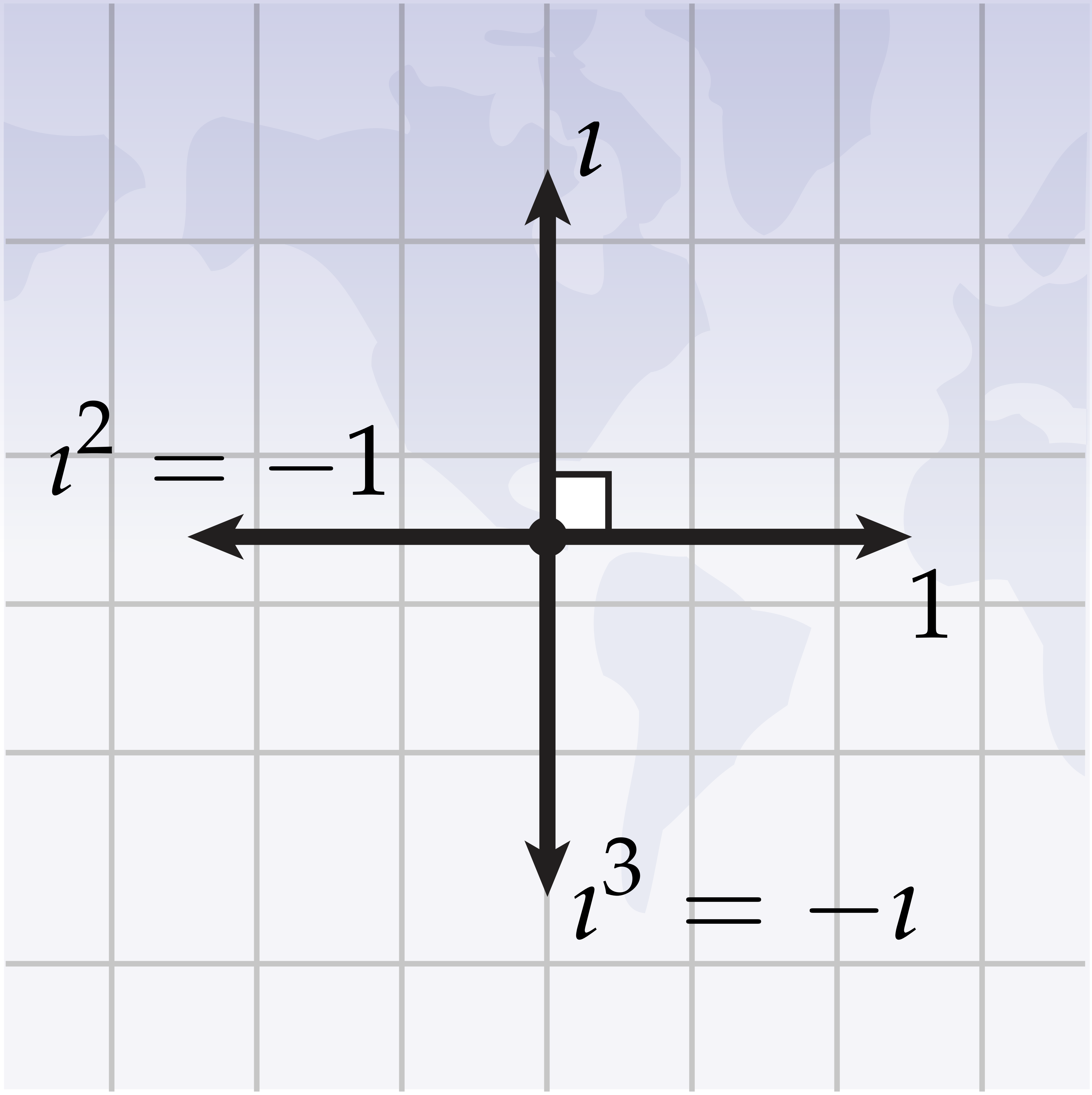
Complex Numbers

$$\cancel{\imath := \sqrt{-1}}$$

nonsense!

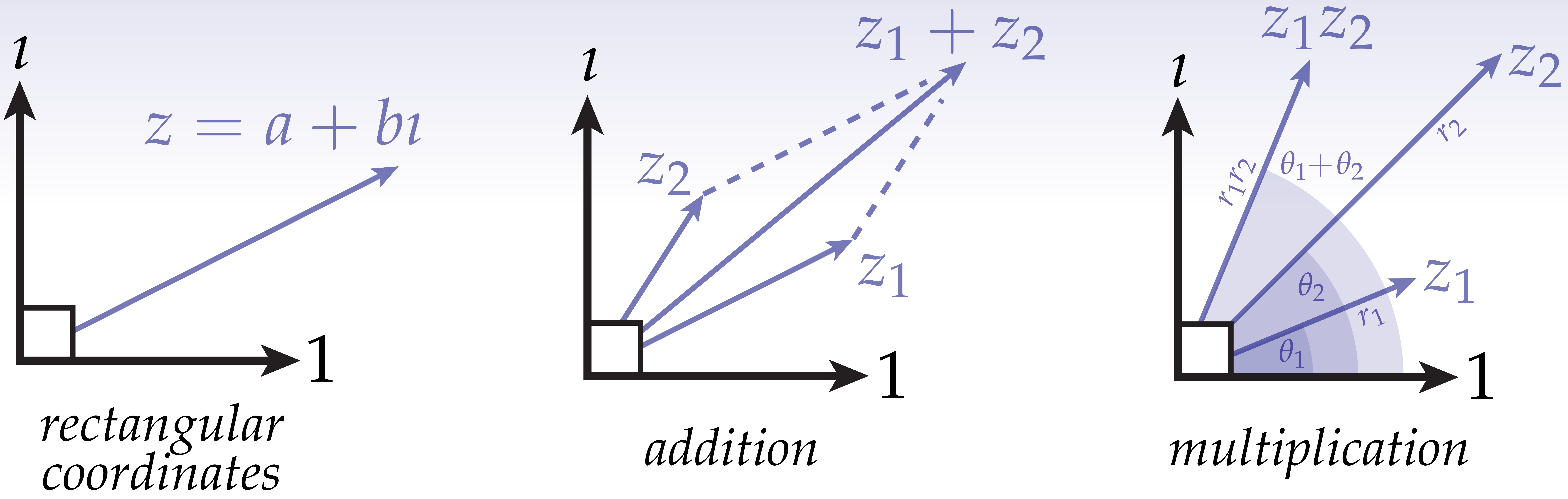
More importantly: obscures geometric meaning.

Imaginary Unit—Geometric Description



Symbol $\imath$ denotes *quarter-turn* in the *counter-clockwise* direction.

Complex Arithmetic — Visualized



Complex Product

- Usual definition:

- Complex product distributes over addition. Hence,

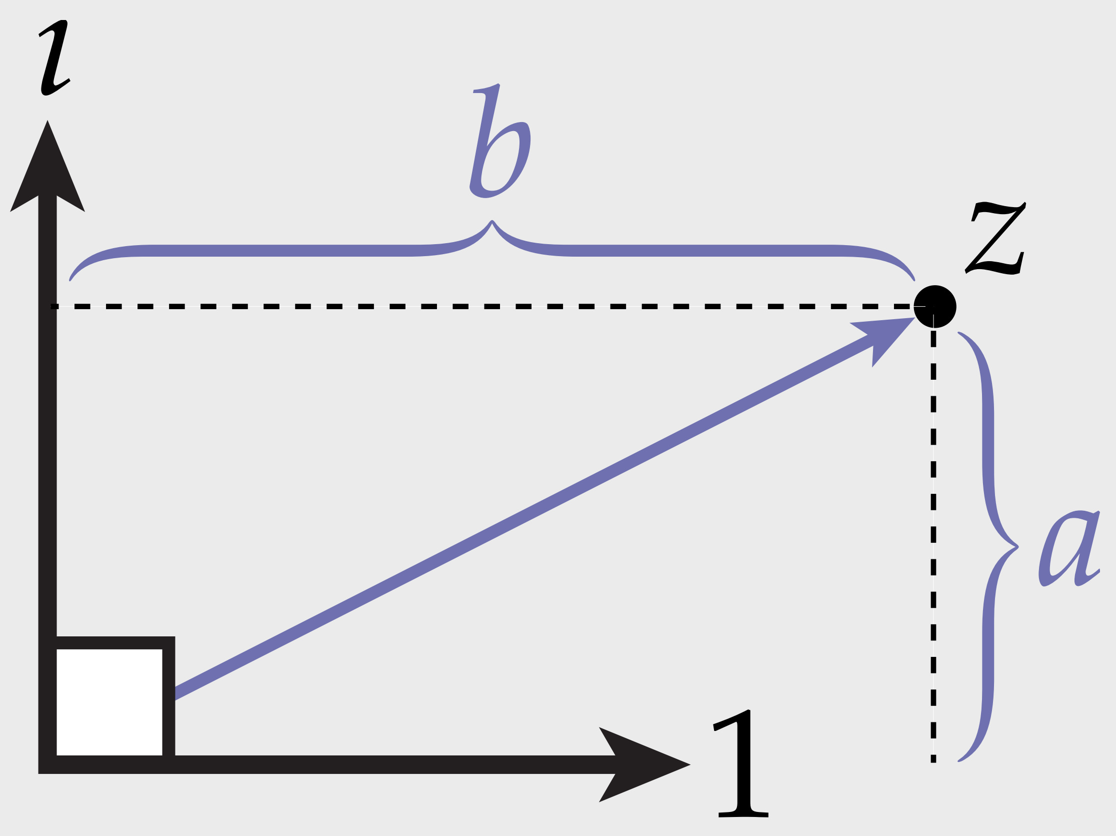
$$
\begin{aligned}
z_1 &:= a + b\imath \\
z_2 &:= c + d\imath
\end{aligned}
$$

$$
\begin{aligned}
z_1 z_2 &= (a + b\imath)(c + d\imath) \\
&= (a + b\imath)c + (a + b\imath)d\imath \\
&= ac + bc\imath + ad\imath + bd\imath^2 \\
&= \boxed{(ac - bd) + (ad + bc)\imath}
\end{aligned}
$$

Ok, terrific… but what does it mean *geometrically?*

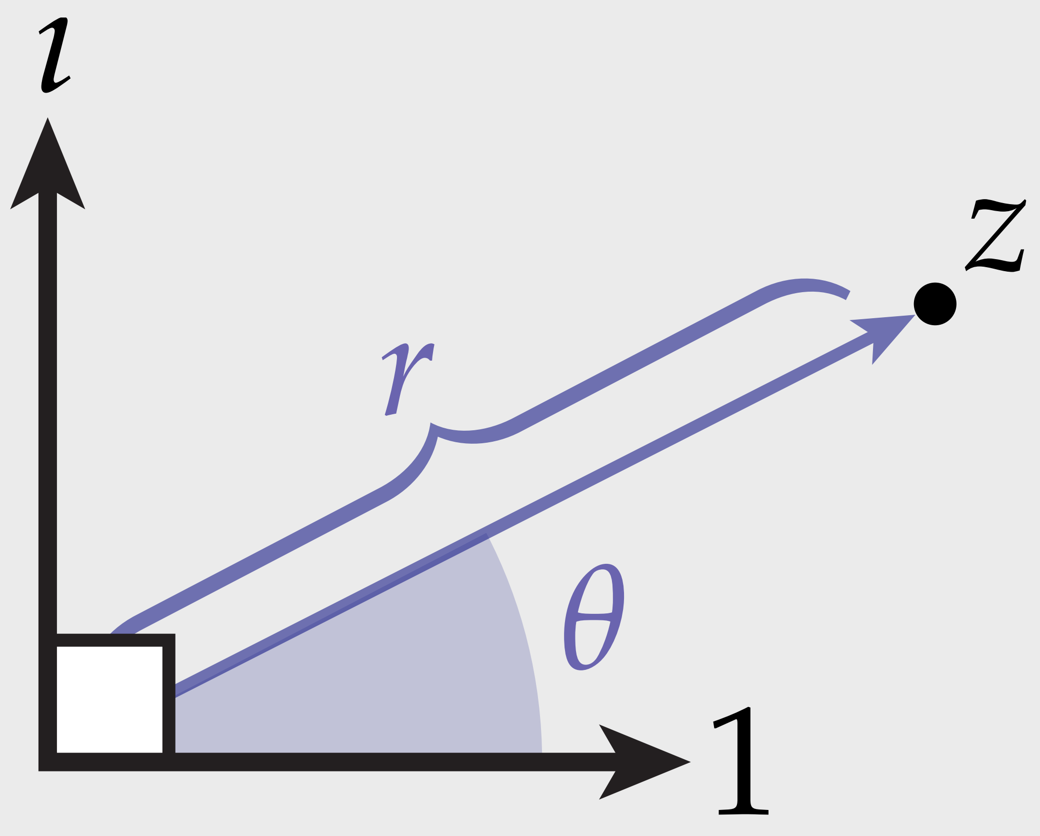
Rectangular vs. Polar Coordinates

RECTANGULAR



$$z = a + b\imath$$

POLAR



$$z = r(\cos\theta + \imath\sin\theta) = re^{\imath\theta}$$

EULER'S IDENTITY

$$\boxed{e^{\imath\theta} = \cos\theta + \imath\sin\theta}$$

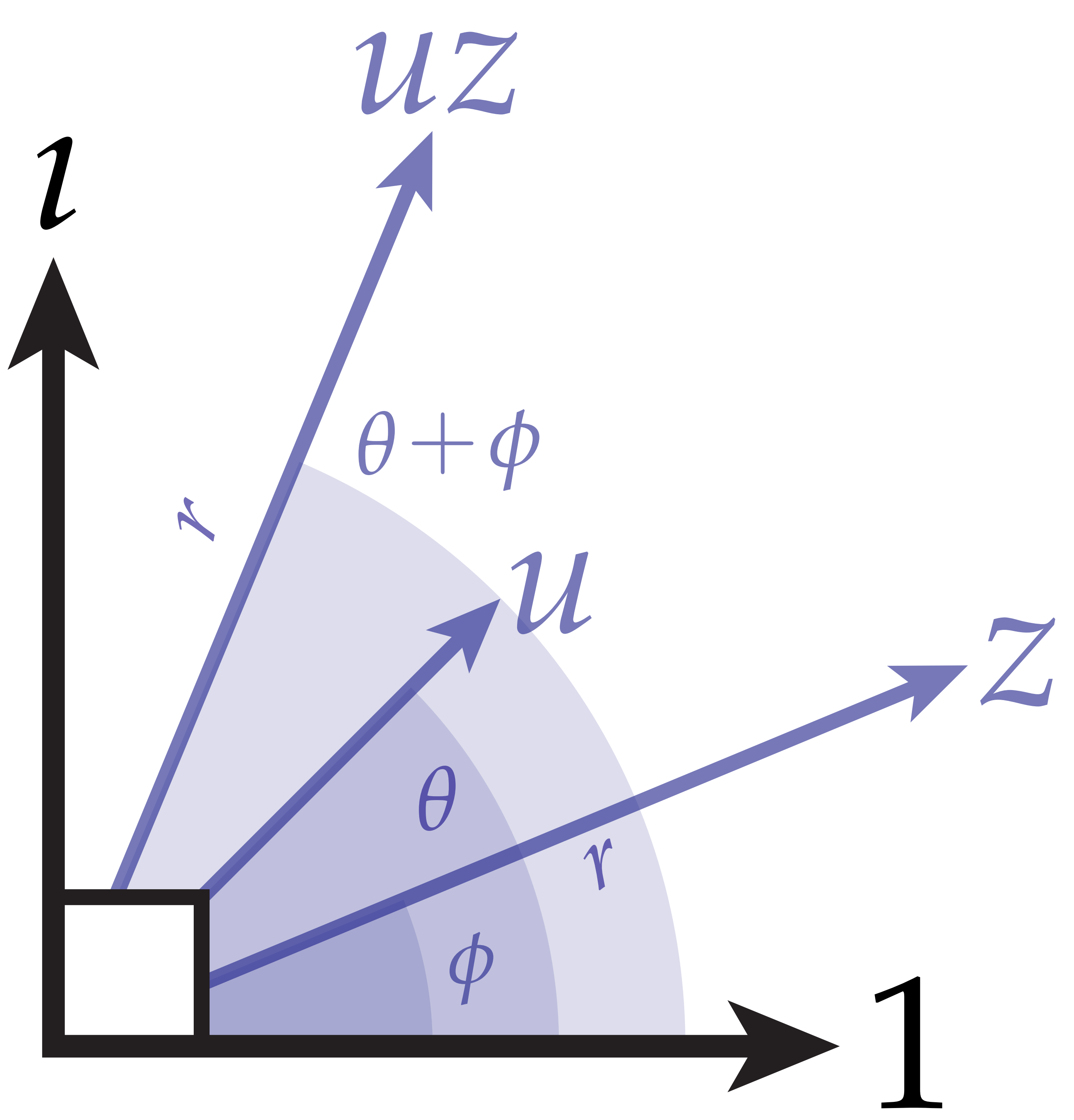
(In practice: just convenient shorthand!)

Rotations with Complex Numbers

- How can we express rotation?
- Let u be any *unit* complex number: $u = e^{\imath\theta}$
- Then for any point $z = re^{\imath\phi}$ we have

$$uz = (e^{\imath\theta})(re^{\imath\phi}) = re^{\imath(\theta+\phi)}$$

(same radius, new angle)

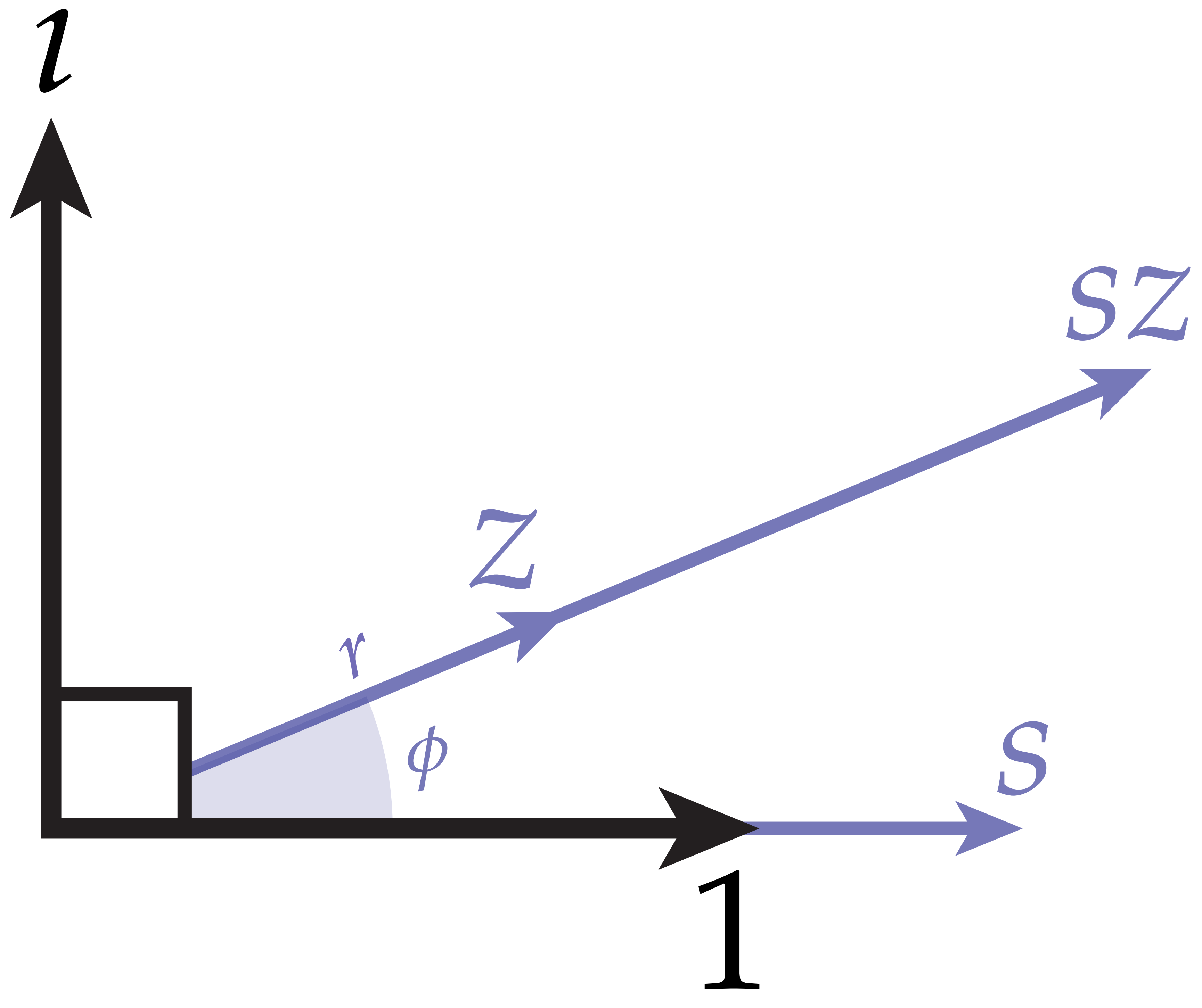


Scaling with Complex Numbers

- How can we express scaling?
- Let s be any *real* complex number: $s = a + 0\imath$
- Then for any point $z = re^{\imath\phi}$ we have

$$sz = (a + 0\imath)(re^{\imath\phi}) = are^{\imath\phi}$$

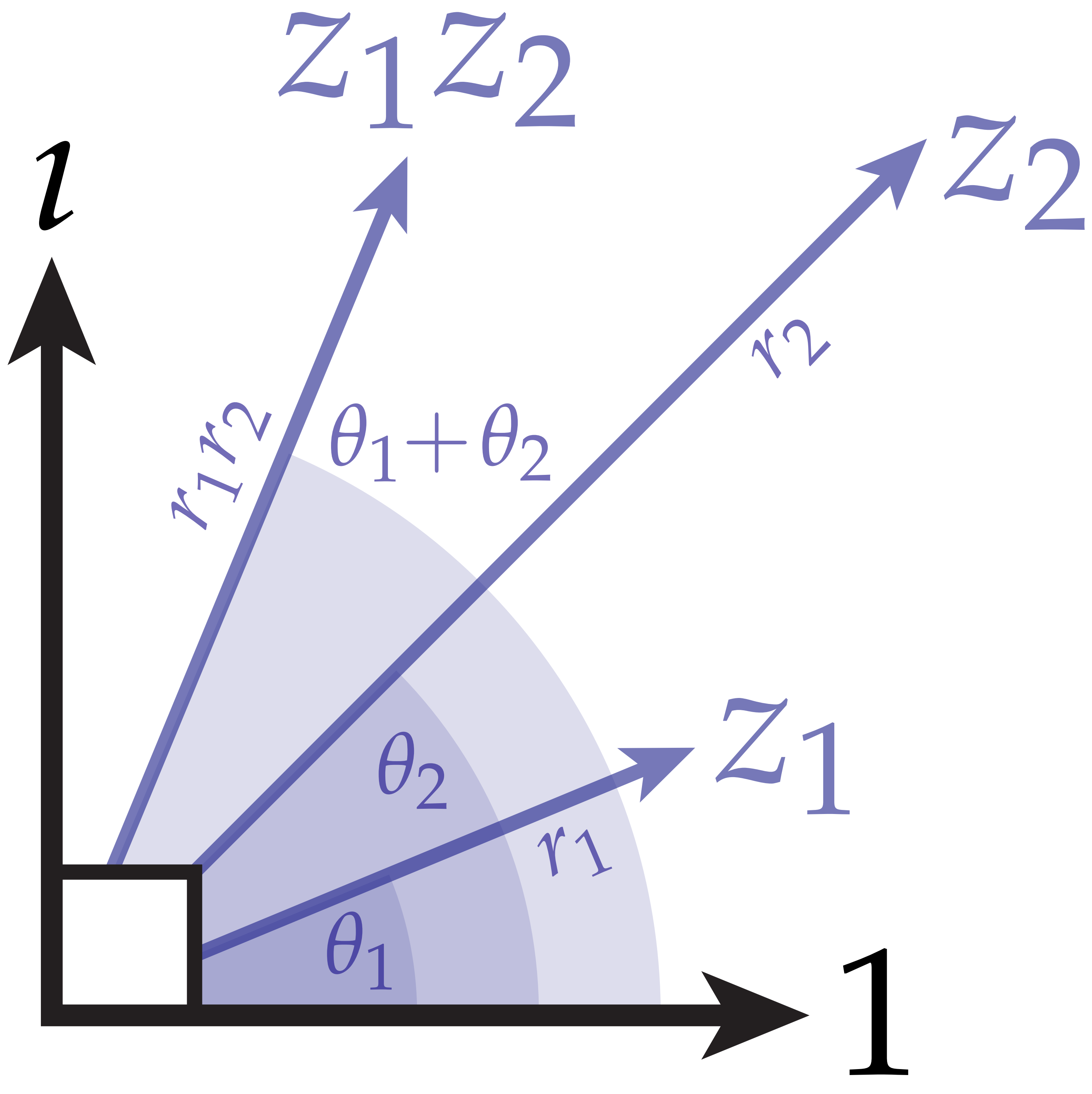
(same angle, new radius)



Complex Product—Polar Form

More generally, consider *any* two complex numbers:

$$\begin{aligned} z_1 &:= r_1 e^{\imath\theta_1} \\ z_2 &:= r_2 e^{\imath\theta_2} \end{aligned}$$



We can express their product as

$$z_1z_2 = (r_1e^{\imath\theta_1})(r_2e^{\imath\theta_2}) = r_1r_2e^{\imath(\theta_1+\theta_2)}$$

- New angle is *sum* of angles
- New radius is *product* of radii

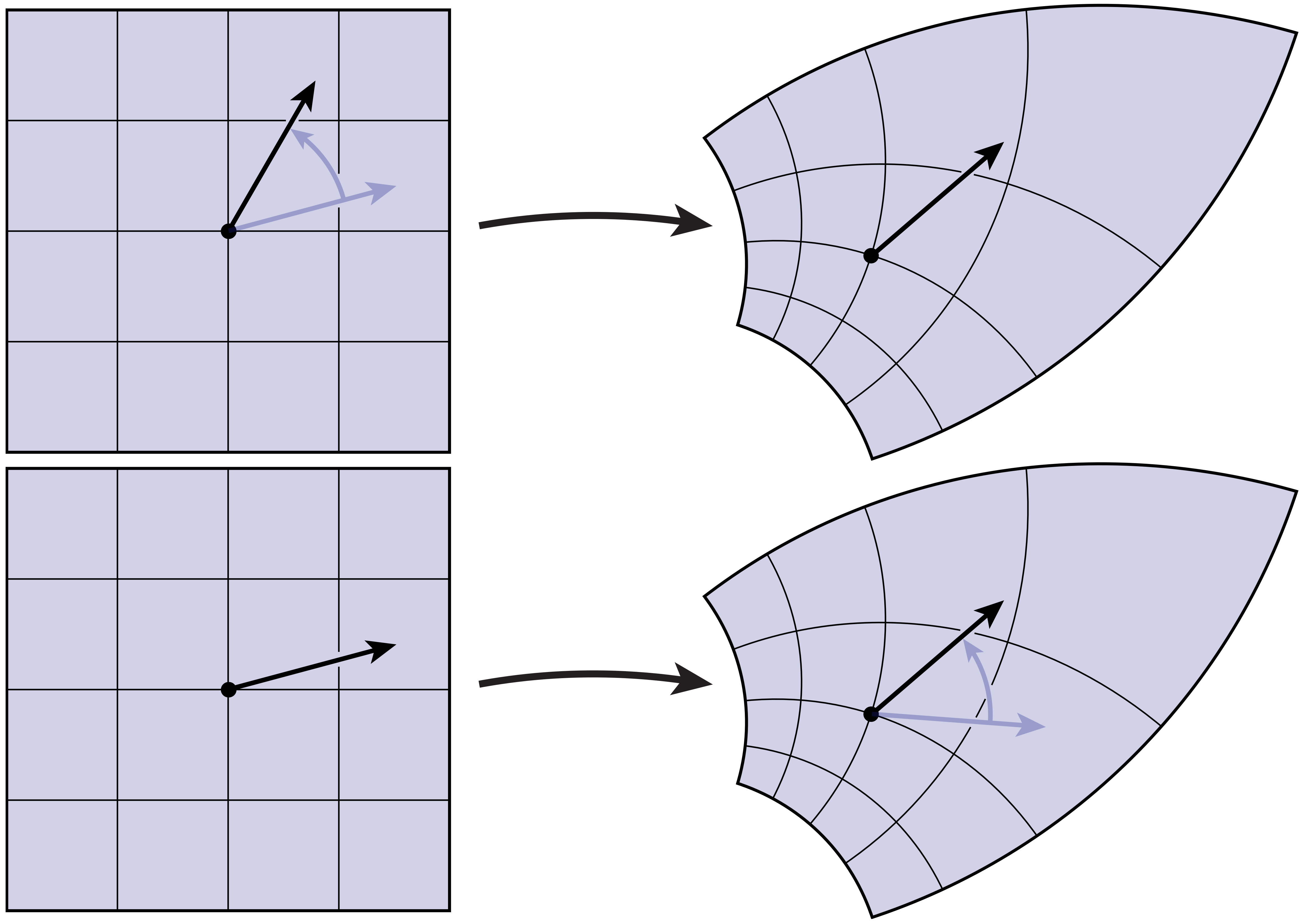
(Now forget the algebra and remember the ***geometry!****)*

Conformal Map, Revisited

- A map is conformal if two operations are equivalent:

1. rotate, then push forward vector

2. push forward vector, then rotate



(How can we write this condition more explicitly?)

Conformal Map, Revisited

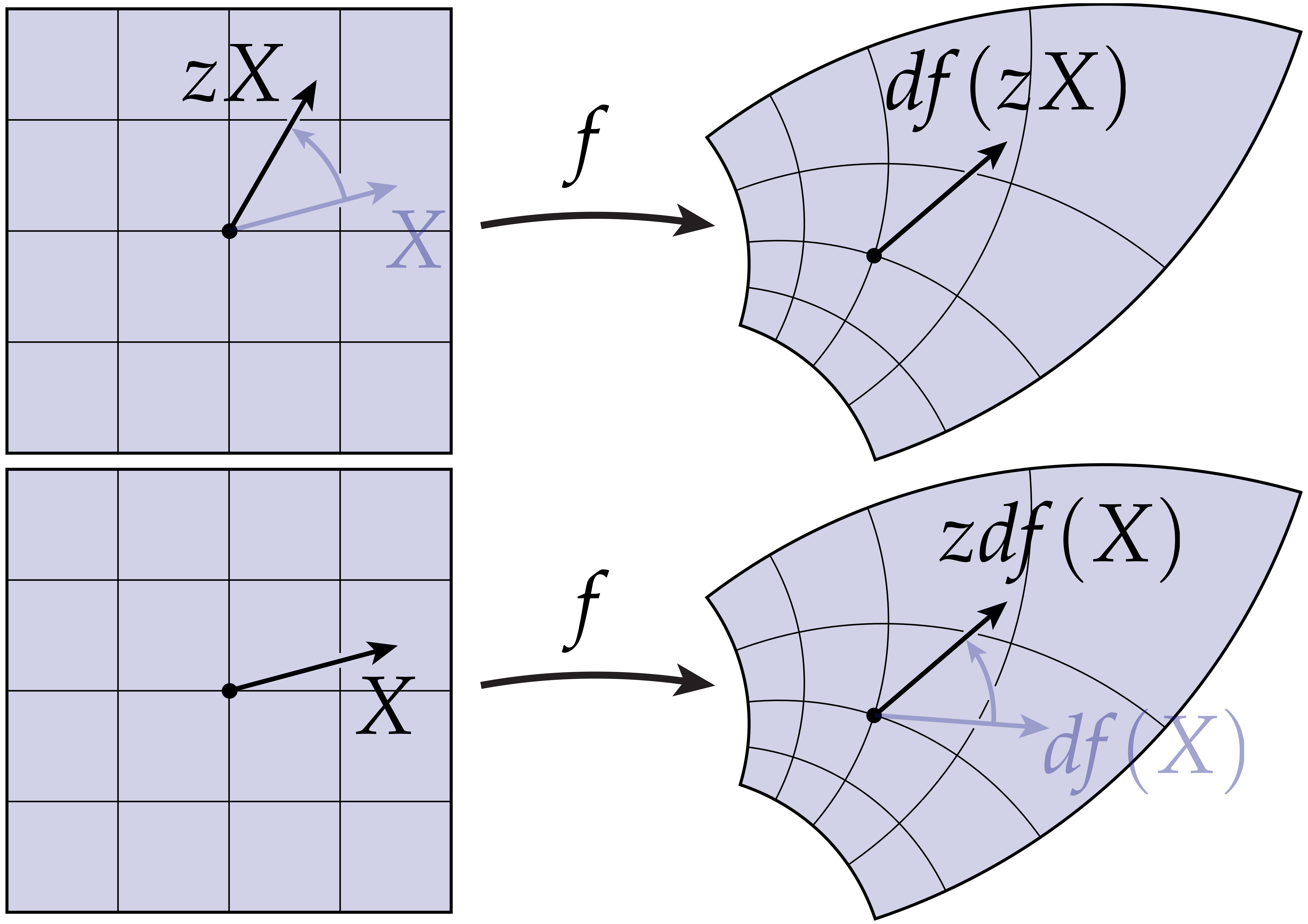
Consider a map $f : \mathbb{C} \to \mathbb{C}$

Then f is conformal as long as

$$df(zX) = zdf(X)$$

for all tangent vectors X and all complex numbers z.

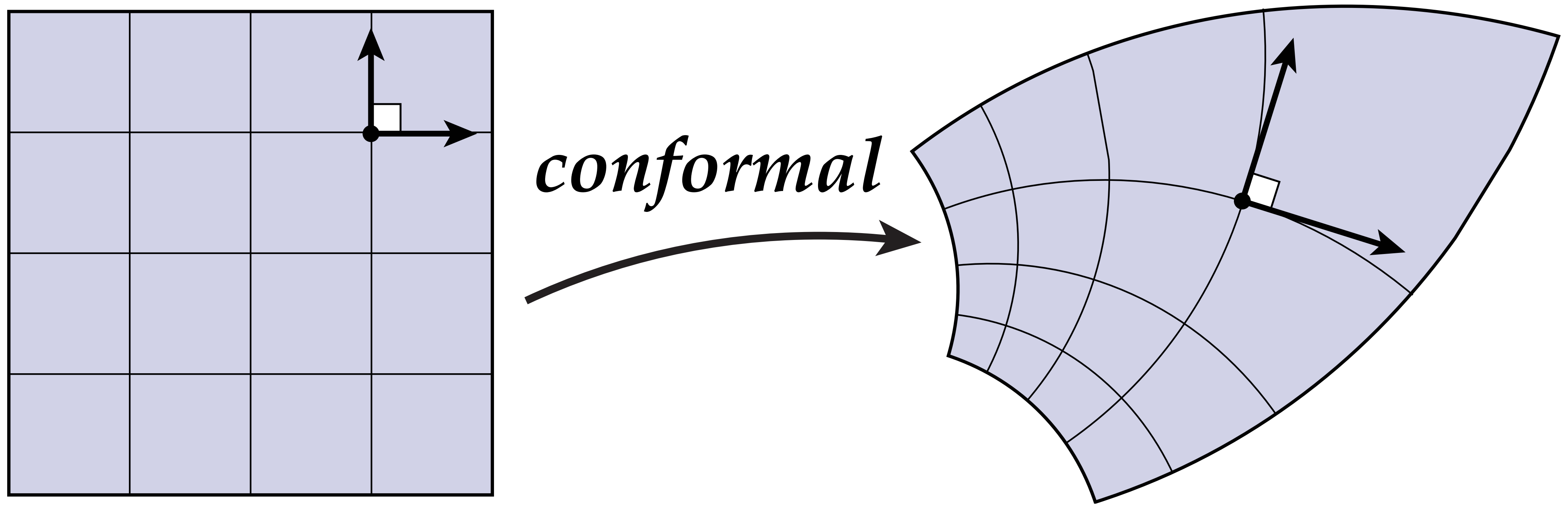
I.e., if it doesn't matter whether you rotate/scale *before* or *after* applying the map.

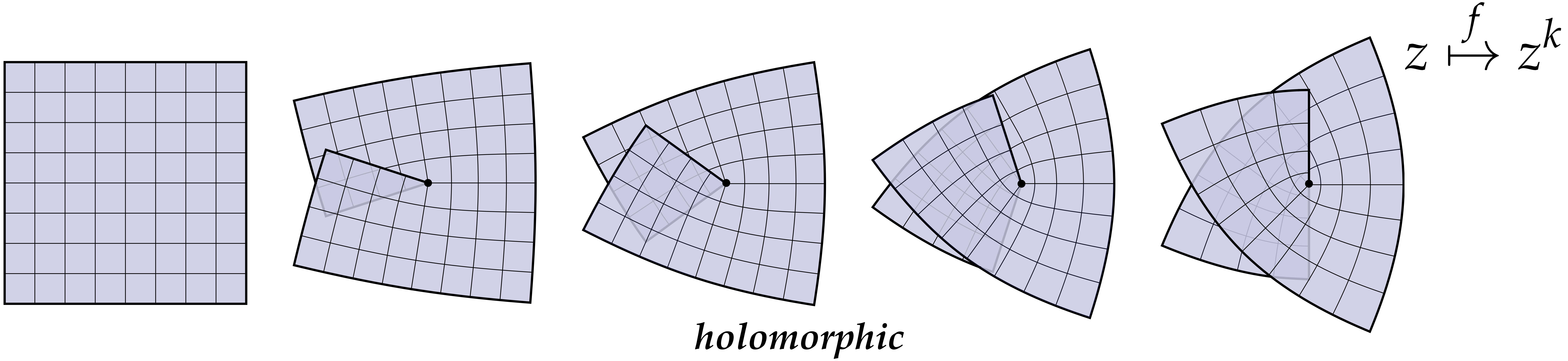


(df is "complex linear")

Holomorphic vs. Conformal

- Important linguistic distinction: a *conformal map* is a holomorphic map that is "nondegenerate", i.e., the differential is never zero.





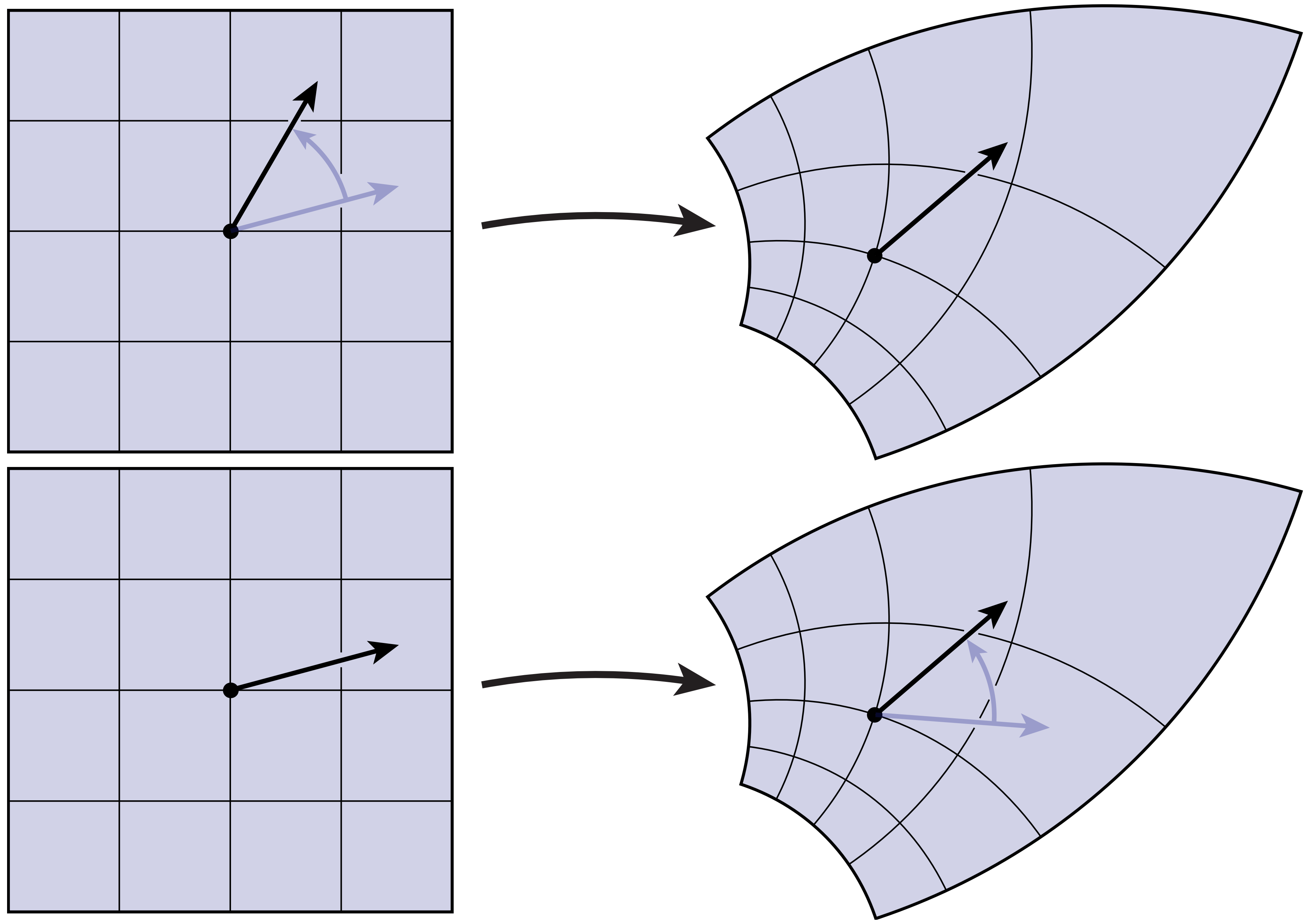
Cauchy-Riemann Equation

Several equivalent ways of writing *Cauchy-Riemann equation*:

$$df(zX) = z\,df(X)$$

$$df(\imath X) = \imath\,df(X)$$

$$\star df = \imath df$$



$$\left.\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial f_2}{\partial y} \\ \frac{\partial f_1}{\partial y} &= -\frac{\partial f_2}{\partial x} \end{aligned}\right\}$$

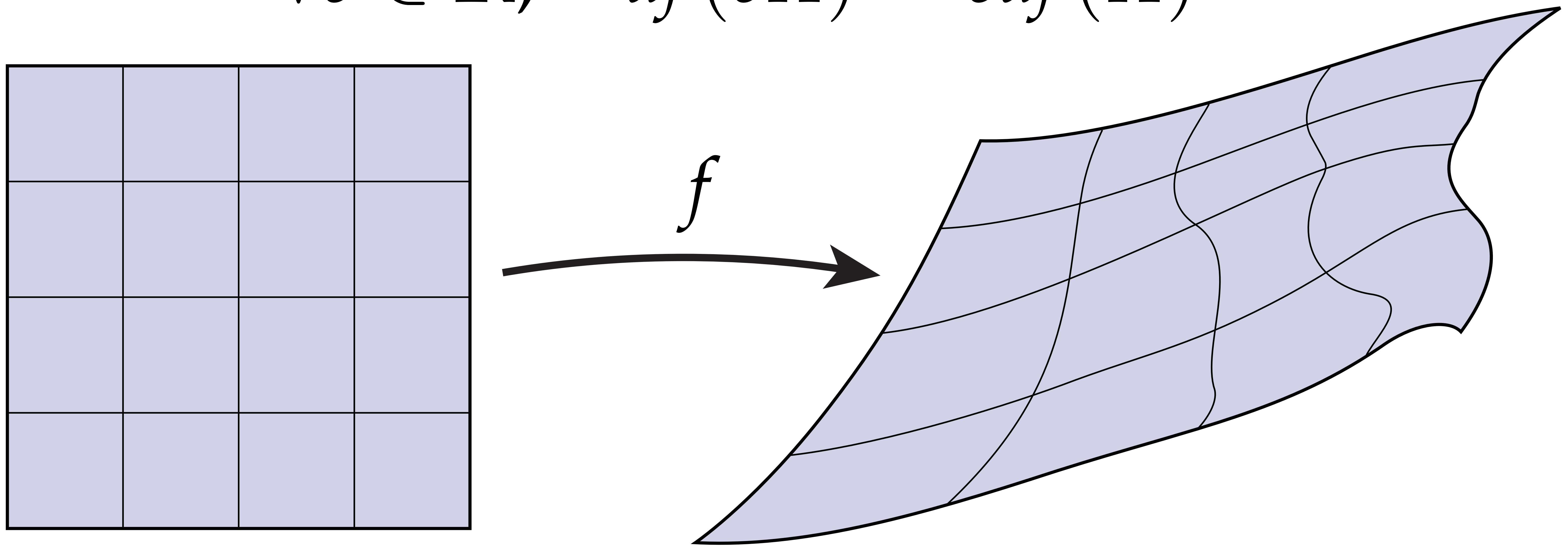
$$\bar{\partial} f = 0$$

All express the same **geometric** idea!

Aside: Real vs. Complex Linearity

What if we just ask for *real* linearity?

$$\forall c \in \mathbb{R}, \quad df(cX) = c\,df(X)$$



No angle preservation.

In fact, maps can be arbitrarily "ugly". Why?

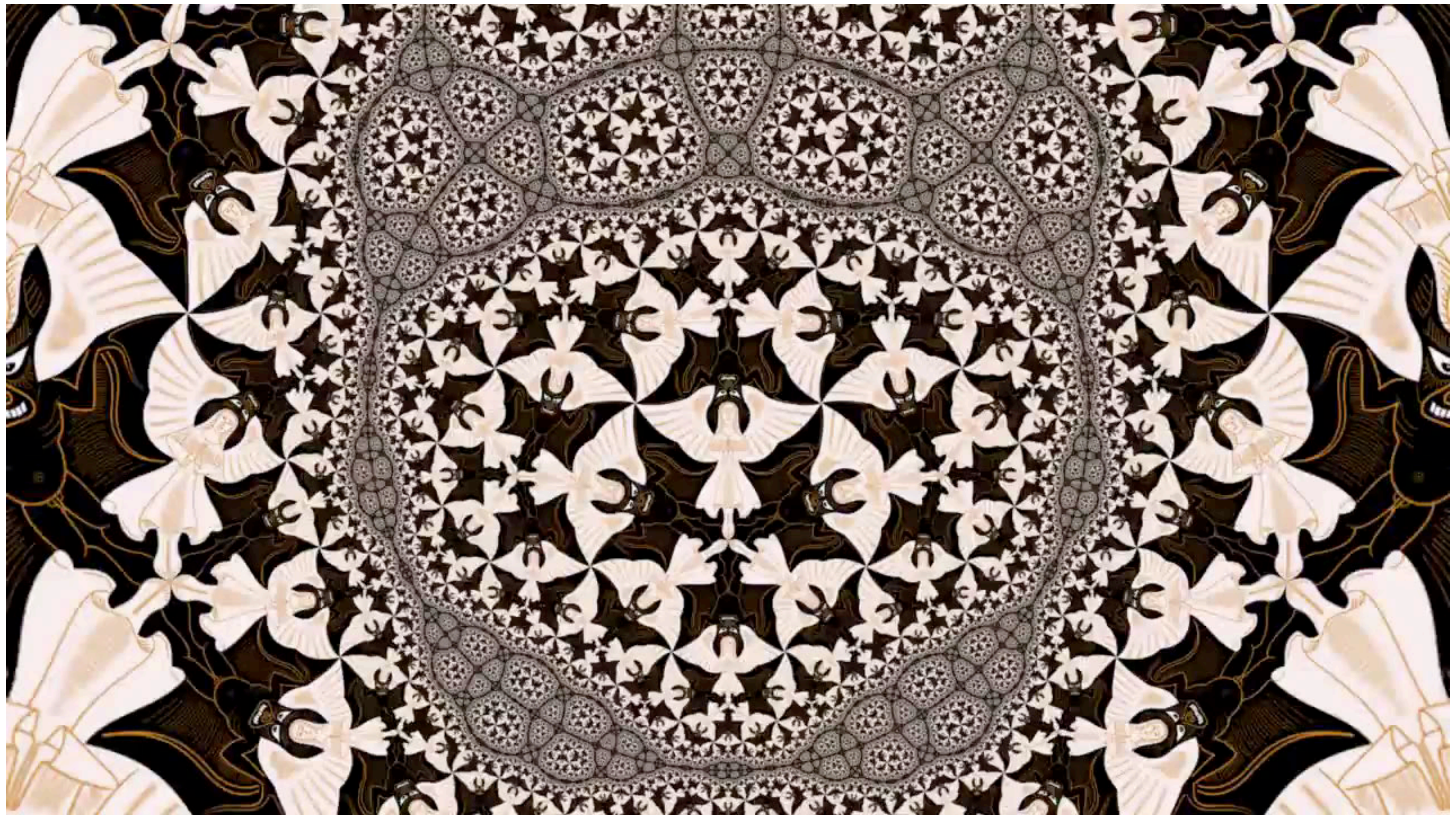
Because *any* differentiable f trivially satisfies this property!

Example—Möbius Transformations (2D)

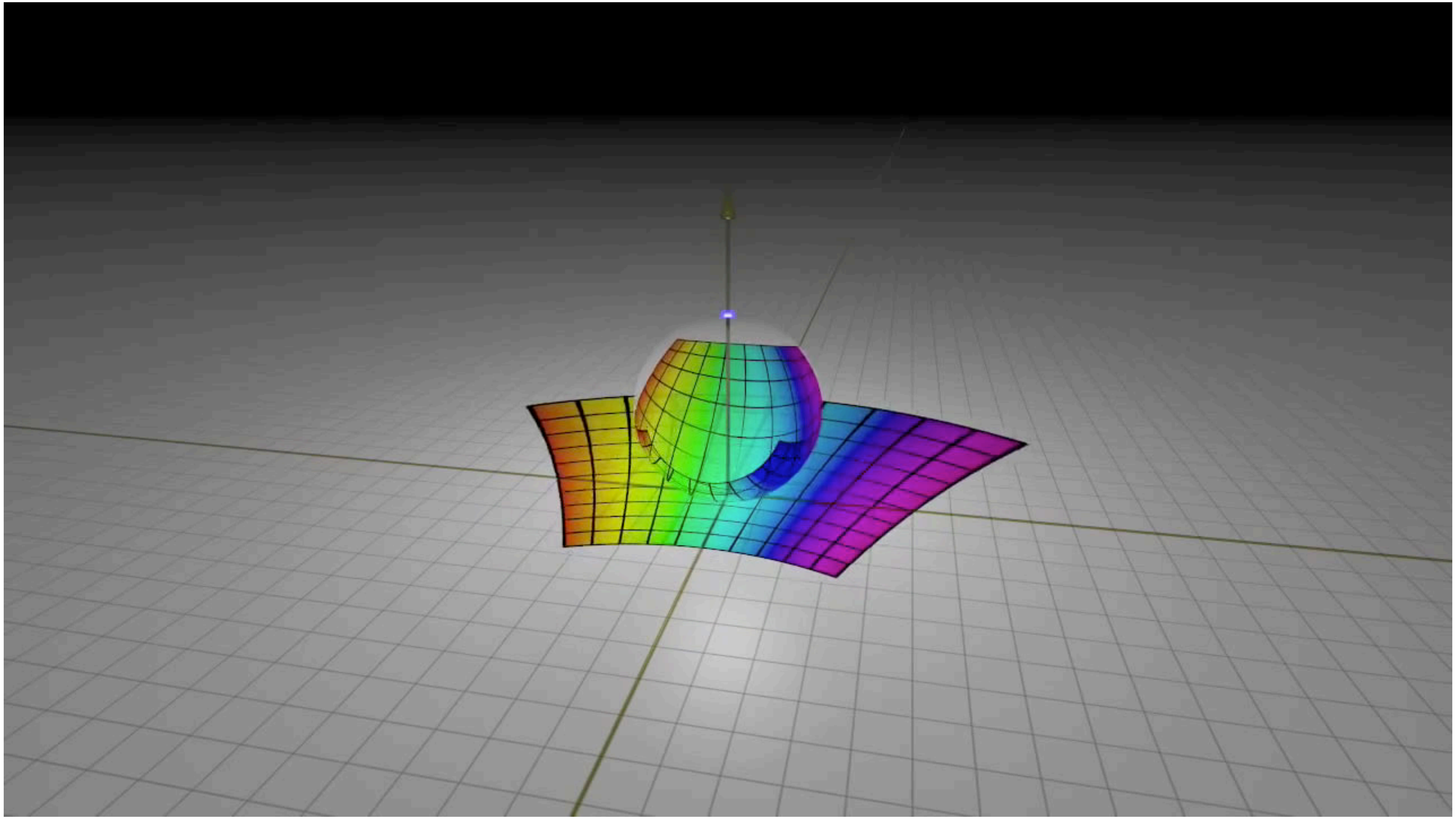
Definition. In 2D, a Möbius transformation is an orientation-preserving map taking circles to circles or lines, and lines to lines or circles. Algebraically, any Möbius transformation can be expressed as a map of the form

$$z \mapsto \frac{az+b}{cz+d}$$

for complex constants $ad \neq bc$.



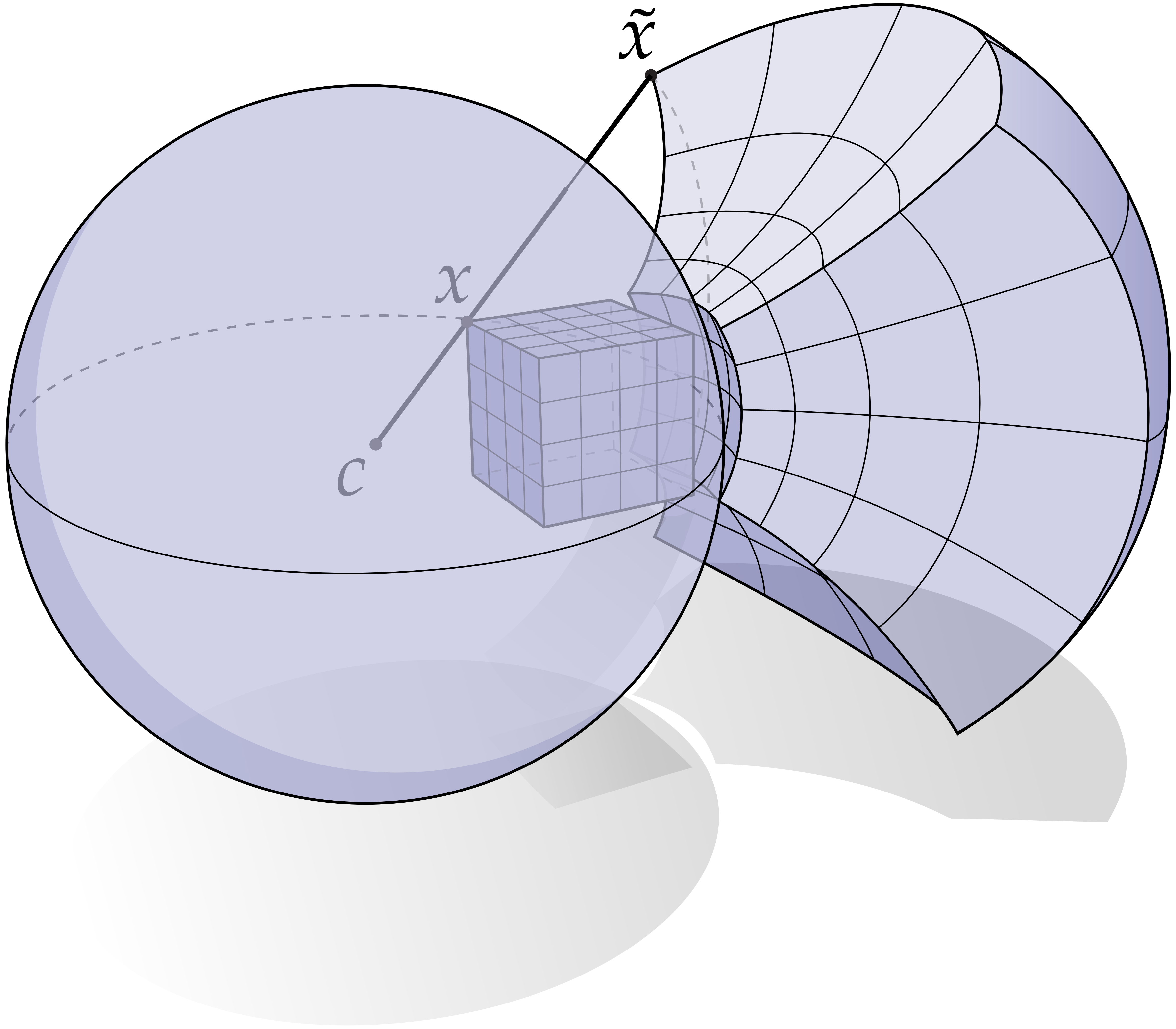
Möbius Transformations "Revealed"



(Douglas Arnold and Jonathan Rogness) https://www.ima.umn.edu/~arnold/moebius/

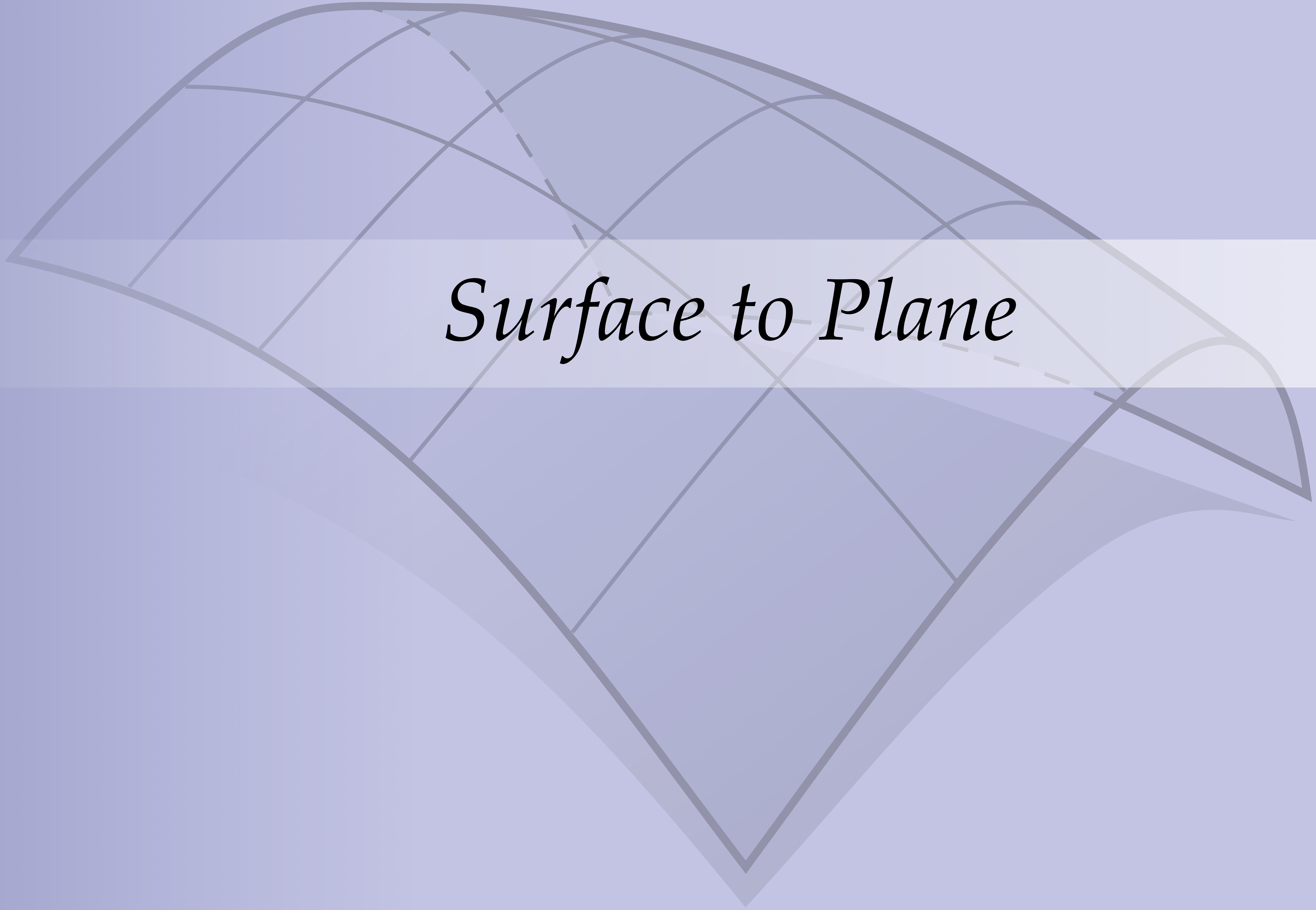
Sphere Inversion (nD)

$$x \mapsto \frac{x-c}{|x-c|^2}$$



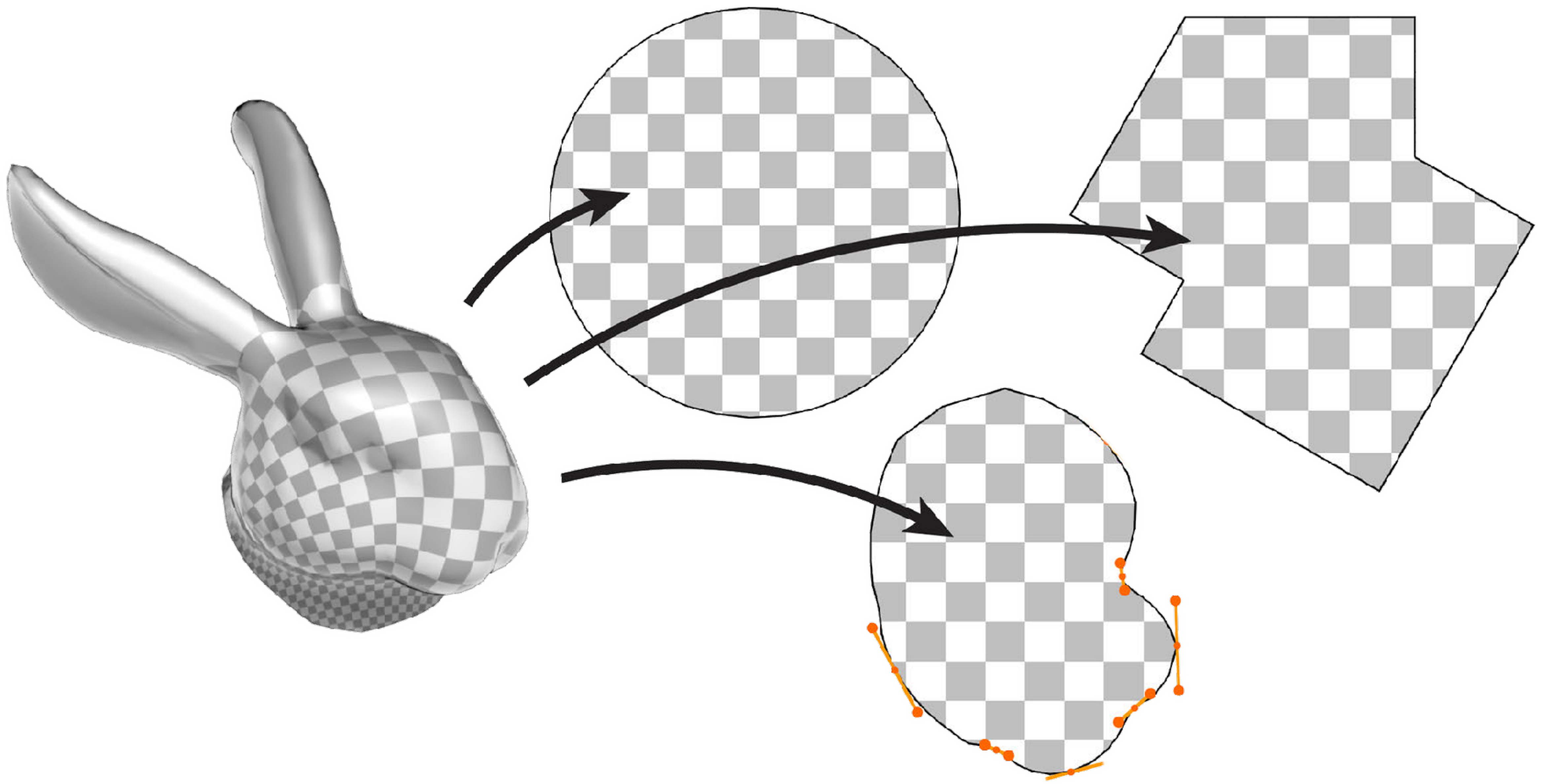
(**Note:** Reverses orientation—*anticonformal* rather than conformal)

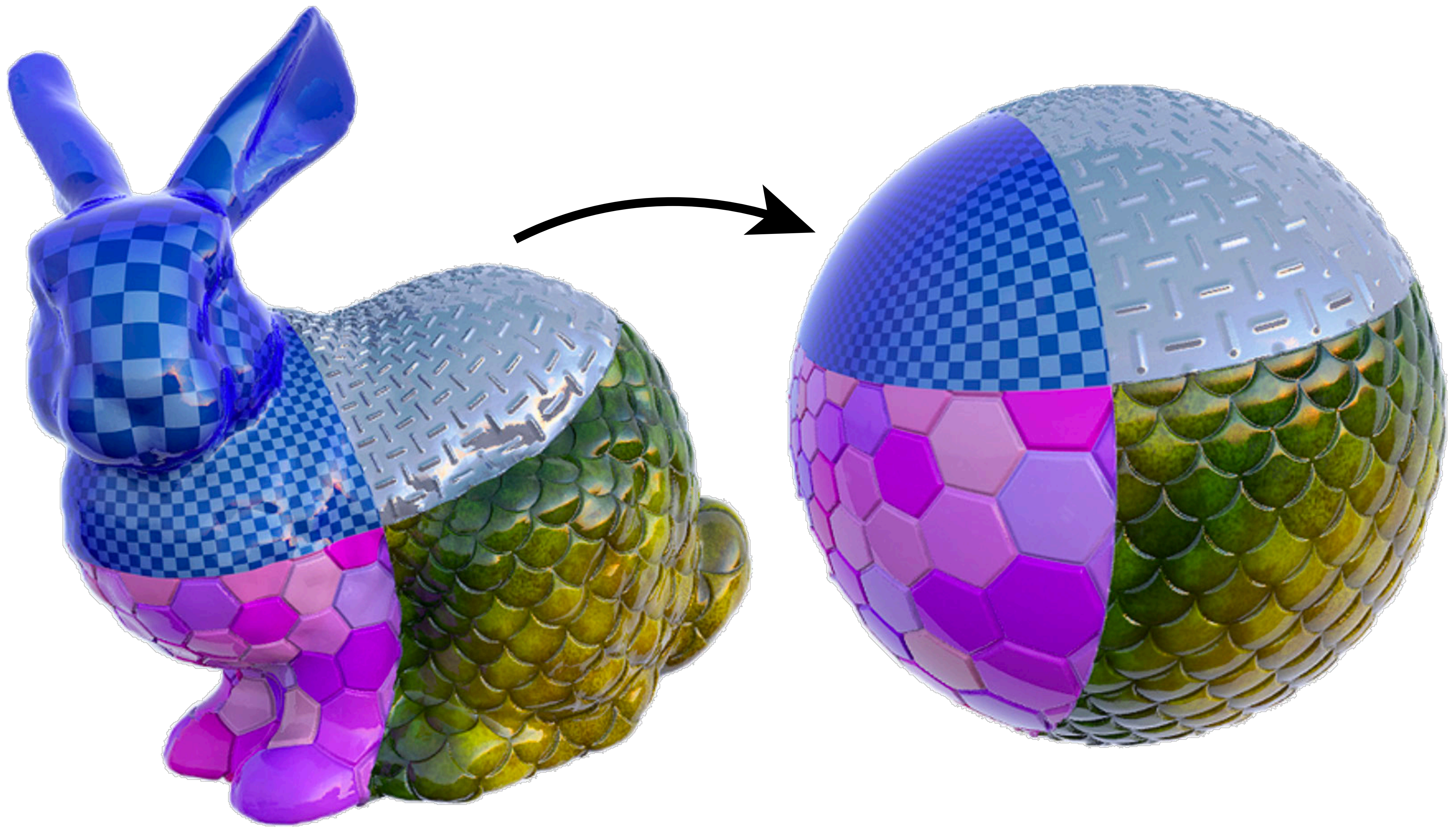
Surface to Plane



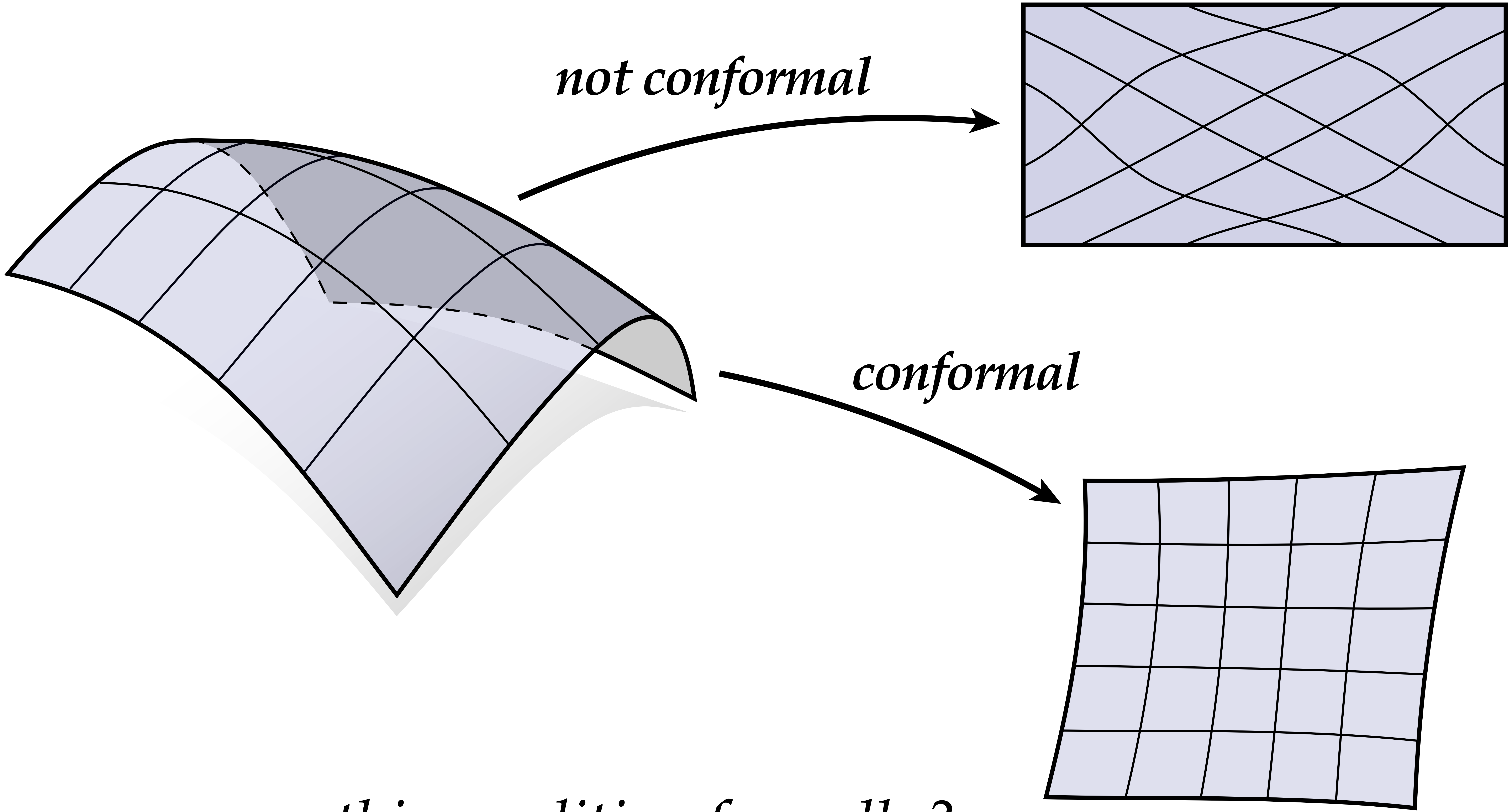
Surface to Plane

- Map curved surface to 2D plane ("conformal flattening")
- Surface does not necessarily sit in 3D
- Slight generalization: target curvature is constant but *nonzero* (*e.g.*, sphere)
- Many different equations: Cauchy-Riemann, Yamabe, …





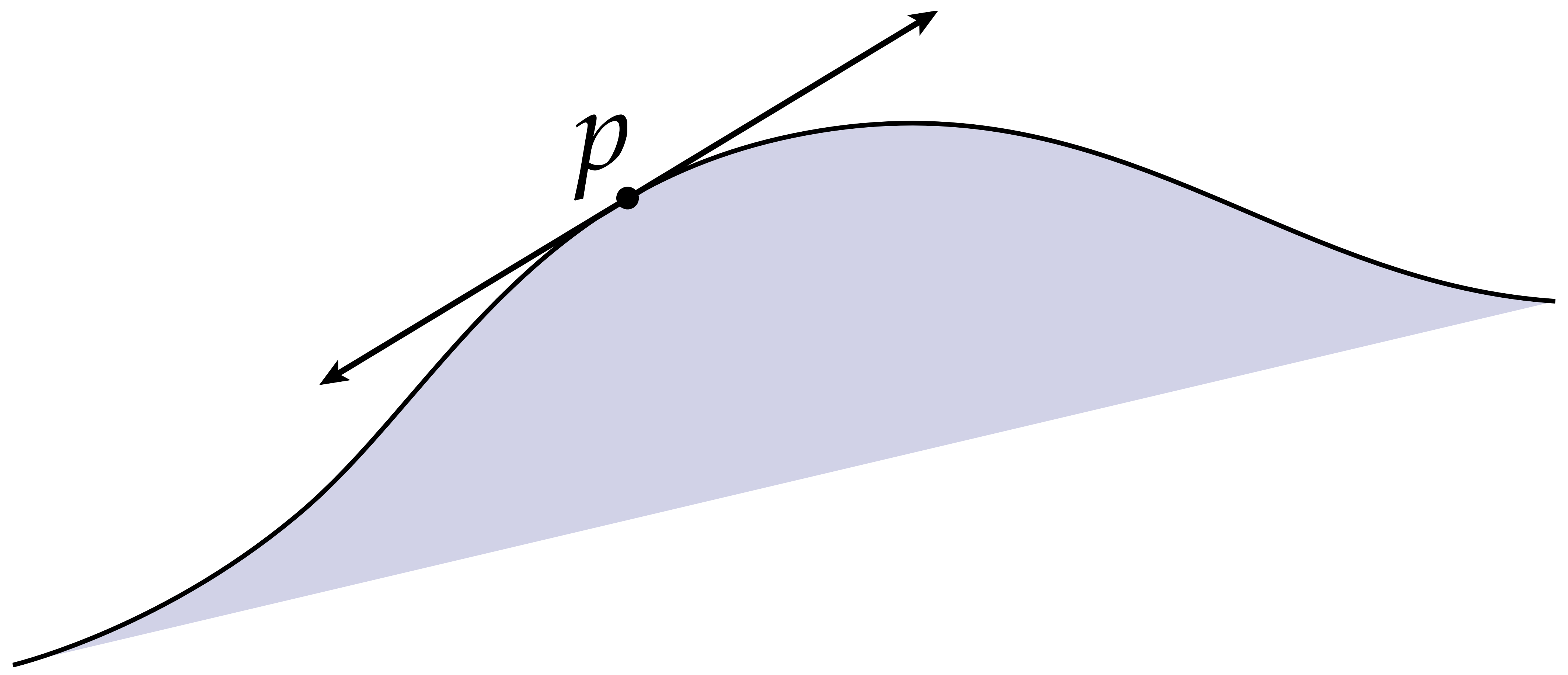
Conformal Maps on Surfaces—Visualized

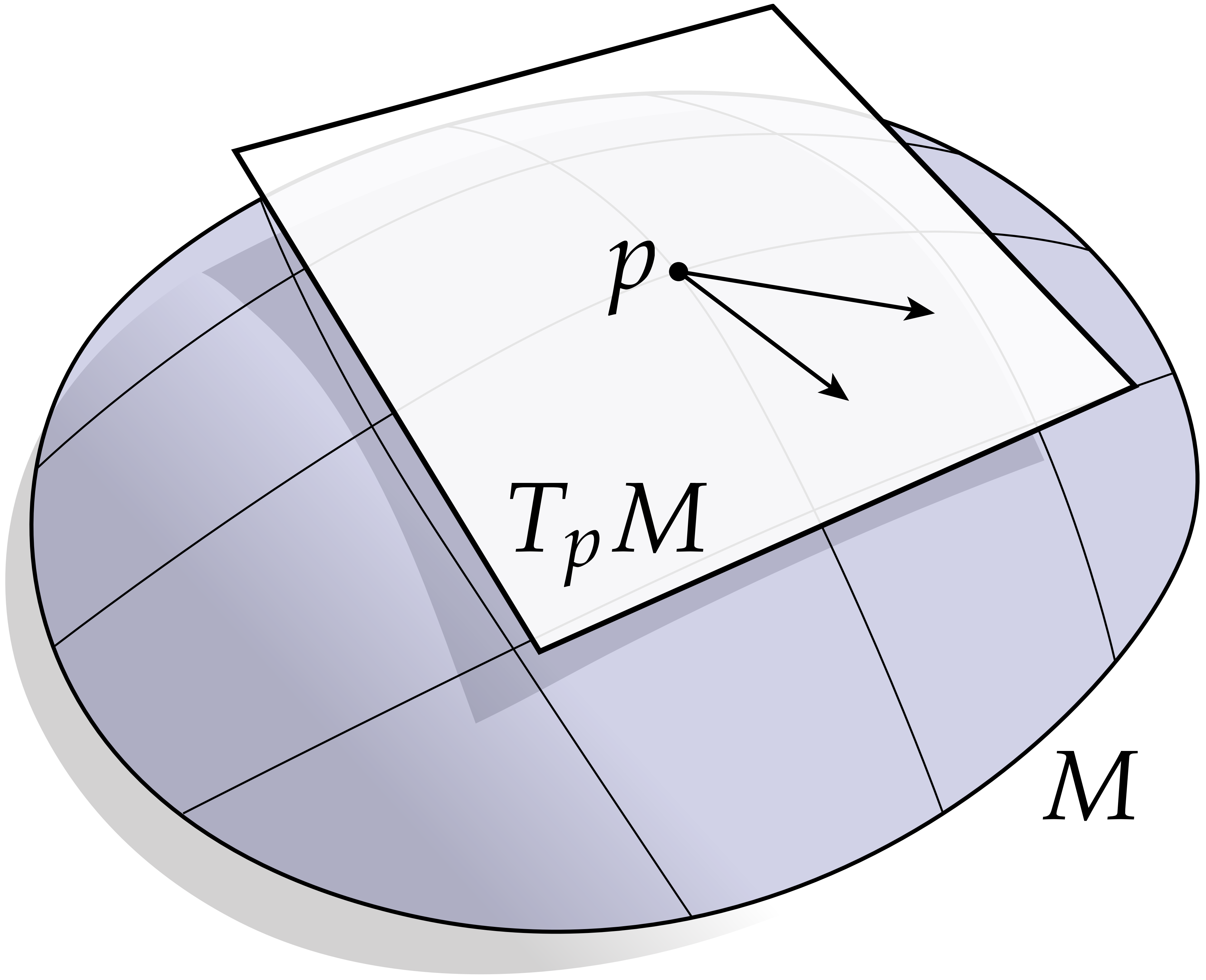


How do we express this condition formally?

Tangent Plane

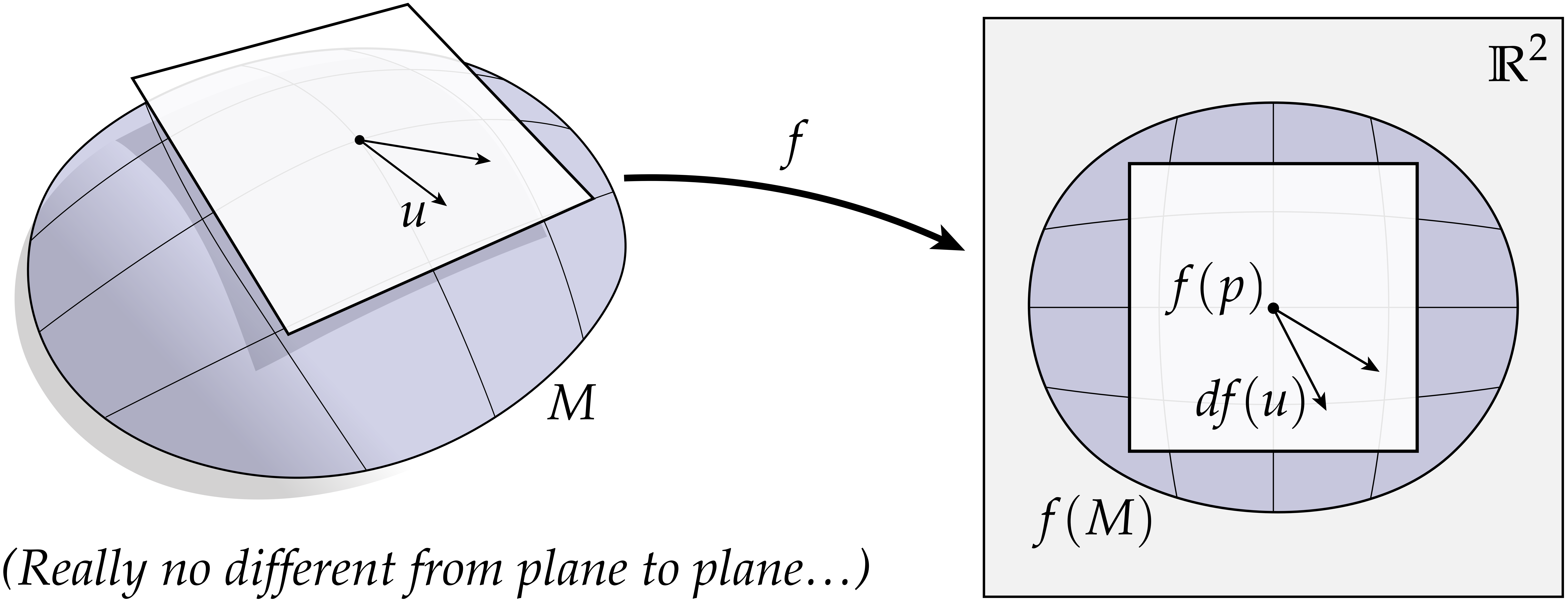
- *Tangent vectors* are those that "graze" the surface
- *Tangent plane* is all the tangent vectors at a given point





Differential of a Map from Surface to Plane

- Consider a map taking each point of a surface to a point in the plane
- *Differential* says how tangent vectors get "stretched out" under this map



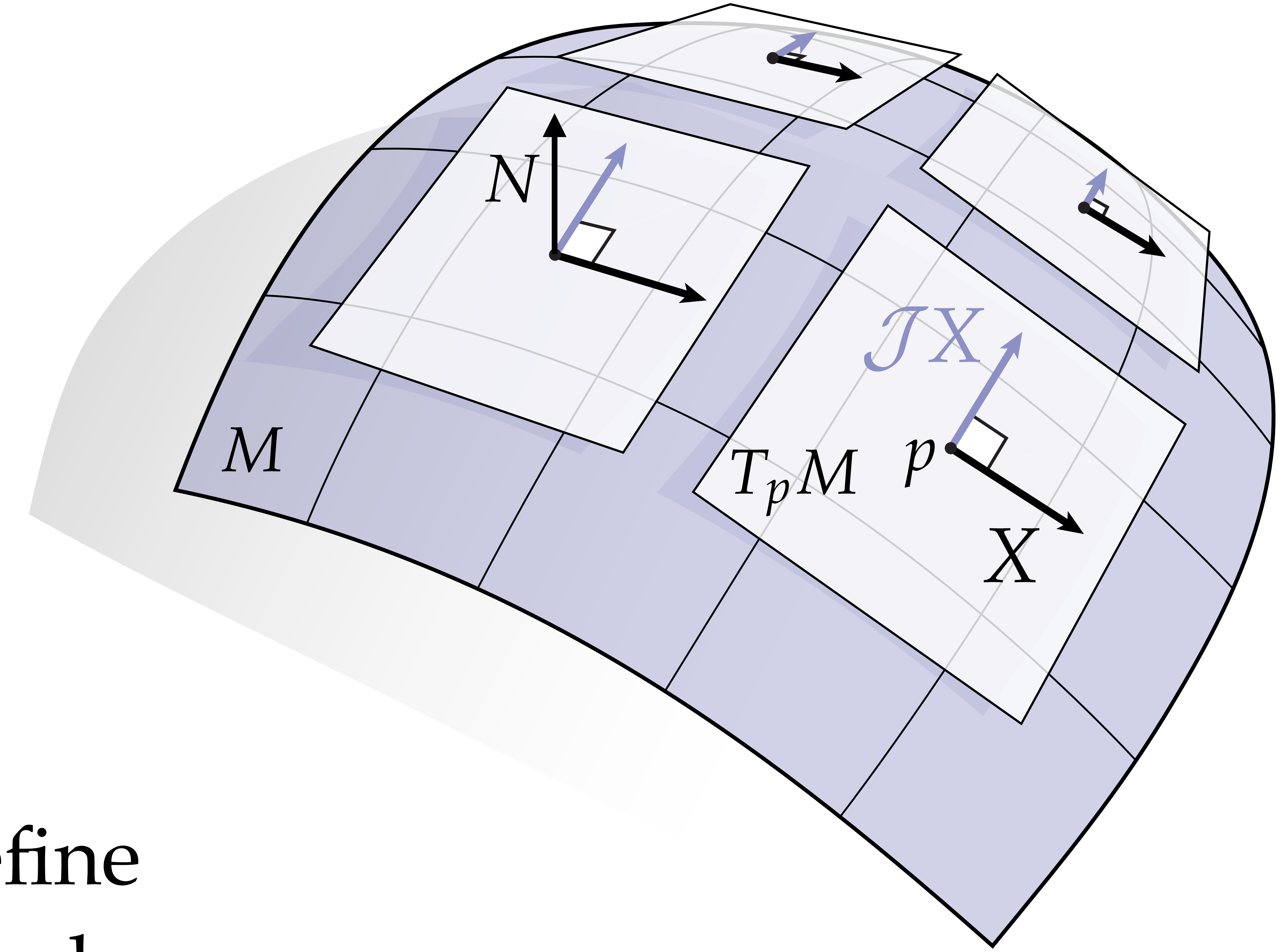
(Really no different from plane to plane…)

Complex Structure

- *Complex structure J* rotates vectors in each tangent plane by 90 degrees
- Analogous to complex unit i
- E.g., $\mathcal{J} \circ \mathcal{J} = -\mathrm{id}$
- For a surface in $\mathbb{R}^3$:

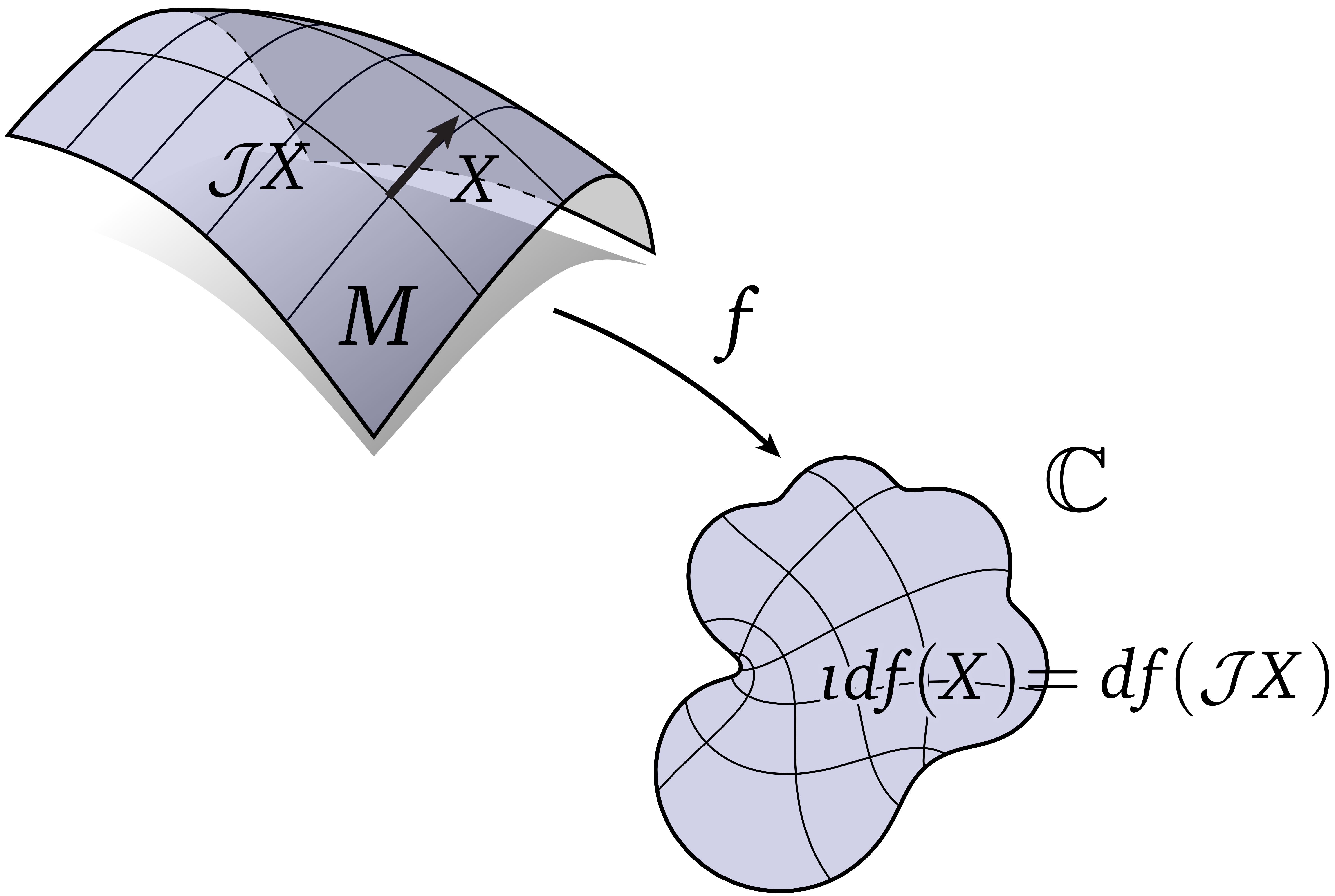
$$\mathcal{J}X = N \times X$$

(where N is unit normal)



Motivation: will enable us to define conformal maps from surface to plane.

Holomorphic Maps from a Surface to the Plane



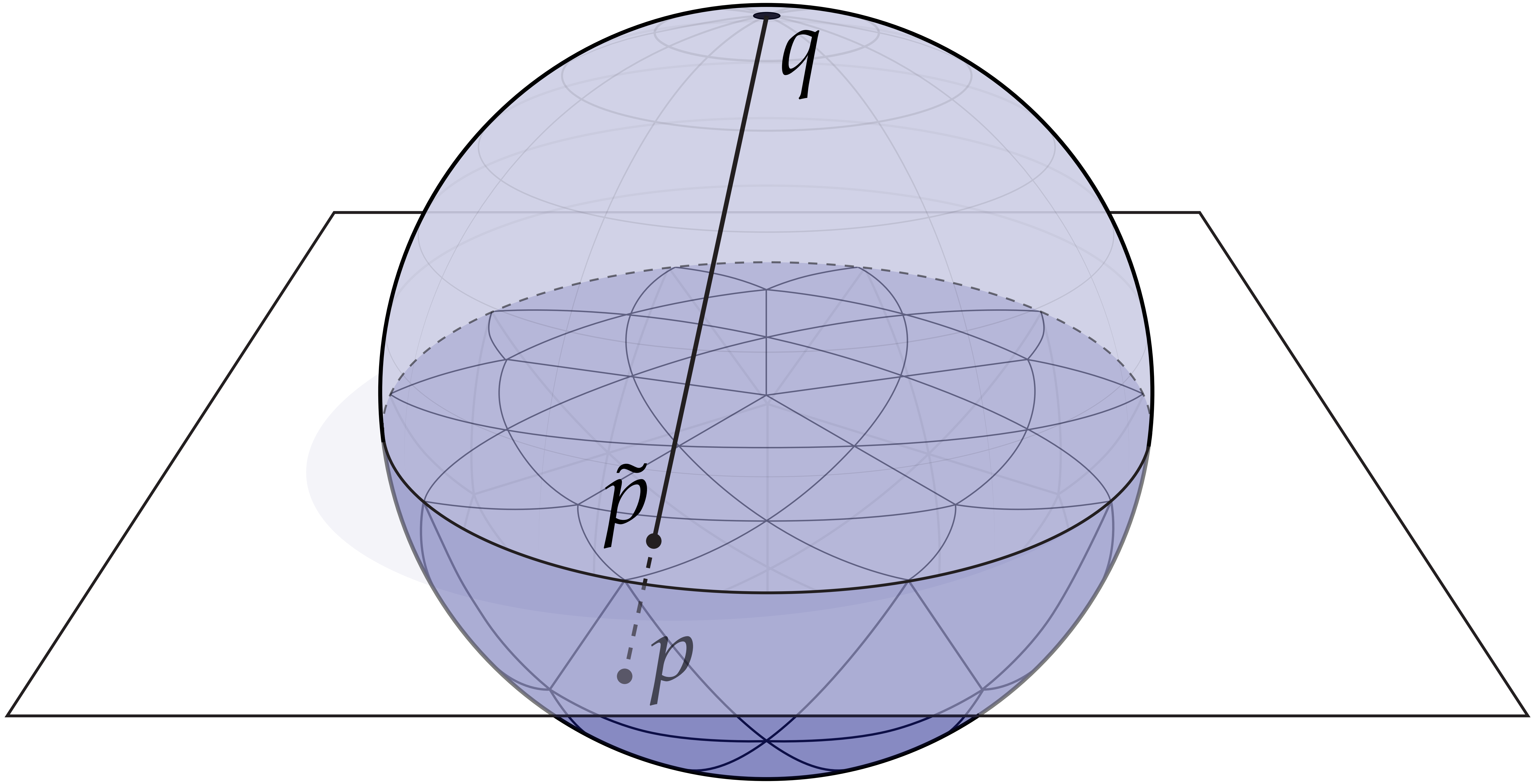
Plane to plane:

$$df(\imath X) = \imath df(X)$$

Surface to plane:

$$df(\mathcal{J}X) = \imath df(X)$$

Example—Stereographic Projection

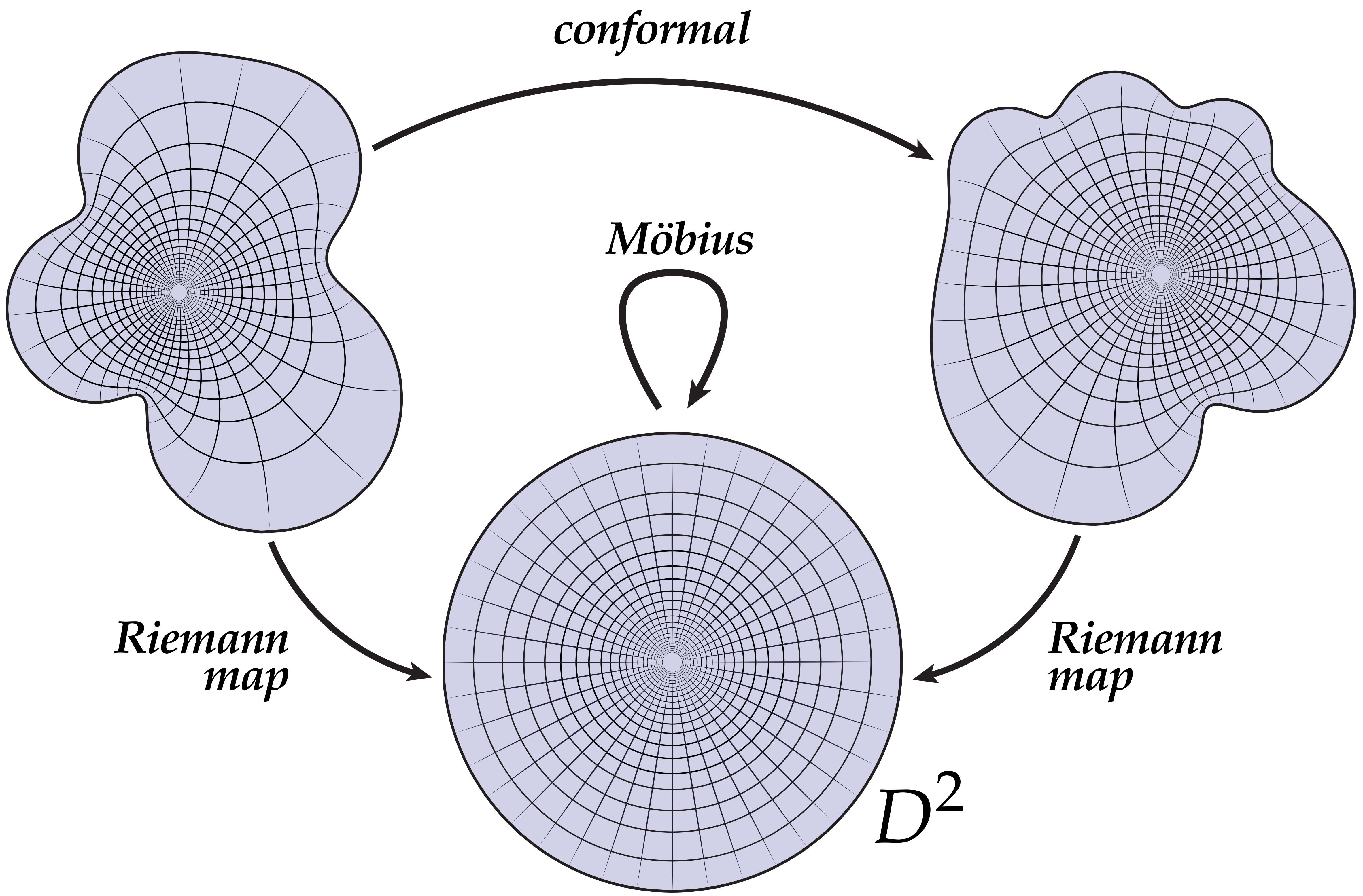


How? Don't memorize some formula—*derive it yourself!*

E.g., What's the equation for a sphere? What's the equation for a ray?

Riemann Mapping Theorem

Theorem (Riemann). Any nonempty simply-connected open proper subset of $\mathbb{C}$ can be conformally mapped to the unit open disk $D^2 := \{z \in \mathbb{C} : |z| < 1\}$.



Fact. The only conformal maps from D^2 to D^2 are Möbius transformations of the form

$$z \mapsto e^{\imath\theta} \frac{z - a}{1 - \bar{a}z}$$

where $a \in D^2$ and $\theta \in S^1$ (three degrees of freedom: inversion center and rotation).

Riemannian Metric

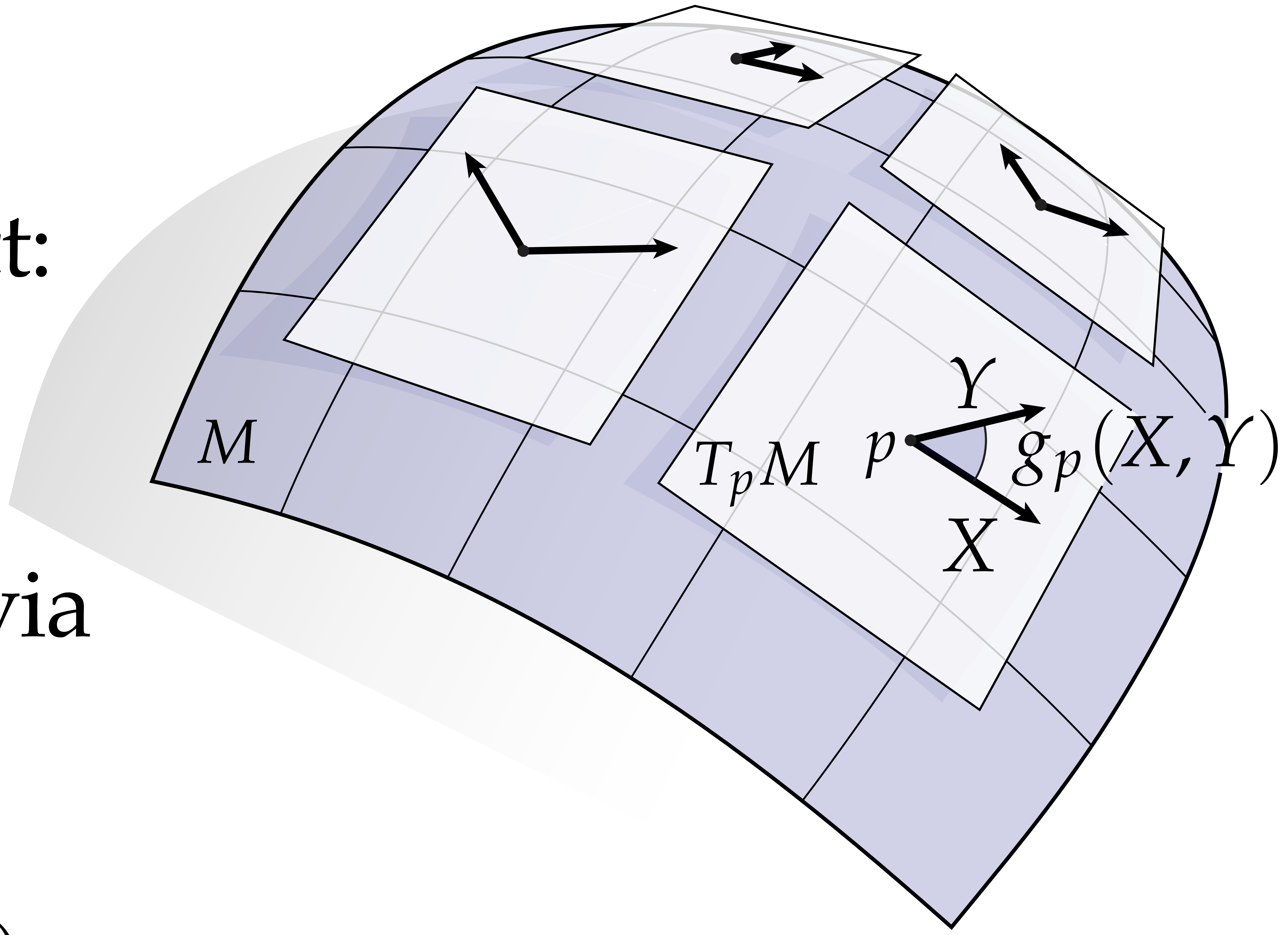
- Can also understand conformal maps in terms of *Riemannian metric*
- Riemannian metric g is simply inner product in each tangent space
- Allows us to measure length, angle, etc.
- E.g., Euclidean metric is just dot product:

$$g_{\mathbb{R}^n}(X,Y) := \sum_i X_i Y_i$$

- In general, length and angle recovered via

$$|X| := \sqrt{g(X,X)}$$

$$\angle(X,Y) := \arccos(g(X,Y)/|X||Y|)$$

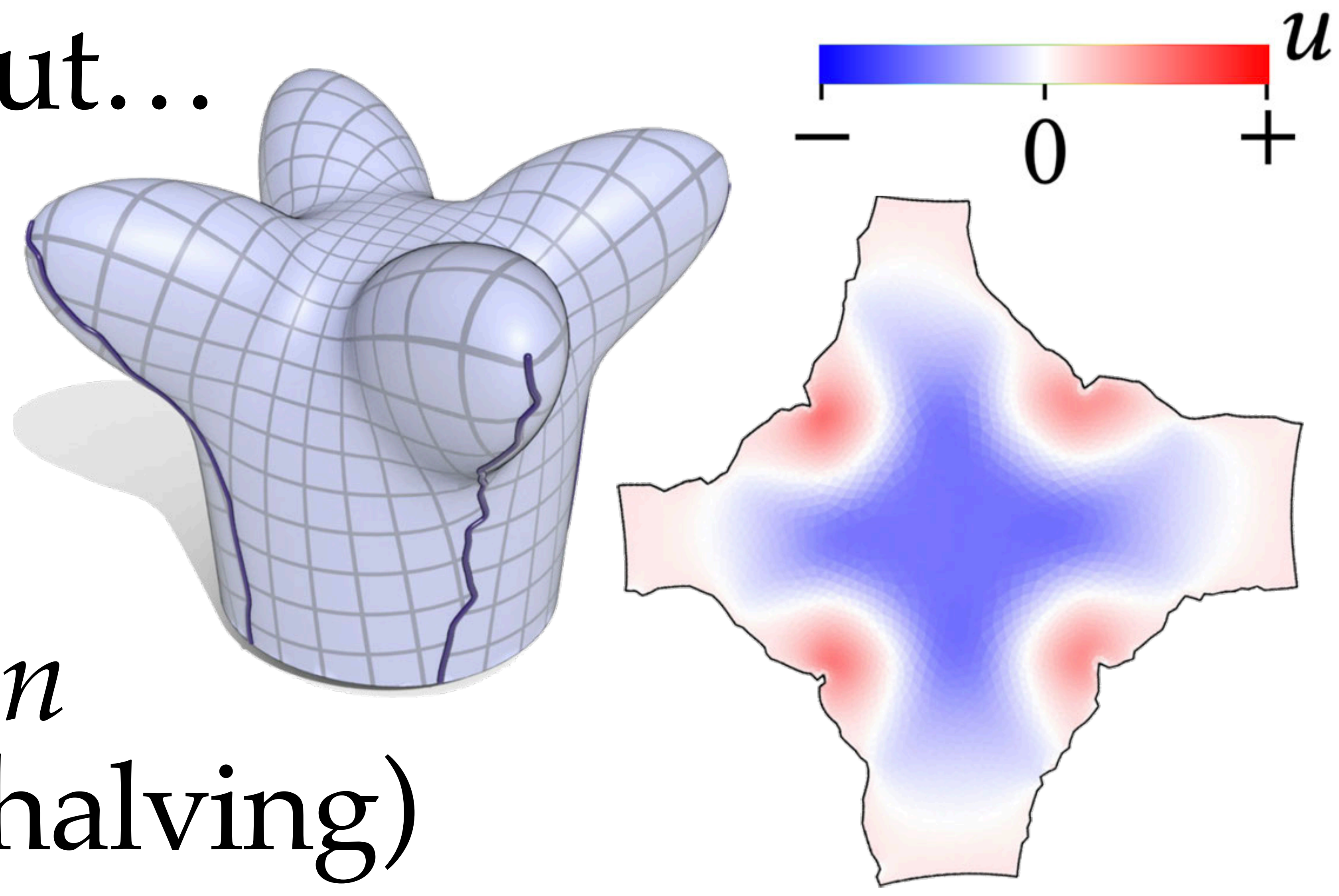


Conformally Equivalent Metrics

- Two metrics are *conformally equivalent* if they are related by a positive **conformal scale factor** at each point p:

$$\boxed{\tilde{g}_p = e^{2u(p)} g_p} \qquad u : M \to \mathbb{R}$$

- Why write scaling as e^{2u}? Initially mysterious, but…
  - ensures scaling is always *positive*
  - factor e^u gives length scaling
  - more natural way of talking about *area distortion* (e.g., doubling in scale "costs" just as much as halving)

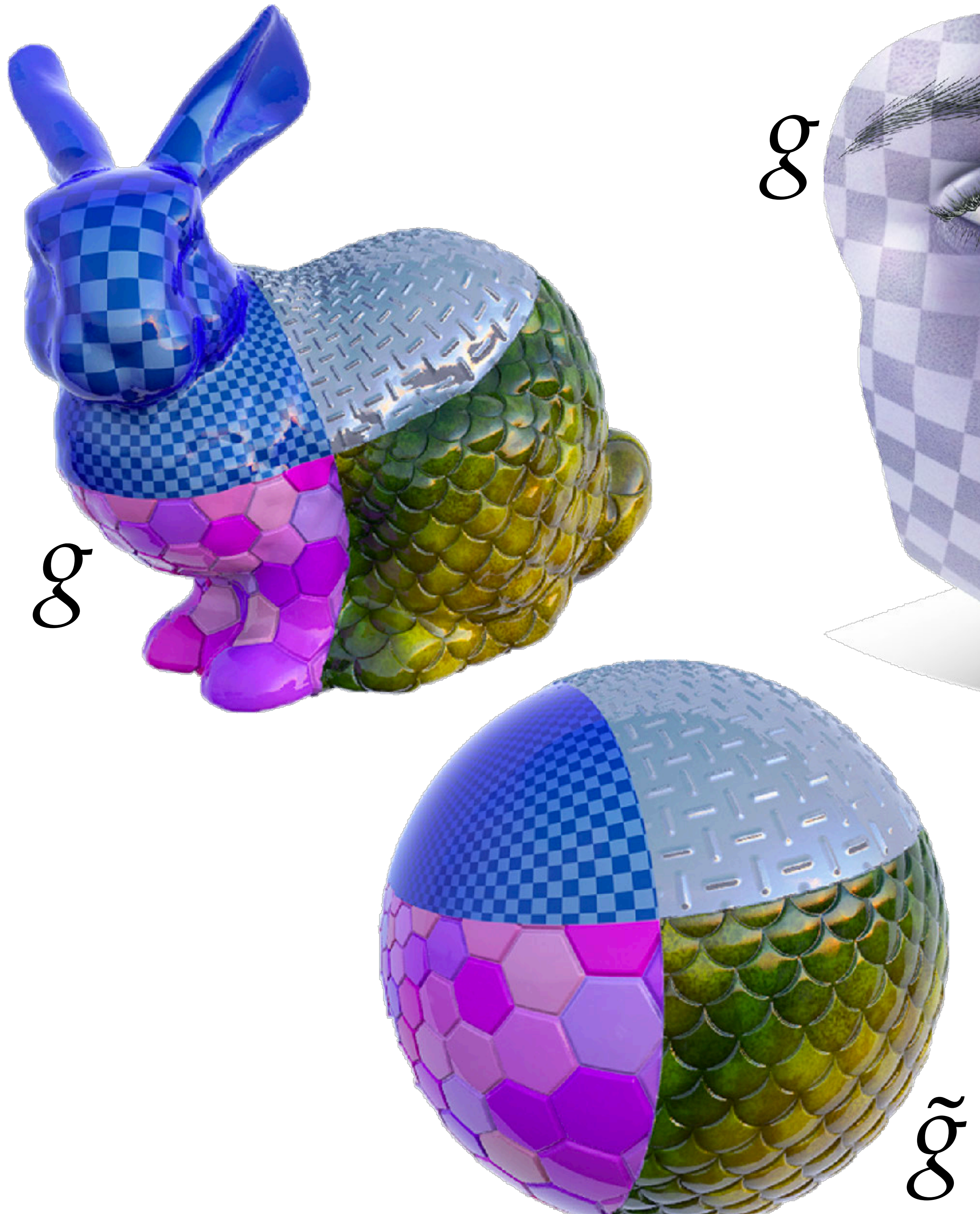


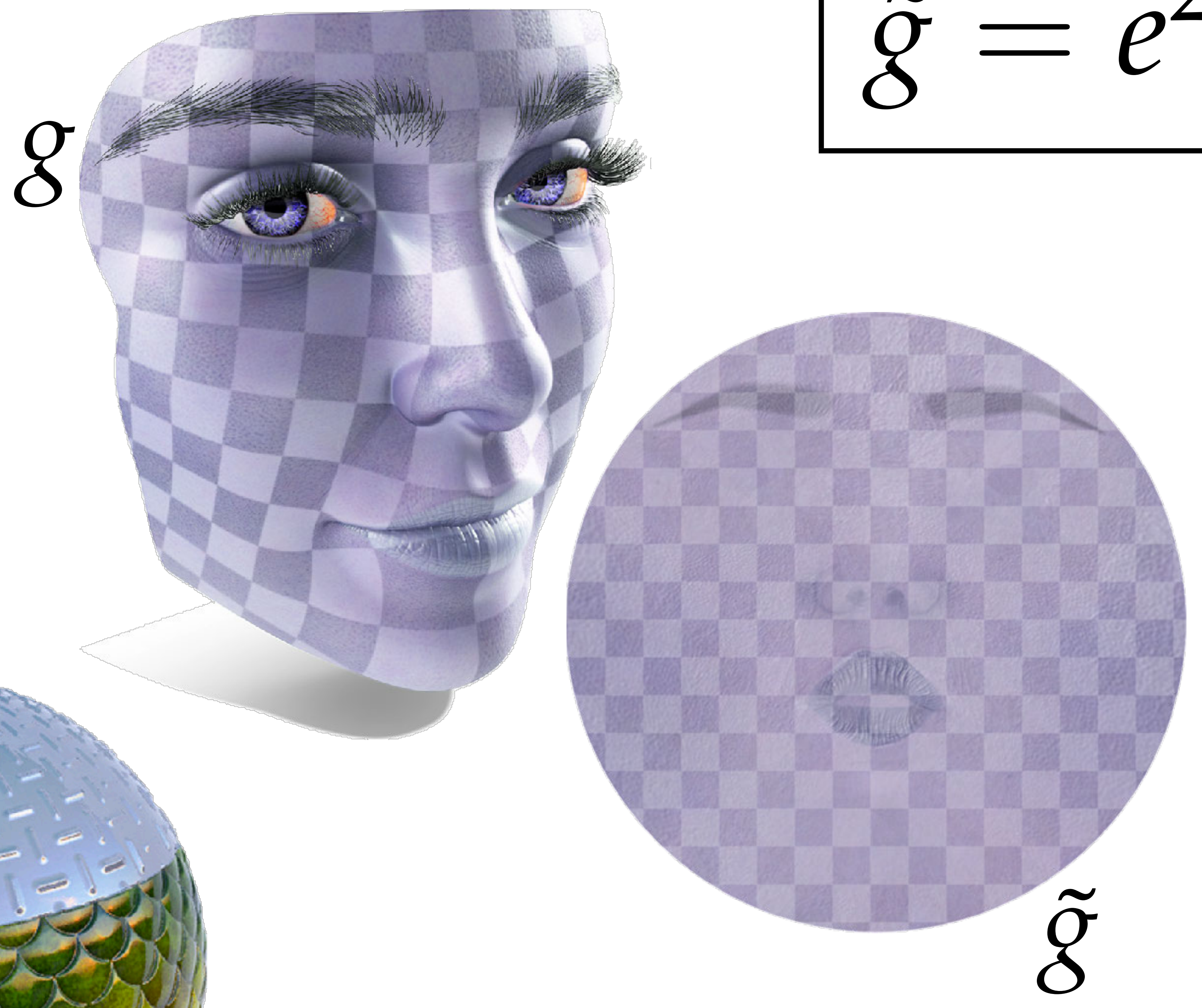
Q: Does this transformation preserve *angles*?

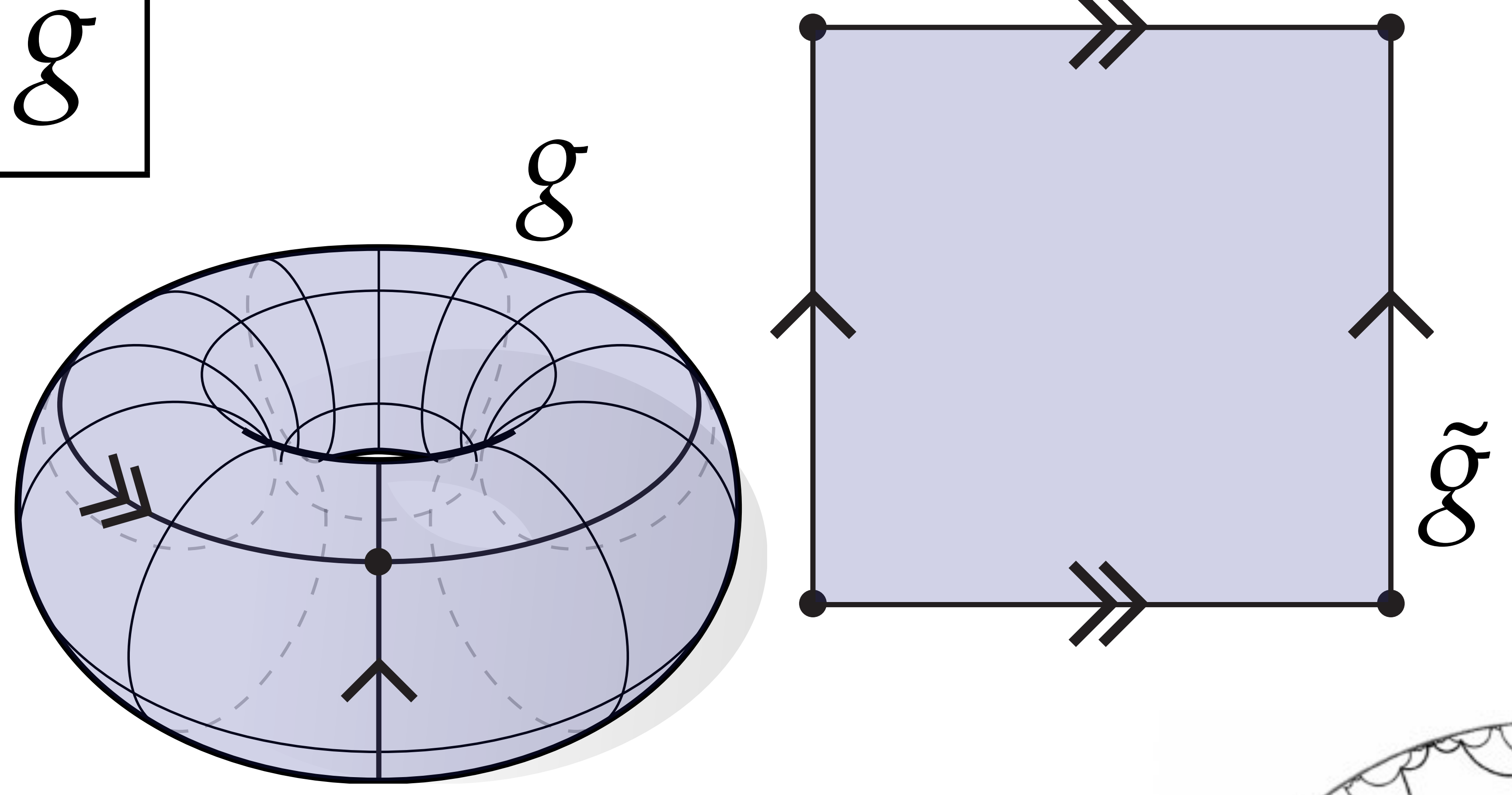
Uniformization Theorem

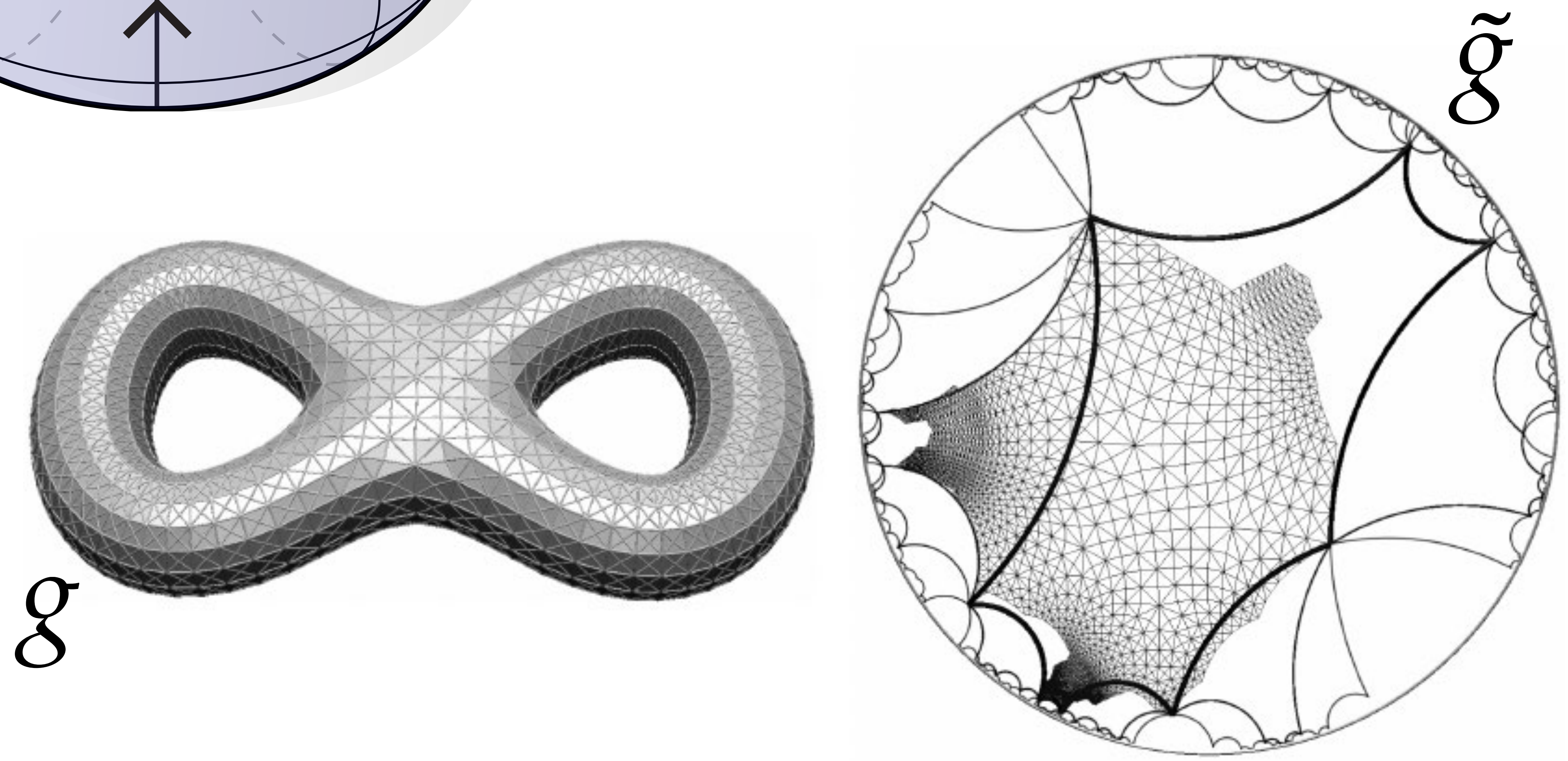
- Roughly speaking, Riemannian metric on any surface is conformally equivalent to one with *constant curvature* (flat, spherical, hyperbolic).

$$\boxed{\tilde{g} = e^{2u} g}$$



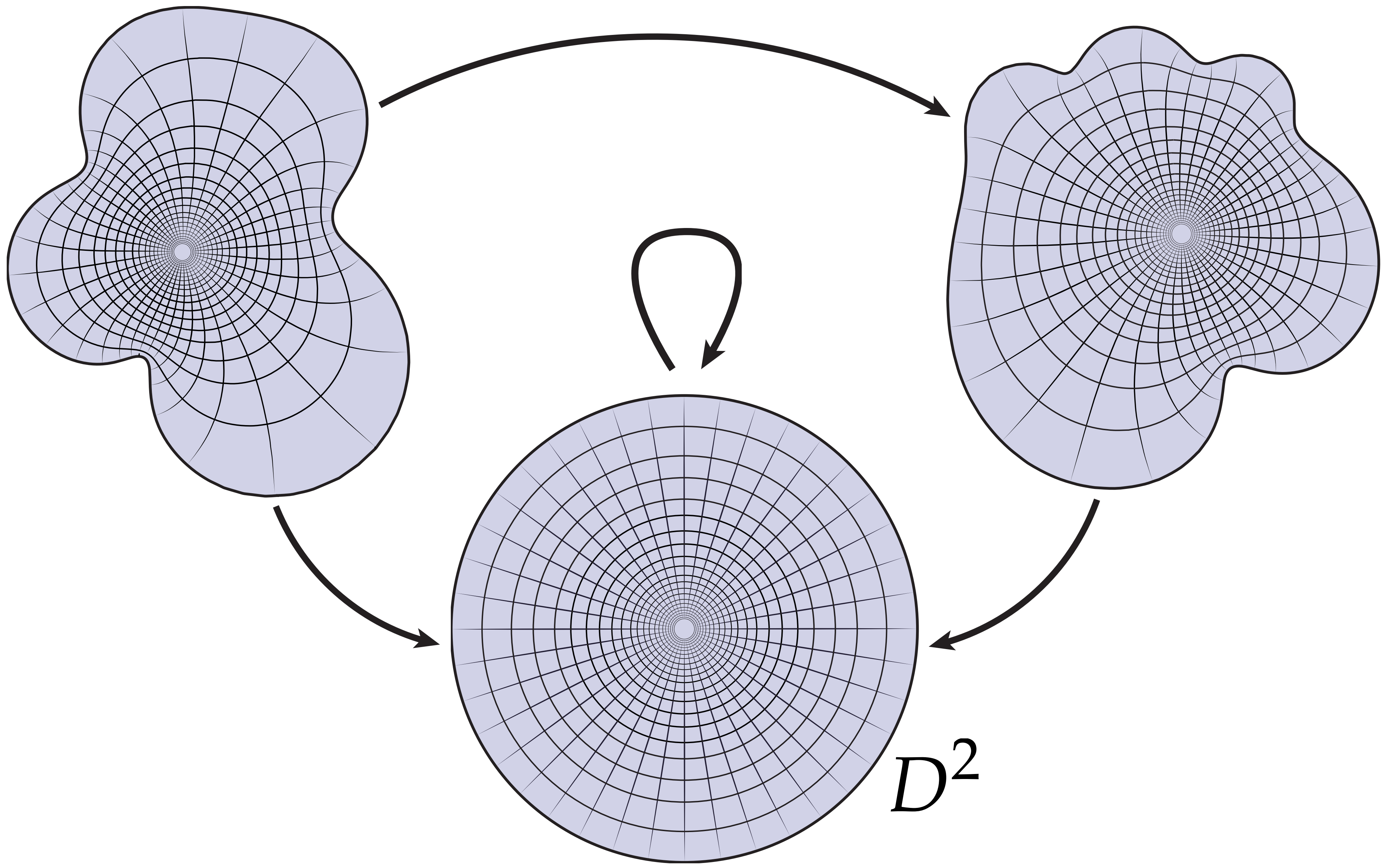


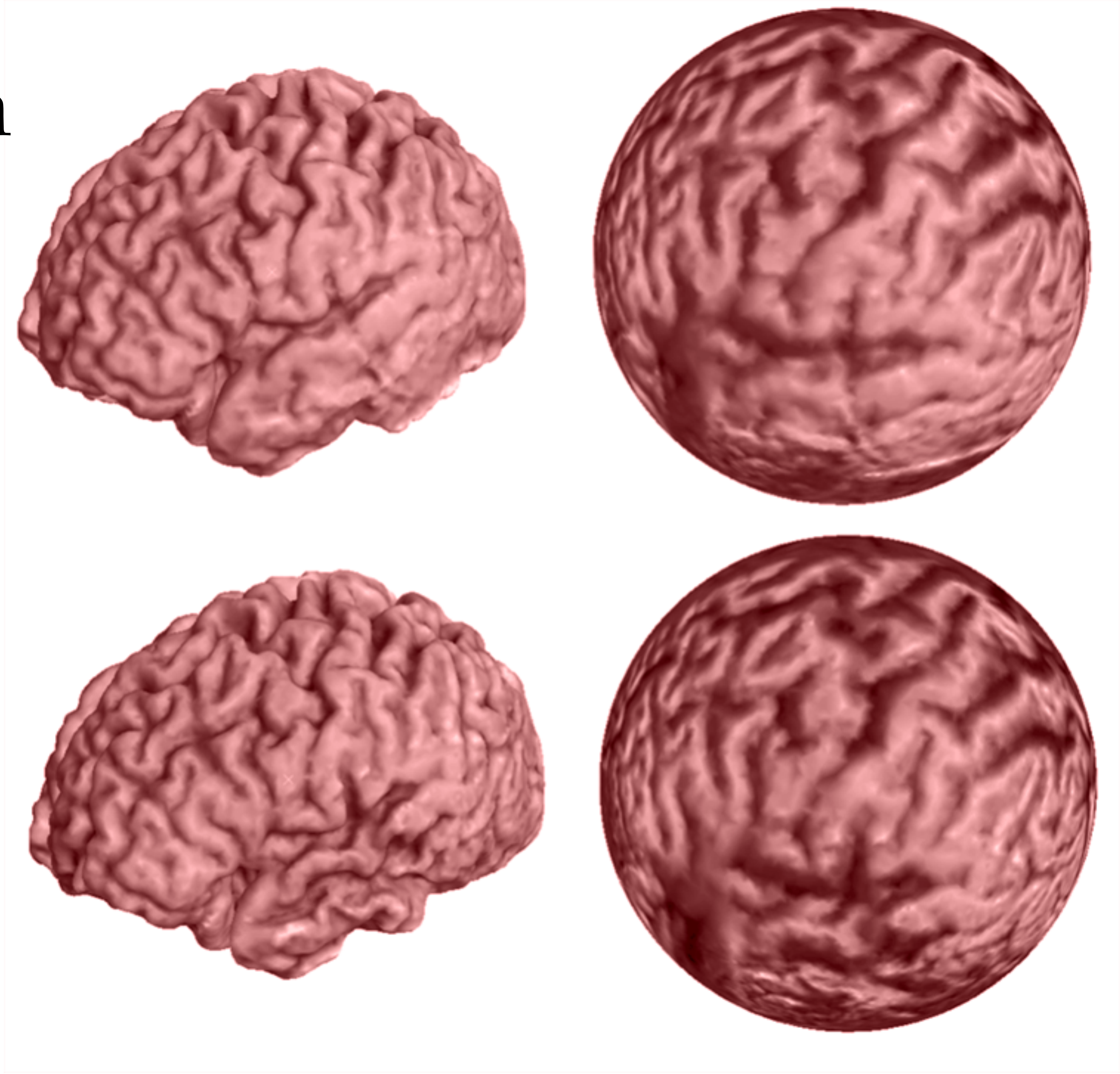




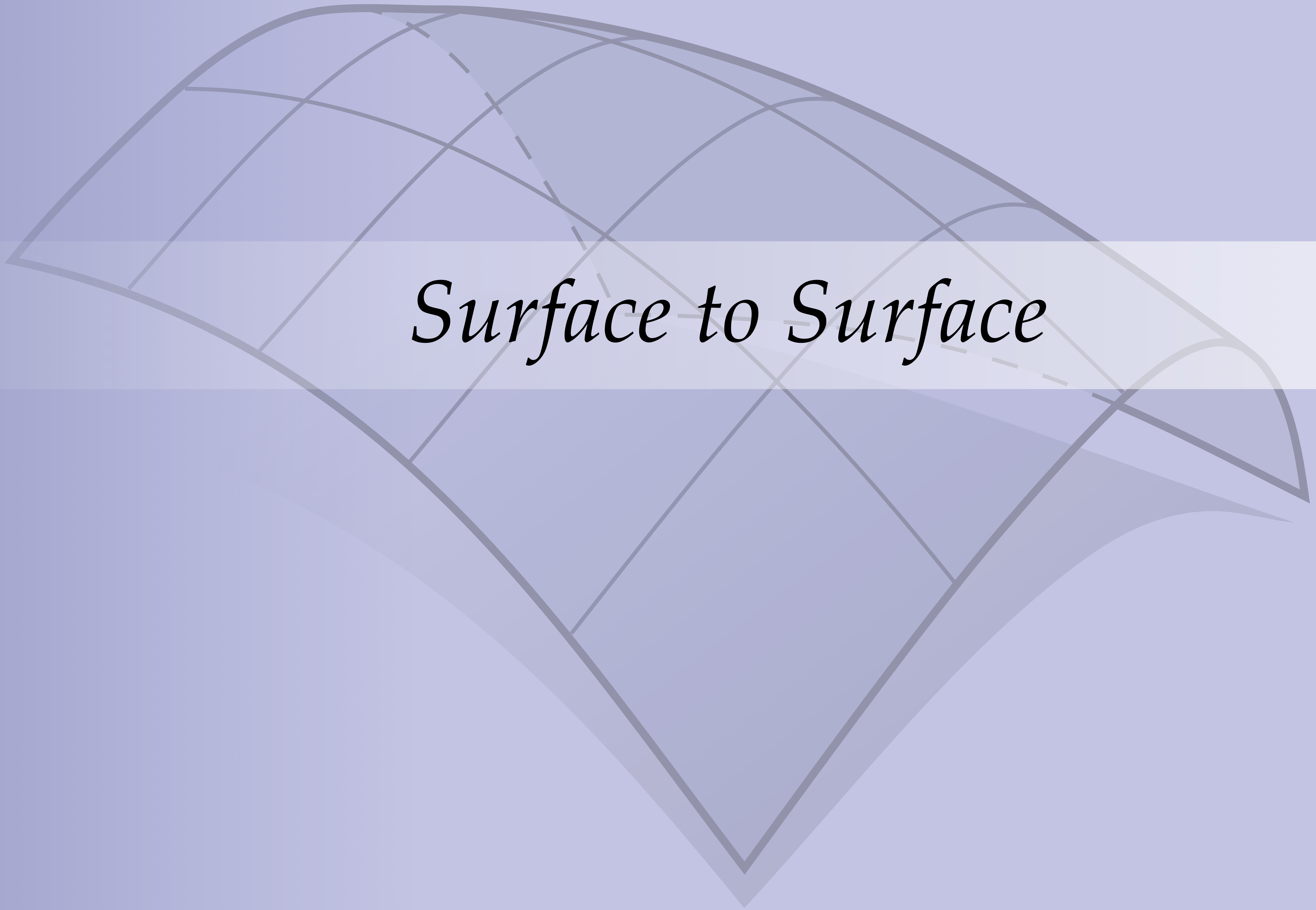
Why is Uniformization Useful?

- Provides canonical domain for solving equations, comparing data, cross-parameterization, etc.
- *Careful:* still have a few degrees of freedom (e.g., Möbius transformations)





Surface to Surface



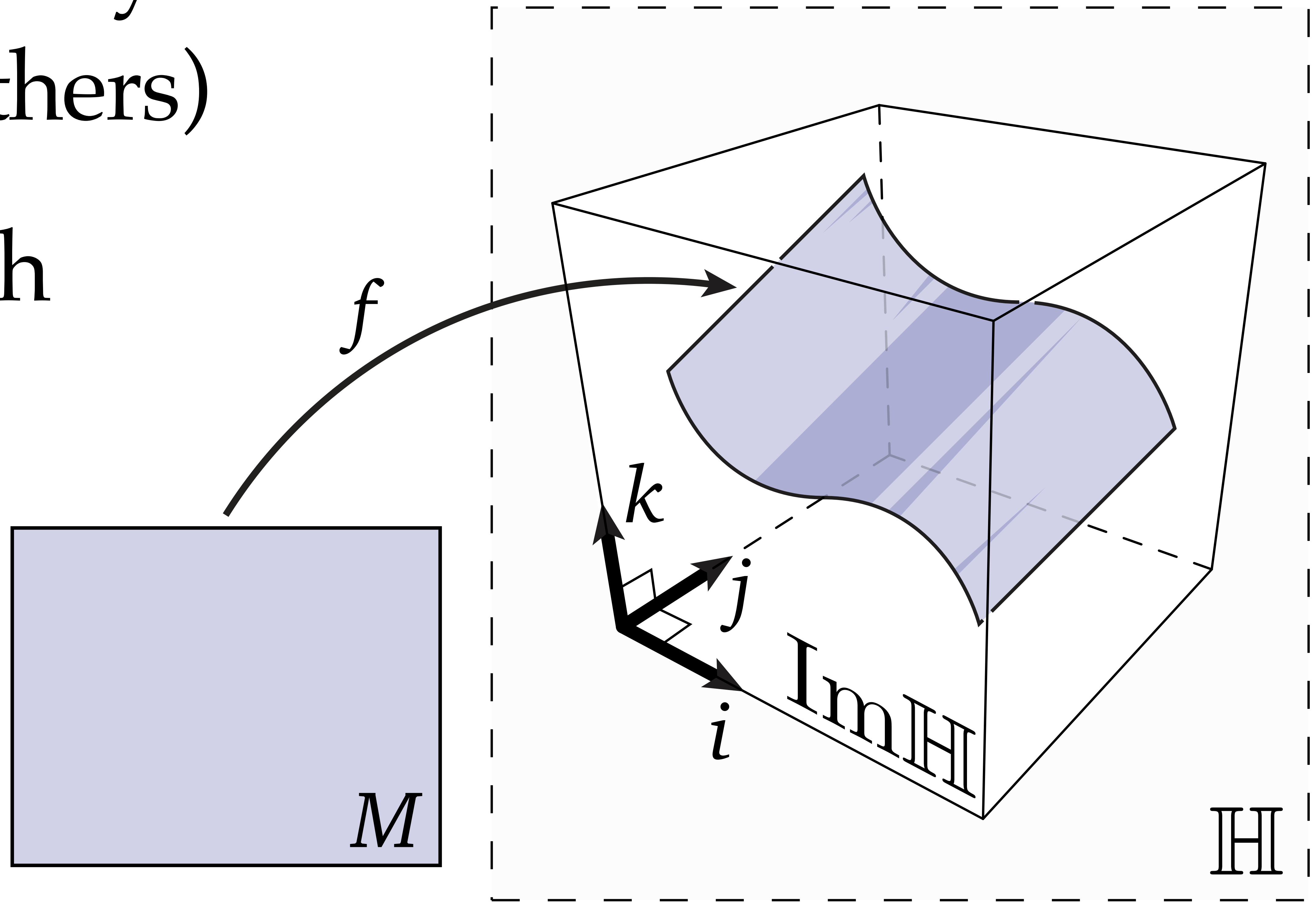
Surface to Surface

- Conformal deformations of surfaces embedded in space
- Both surfaces can have arbitrary curvature (not just sphere, disk, *etc.*)
- Opens door to much broader geometry processing applications
- Very recent theory & algorithms (~1996/2011)
- Key equation: *time-independent Dirac equation*

Won't say too much today… see https://youtu.be/UQC_emOPVK8

Geometry in the Quaternions

- Just as complex numbers helped with 2D transformations, *quaternions* provide natural language for 3D transformations
- Recent use of quaternions as alternative way of analyzing surfaces (Pedit, Pinkall, and others)
- Basic idea: points (a,b,c) get replaced with *imaginary* quaternions ai + bj + ck
- Surface is likewise an imaginary map f

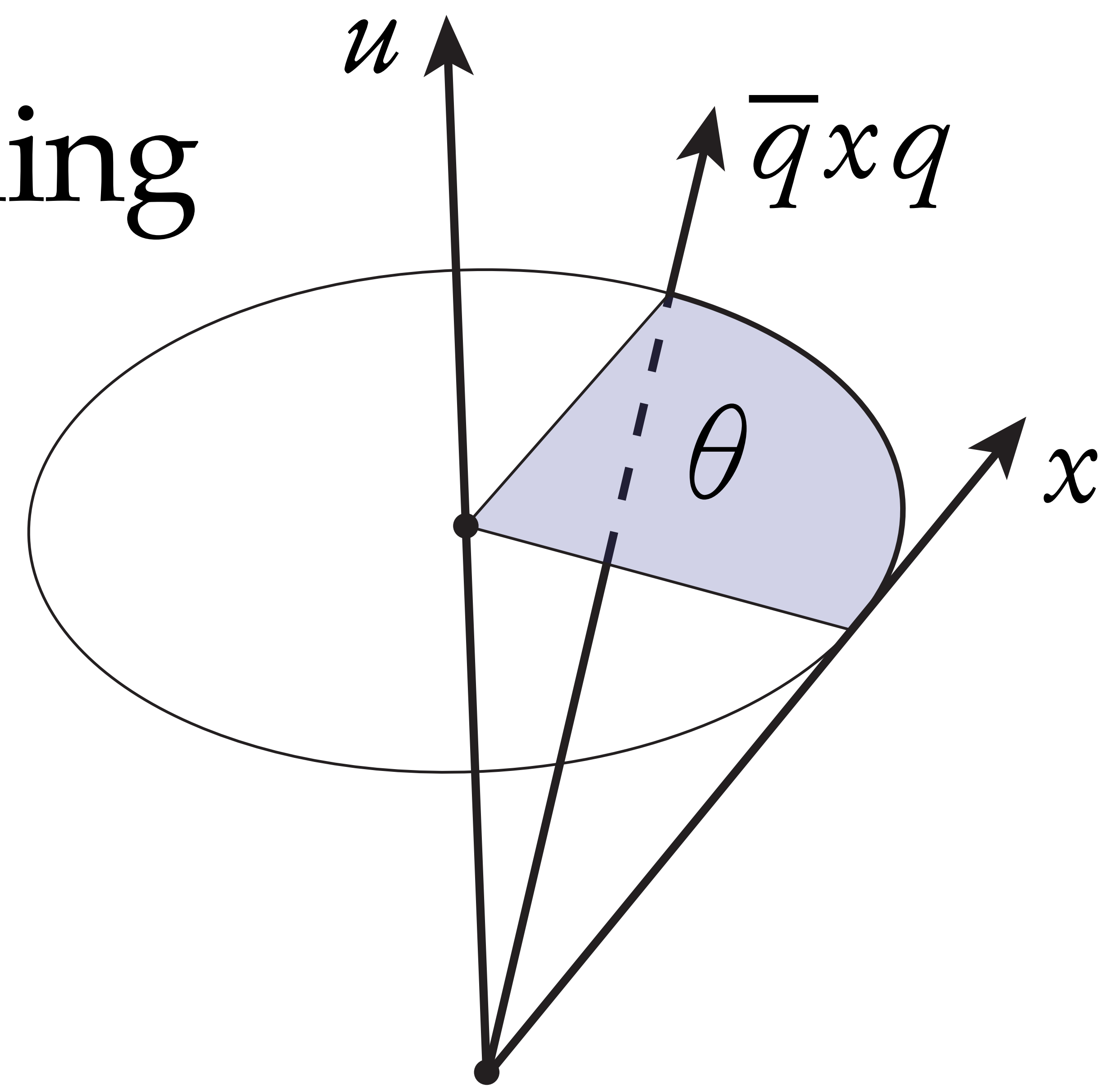


Stretch Rotations

- How do we express rotation using quaternions?
- Similar to complex case, can rotate a vector x using a unit quaternion q:

$$\underbrace{\tilde{x}}_{\textbf{rotated}} = \bar{q}\underbrace{x}_{\textbf{original}}q$$

- If q has non-unit magnitude, we get a rotation and scaling
- Should remind you of conformal map:
  scaling & *rotation* (but no shear)



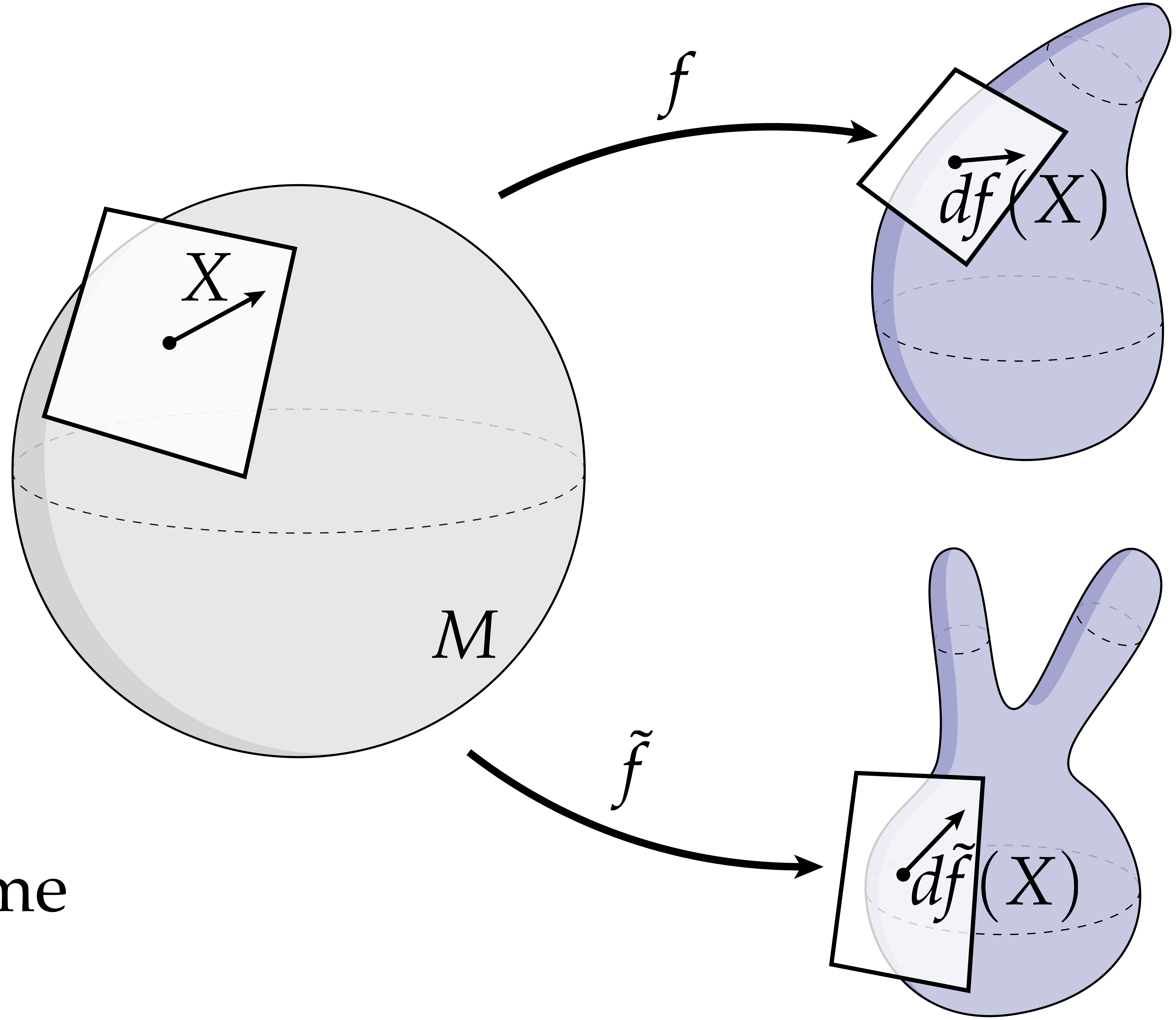
Spin Equivalence

- From here, not hard to express conformal deformation of surfaces

- Two surfaces f_0, f are *spin equivalent* if their tangent planes are related by a pure scaling and rotation at each point:

$$d\tilde{f}(X) = \bar{\psi}\, df(X)\psi$$

for all tangent vectors X and some stretch rotation $\psi : M \to \mathbb{H}$



Dirac Equation

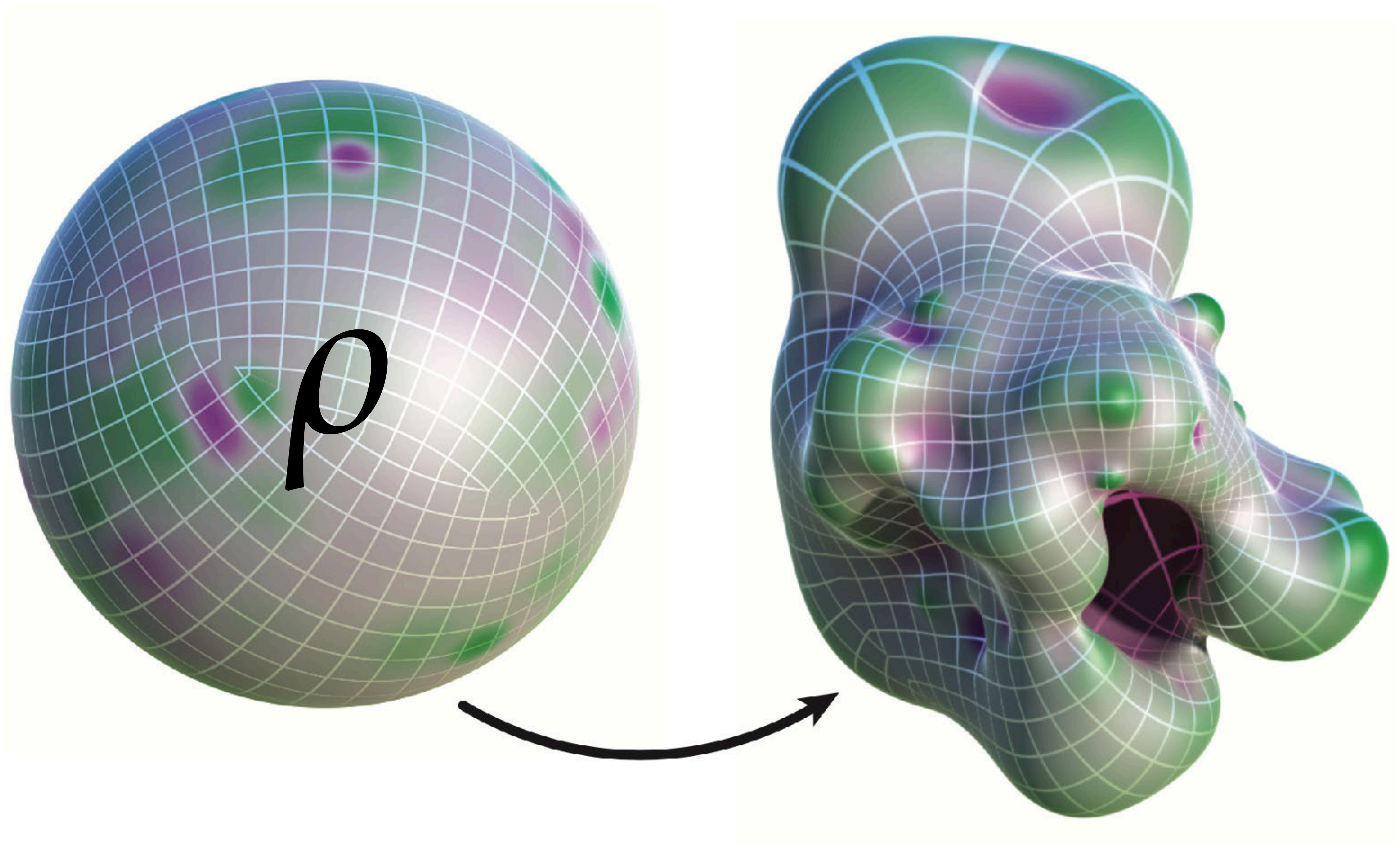
- From here, one can derive the fundamental equation for conformal surface deformations, a *time-independent Dirac equation*

change in curvature

$$\boxed{(D - \rho)\psi = 0}$$

quaternionic Dirac operator **stretch rotation**

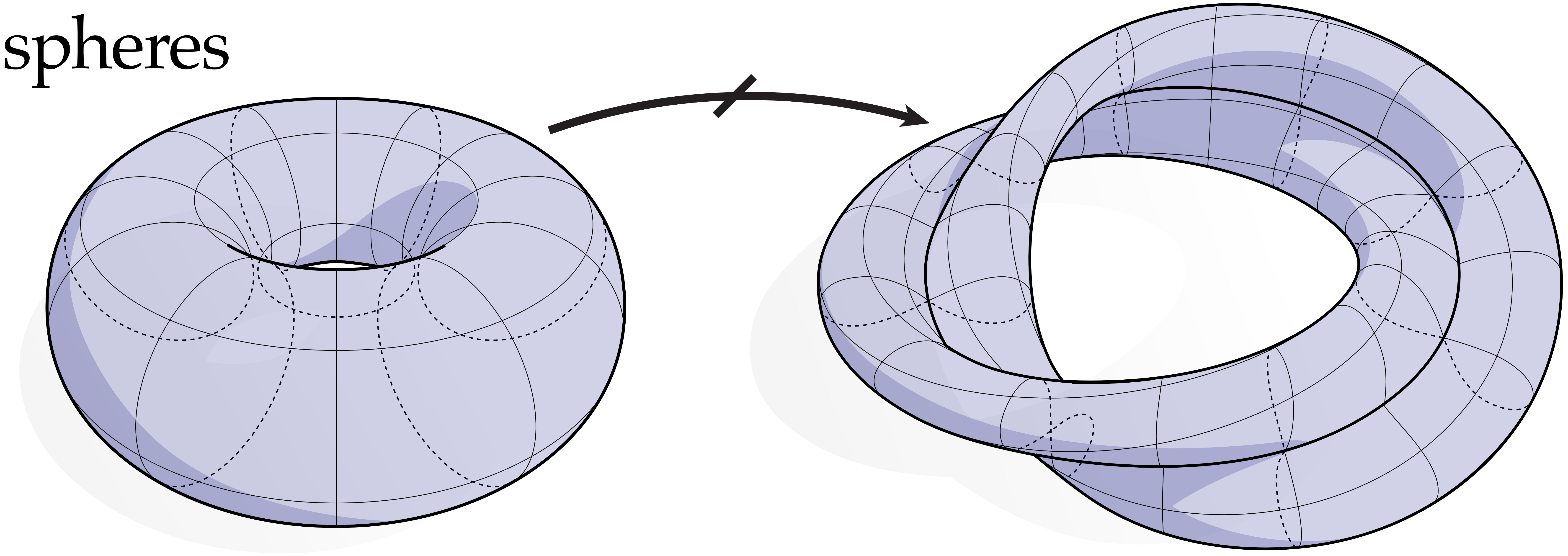
$$D\psi := -\frac{df \wedge d\psi}{|df|^2}$$



CRANE, PINKALL, SCHRÖDER, *"Spin Transformations of Discrete Surfaces"* (2011)

Spin vs. Conformal Equivalence

- Two surfaces that are spin equivalent are also conformally equivalent: tangent vectors just get *rotated* and *scaled*! (no shearing)
- Are conformally equivalent surfaces always spin equivalent?
  - **No** in general, *e.g.*, tori that are not *regularly homotopic* (below)
  - **Yes** for topological spheres

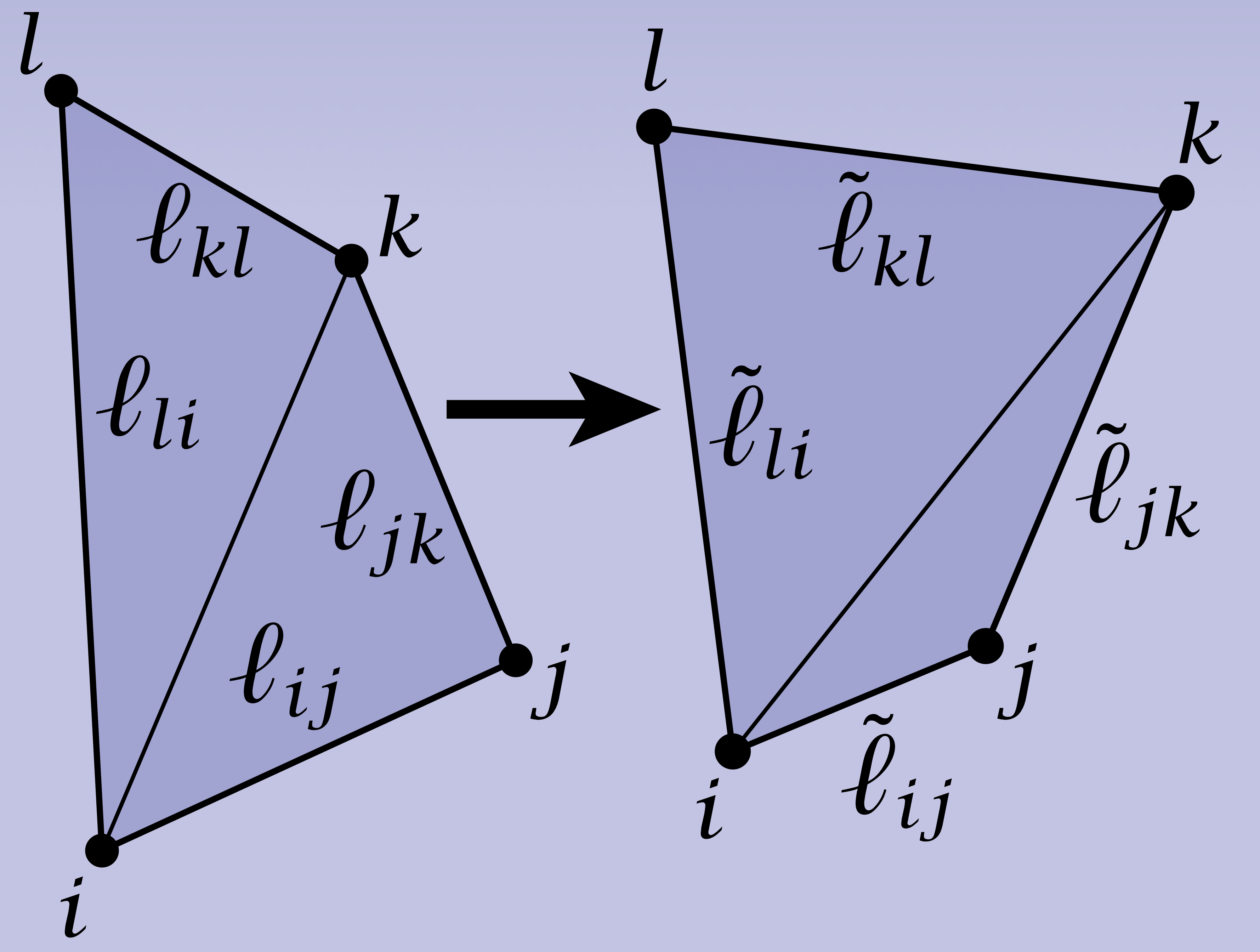


Why Not Just Optimize Angles?

- Forget the mathematics—why not just optimize mesh to preserve angles?
- As discussed before, angle preservation is *too rigid!*
- E.g., convex surface *uniquely* determined by angles (up to rigid motion)

original **deformed** **optimize angles**

Part III: Discretization



Conformal Geometry Processing

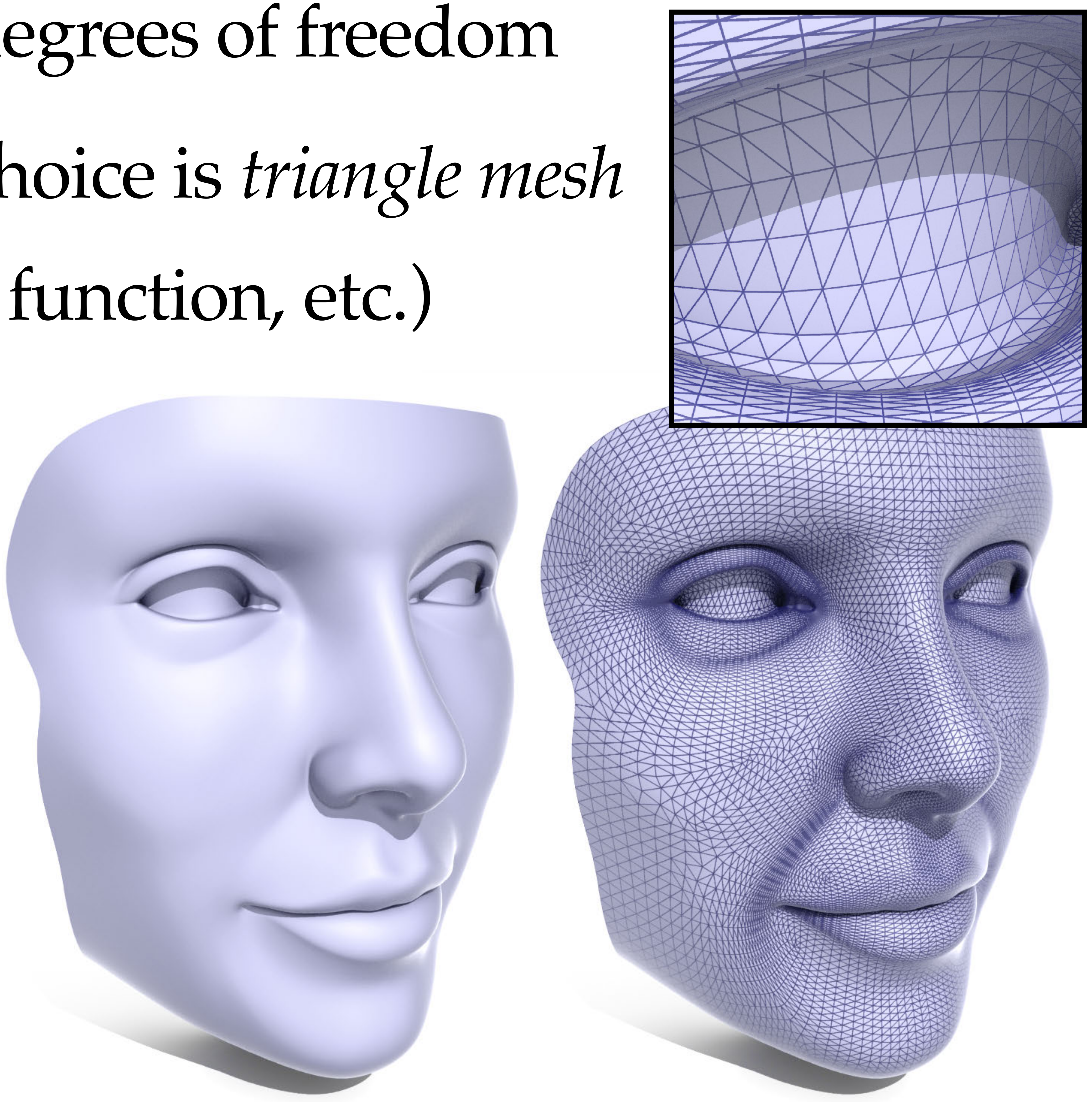
Keenan Crane • Symposium on Geometry Processing Grad School • Summer 2017

Surfaces as Triangle Meshes

- For computation, need *finitely many* degrees of freedom
- Many ways to discretize—common choice is *triangle mesh*
  - No restrictions on geometry (height function, etc.)
  - Any polygon can be triangulated
  - Simple formulas (e.g., per triangle)
  - Efficient computation (sparse)

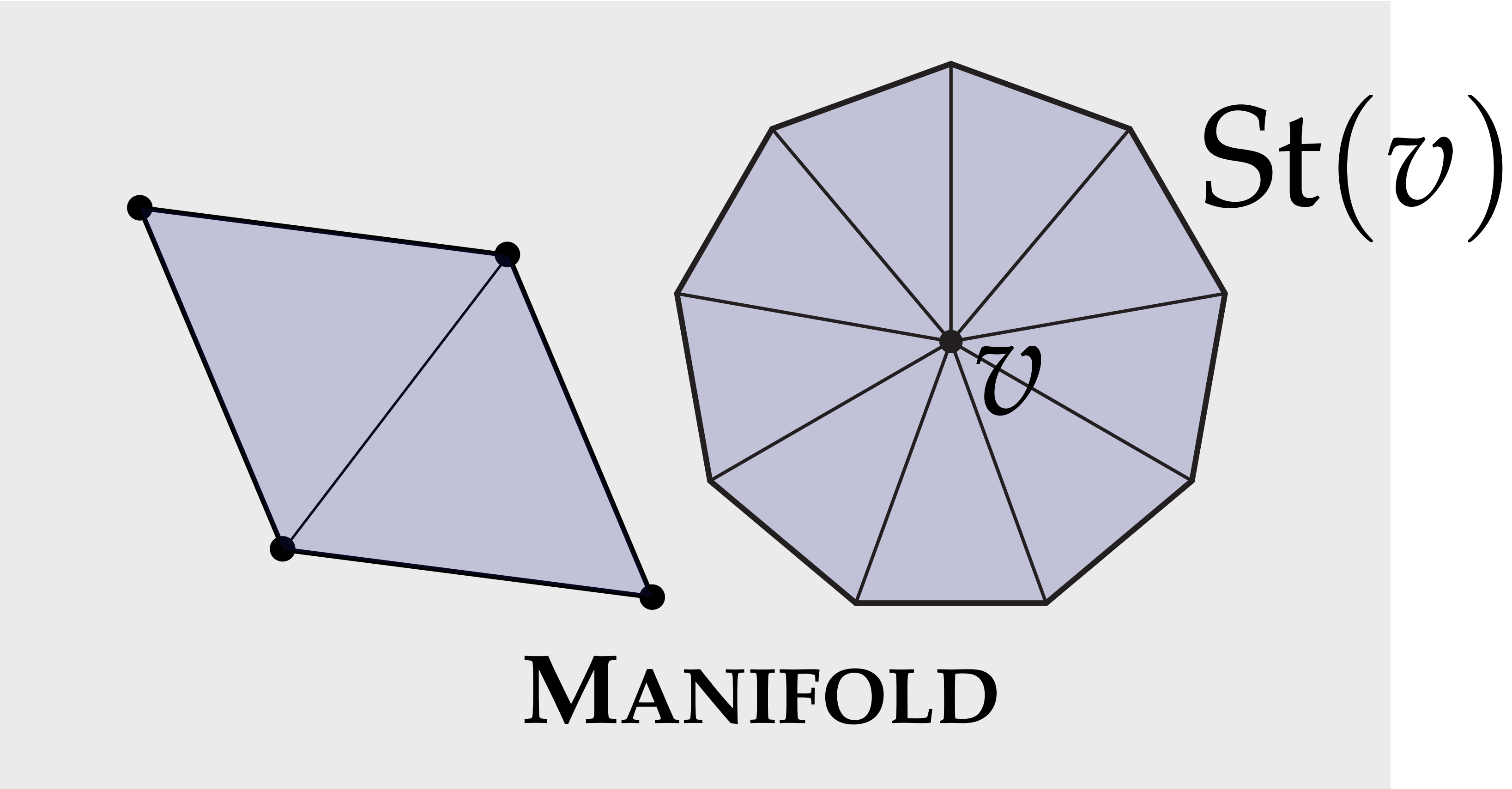
$$K = (V, E, F)$$

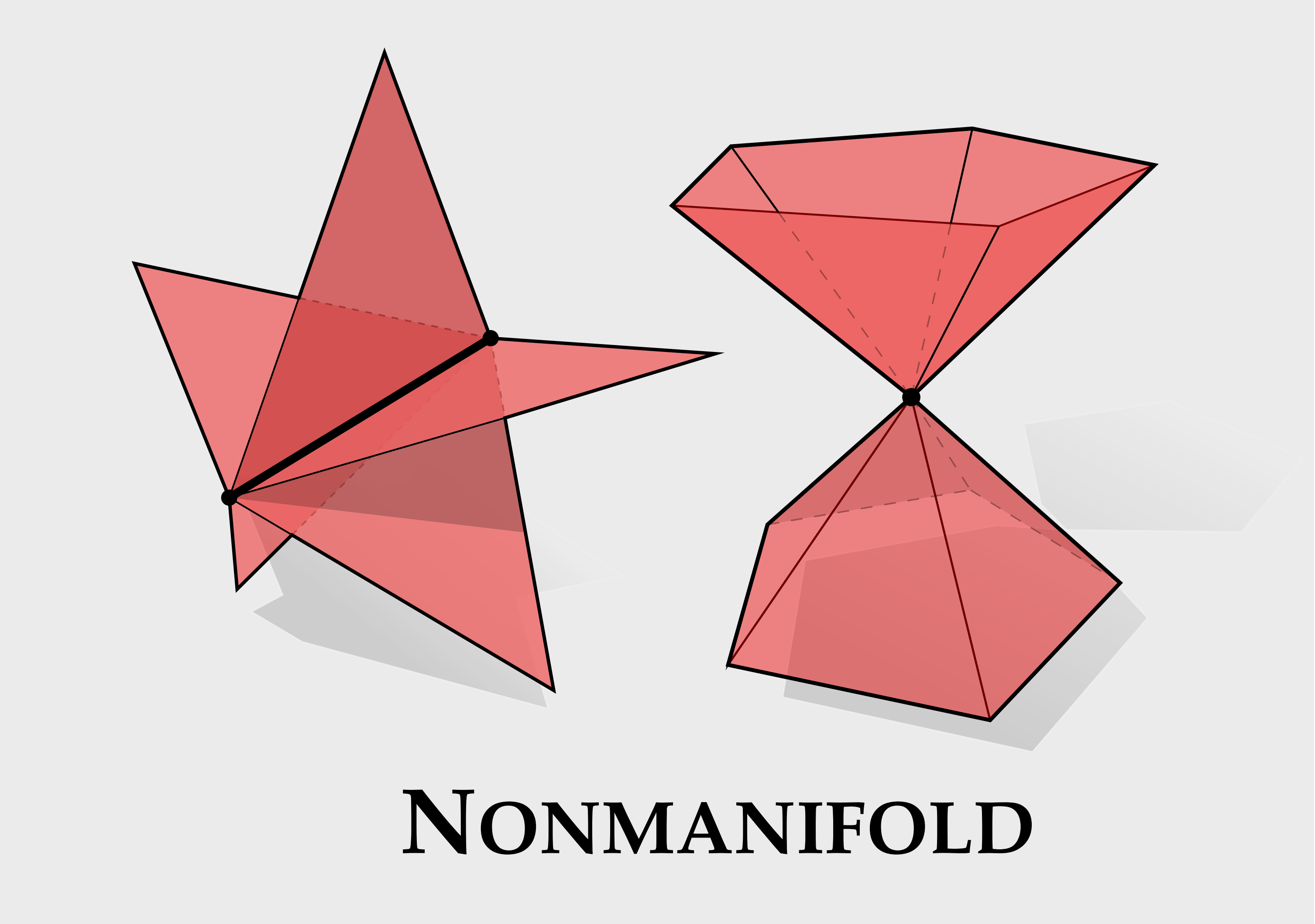
mesh **vertices** **edges** **faces**



Manifold Triangle Mesh

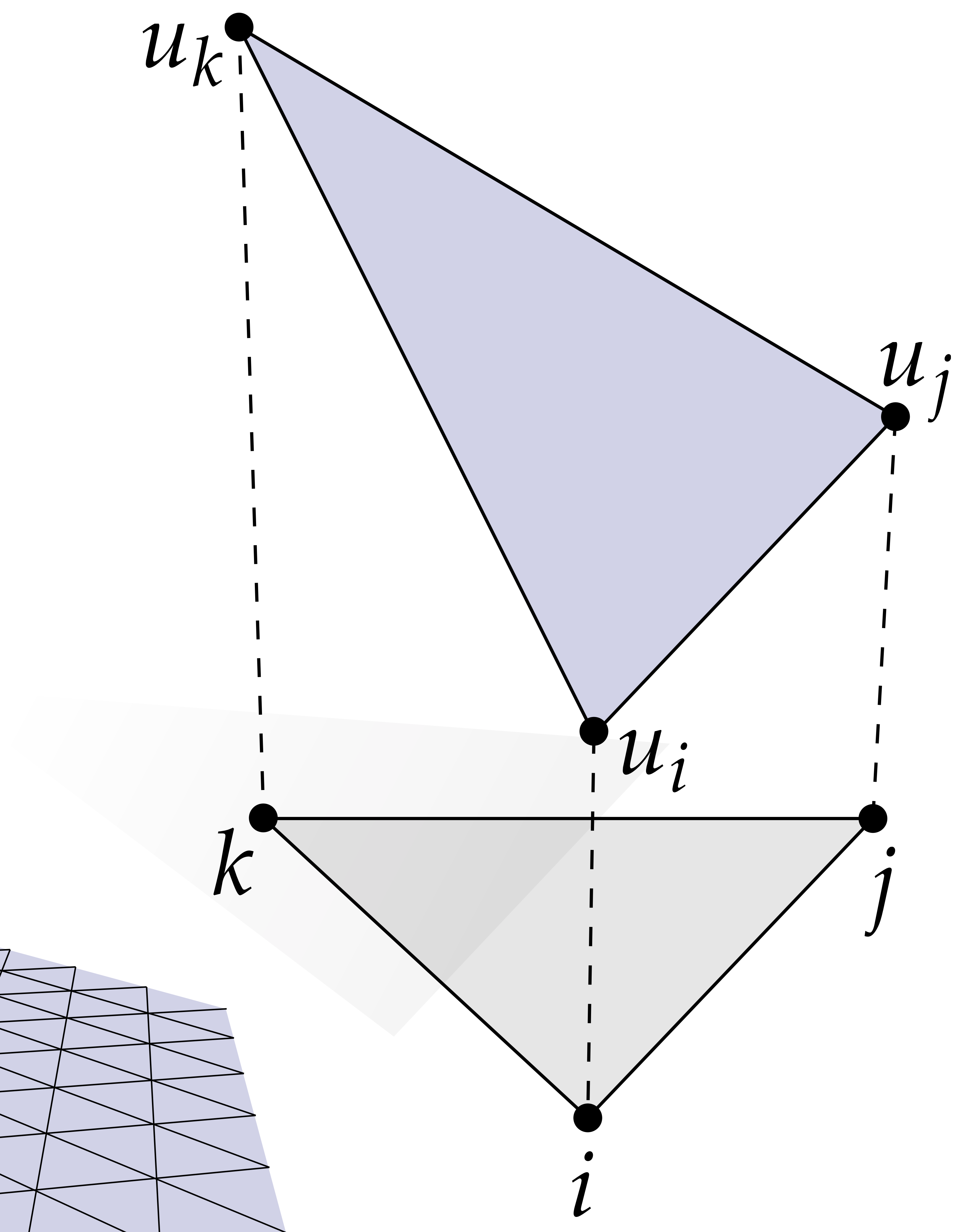
- Images: assume every pixel has four neighbors (keeps things simple!)
- Likewise, assume meshes are *manifold*
  - edges contained in no more than two faces
  - vertex contained in "fan" of triangles
  - formally: every *vertex star* St(v) is a disk
- Keeps formulas simple
- Fewer special cases in code
- Easier to translate between smooth / discrete

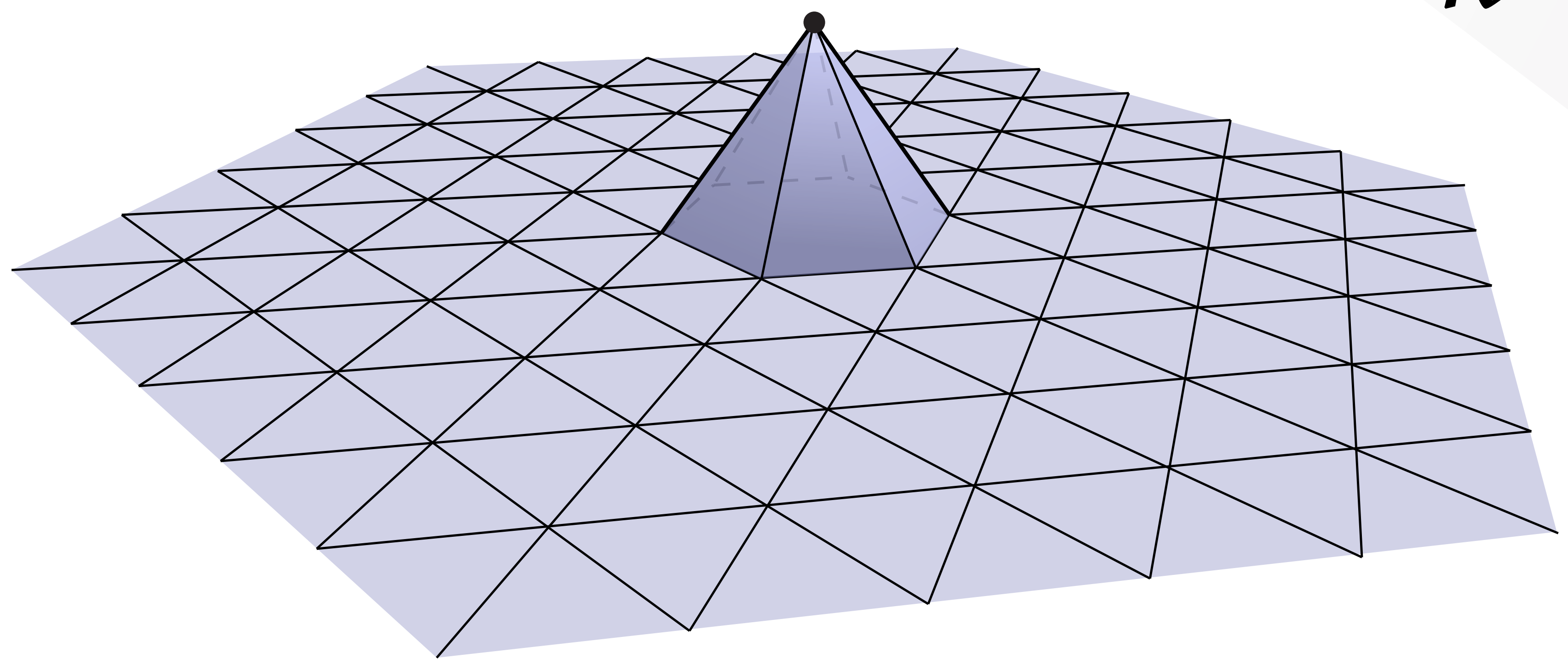




Piecewise Linear Function

- Typical way to encode any function u on a triangle mesh
- Store one value u_i per vertex i
- "Extend" values linearly over each triangle
- More sophisticated schemes possible, but this one will take you surprisingly far…





"Discretized" vs. "Discrete"

- Two high-level approaches to conformal maps on triangle meshes:

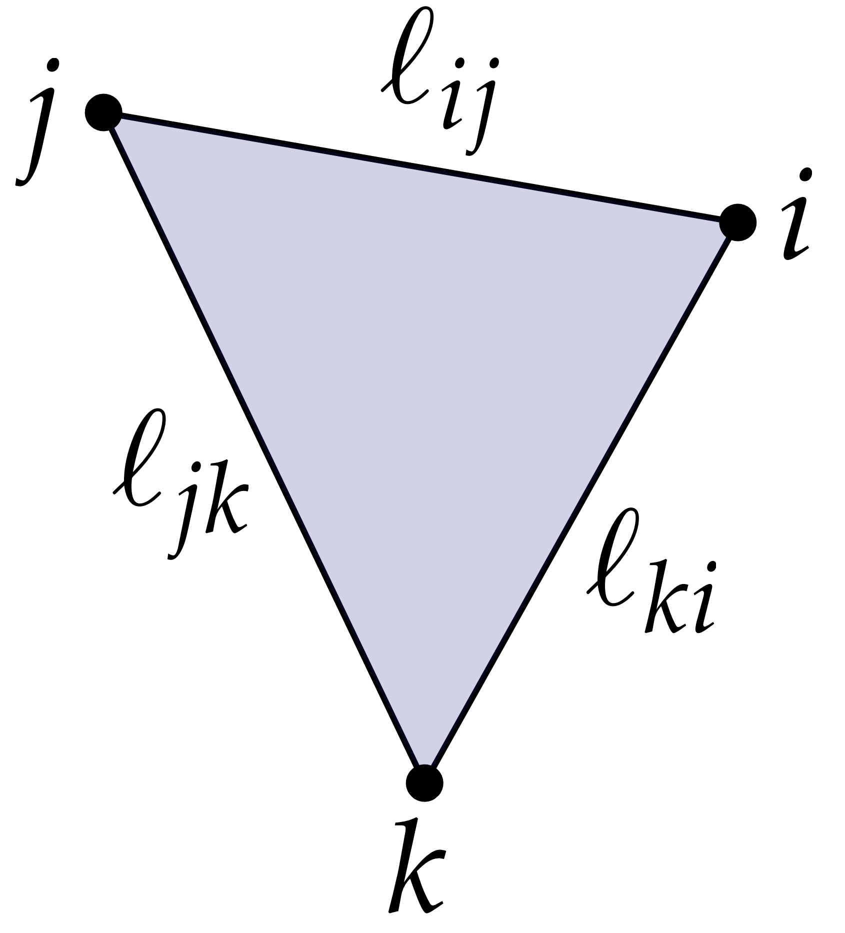
DISCRETIZED	**DISCRETE**
properties satisfied only in limit of refinement (e.g., angle preservation)	quantities preserved exactly no matter how coarse (*e.g., length cross ratios*)
traditional perspective of scientific computing / finite element analysis	more recent perspective of *discrete differential geometry (DDG)*
often (but not always) leads to easy linear problems	can require slightly more difficult computation (e.g., convex optimization)
most of the algorithms we'll consider (*e.g.*, LSCM)	only a few algorithms: circle packing, CETM, inversive distance

Discrete Metric

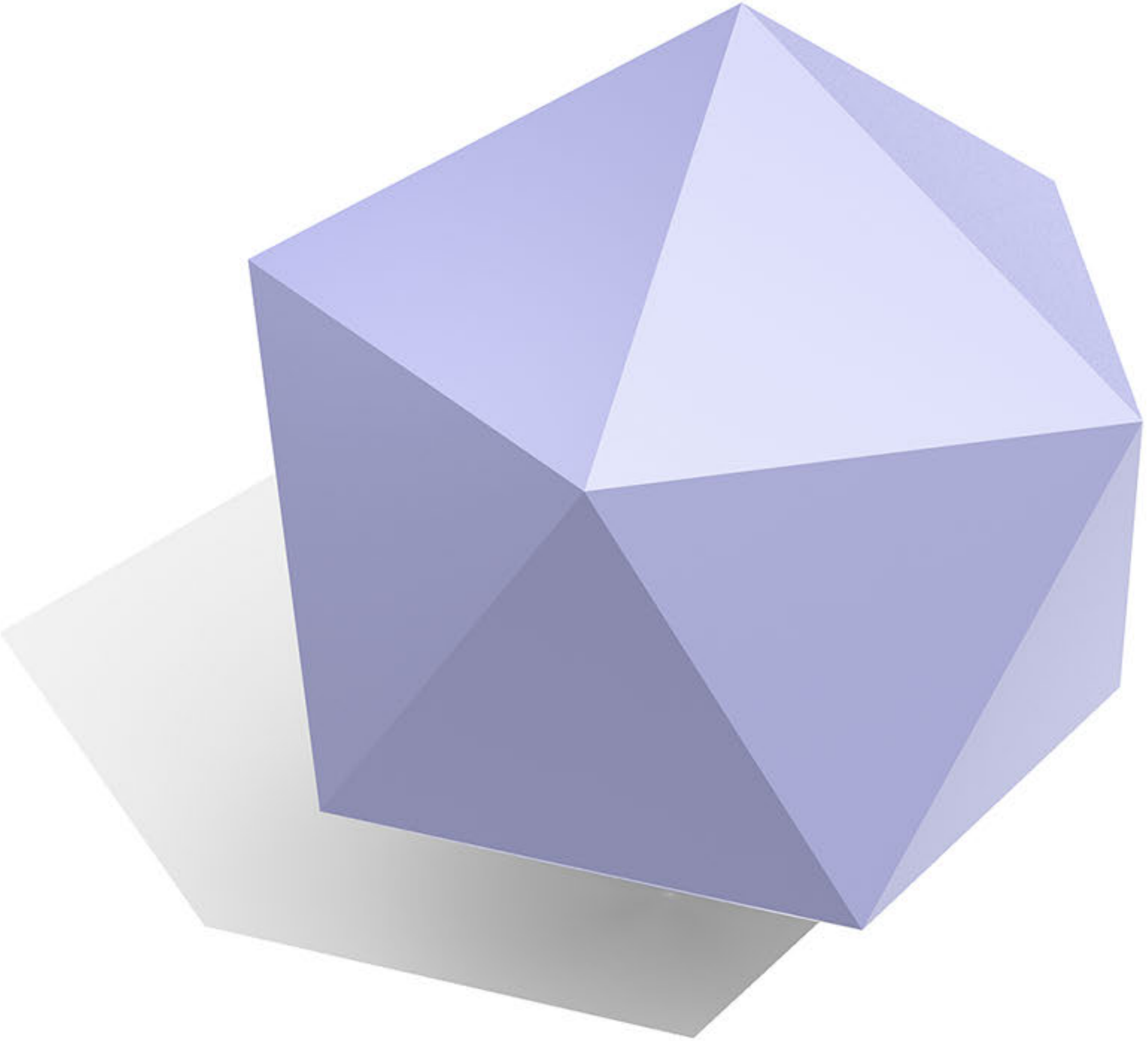
- "Discrete" point of view: try to **exactly** capture smooth relationship

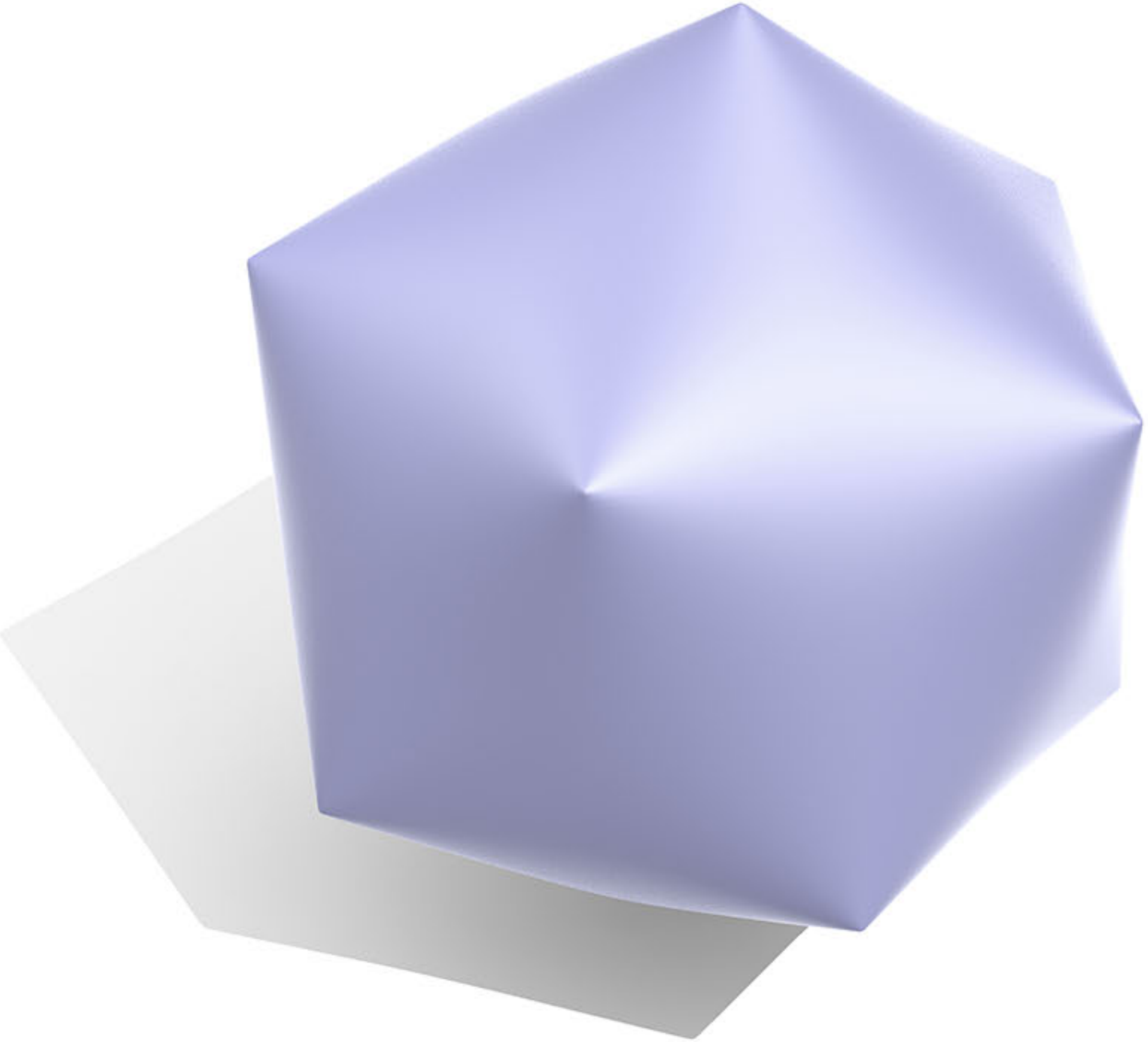
$$\tilde{g} = e^{2u} g$$

- What is a discrete metric?
  - Smooth metric allowed us to measure lengths: $|X| = \sqrt{g(X,X)}$
  - Discrete metric is simply length assigned to each edge: $\ell : E \to \mathbb{R}_{>0}$
  - Must also satisfy triangle inequality: $\ell_{ij} \leq \ell_{jk} + \ell_{ki}$
  - Can then be extended to Euclidean metric per triangle



Discrete Metric — Visualized





(a.k.a. *"cone metric"*)

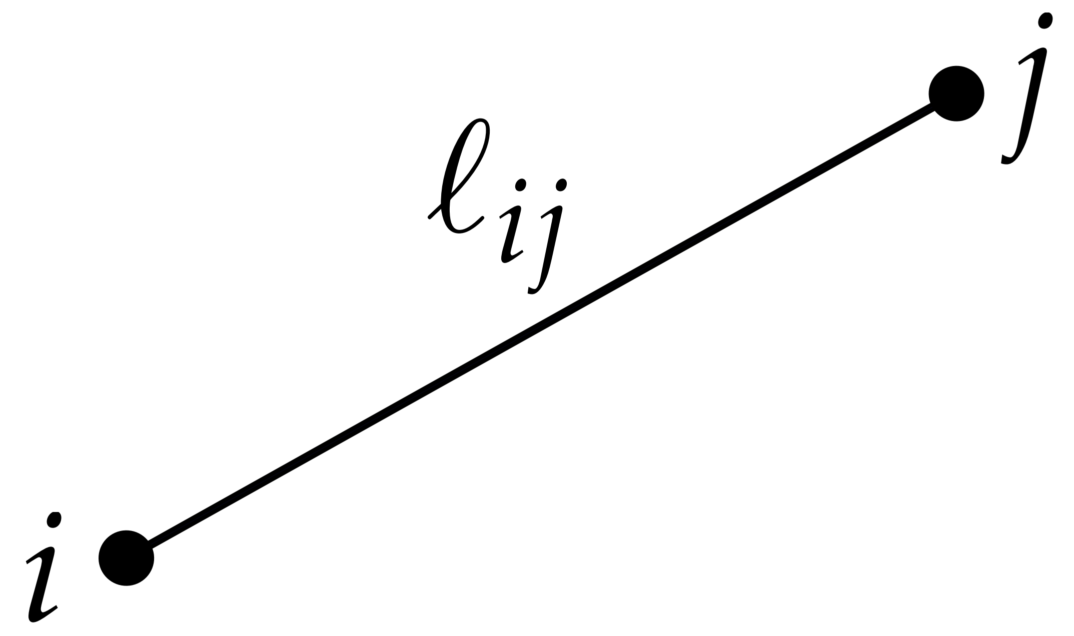
Conformal Equivalence of Triangle Meshes

- "Discrete" point of view: try to **exactly** capture smooth relationship

$$\tilde{g} = e^{2u} g$$

- Discrete analogue: two discrete metrics are conformally equivalent if there is a function u at vertices such that

$$\boxed{\tilde{\ell}_{ij} = e^{(u_i+u_j)/2} \ell_{ij}}$$

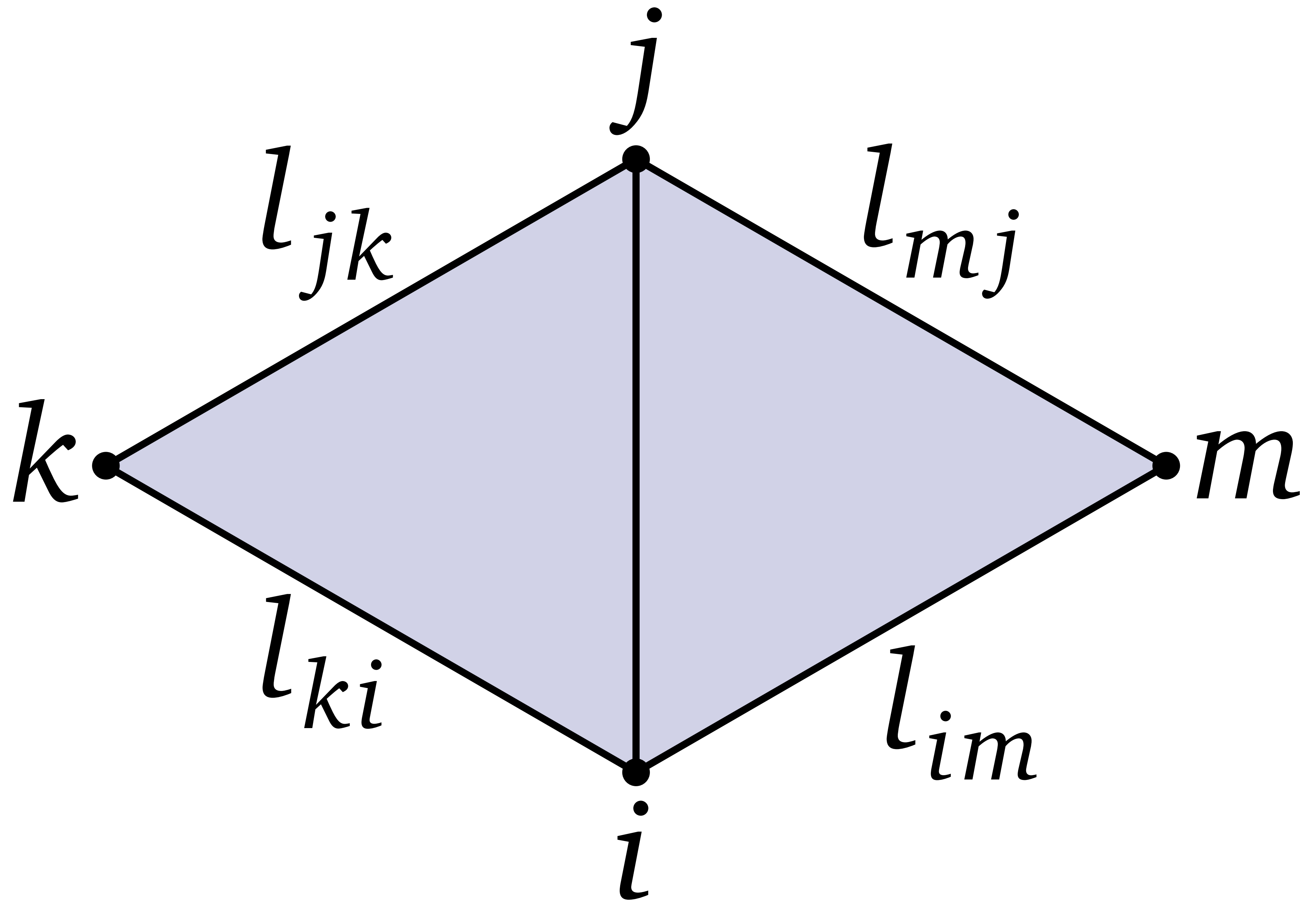


- Initially looks like naïve numerical approximation
- Turns out to provide complete discrete theory that (exactly) captures much of the behavior found in the smooth setting.

Preservation of Length Cross Ratios

Fact. (Springborn-Schröder-Pinkall)

If two discrete metrics are conformally equivalent, then they exhibit the same *length cross ratios*.



length cross ratio

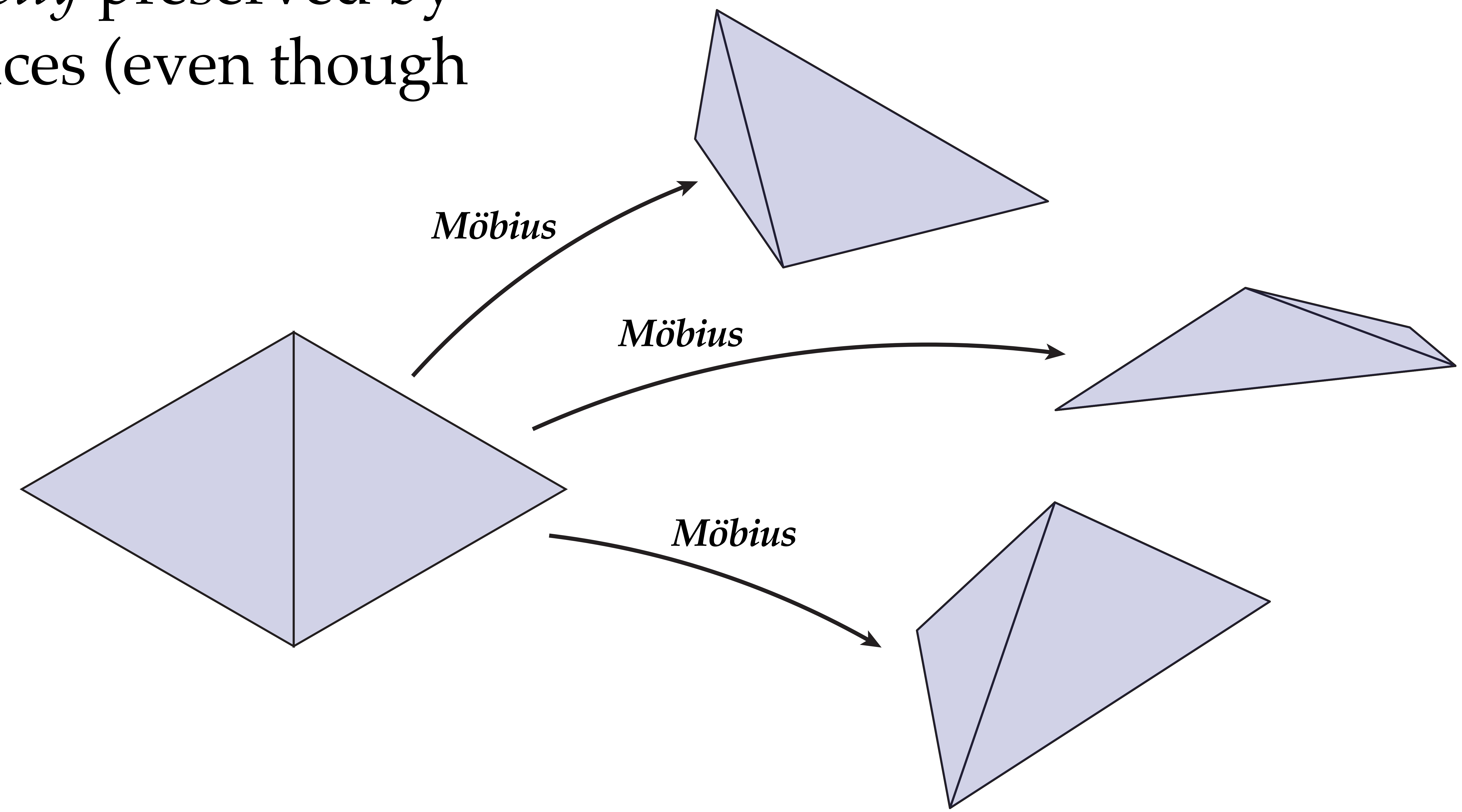
$$\mathfrak{c}_{ij} := \frac{l_{im}}{l_{mj}} \frac{l_{jk}}{l_{ki}}$$

$$\boxed{\mathfrak{c} \equiv \tilde{\mathfrak{c}}}$$

discrete conformal equivalence

Möbius Invariance of CETM

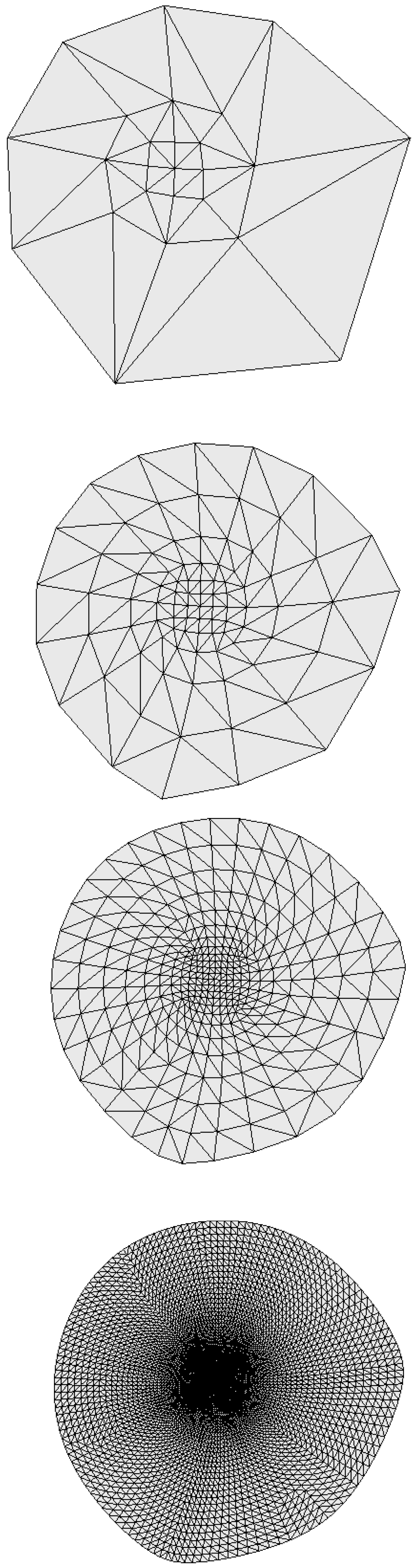
Fact. Length cross ratios are *exactly* preserved by Möbius transformations of vertices (even though *angles* are not!)

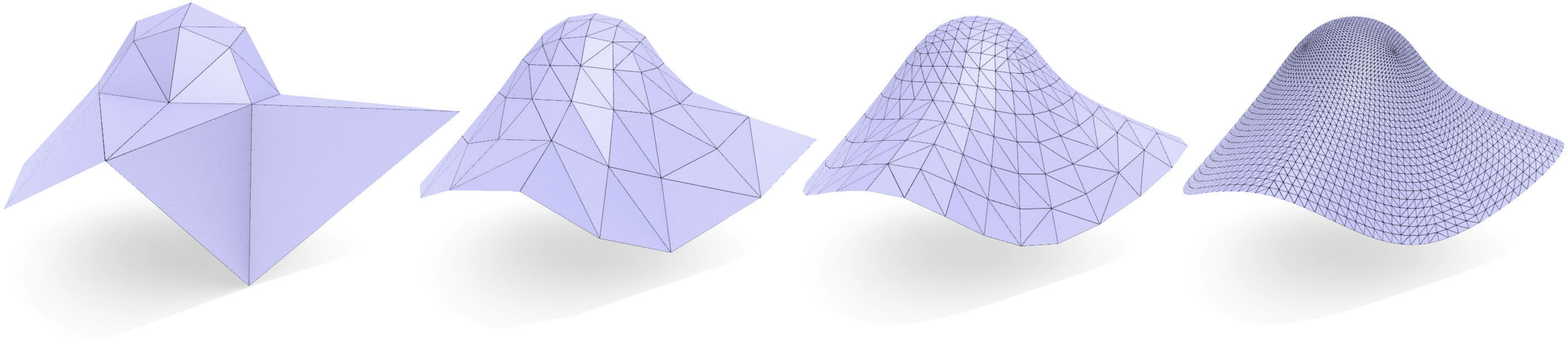


Key idea: discrete theory may not always capture "most obvious" properties (like angles); should try to think more broadly: *"what other characterizations are available?"*

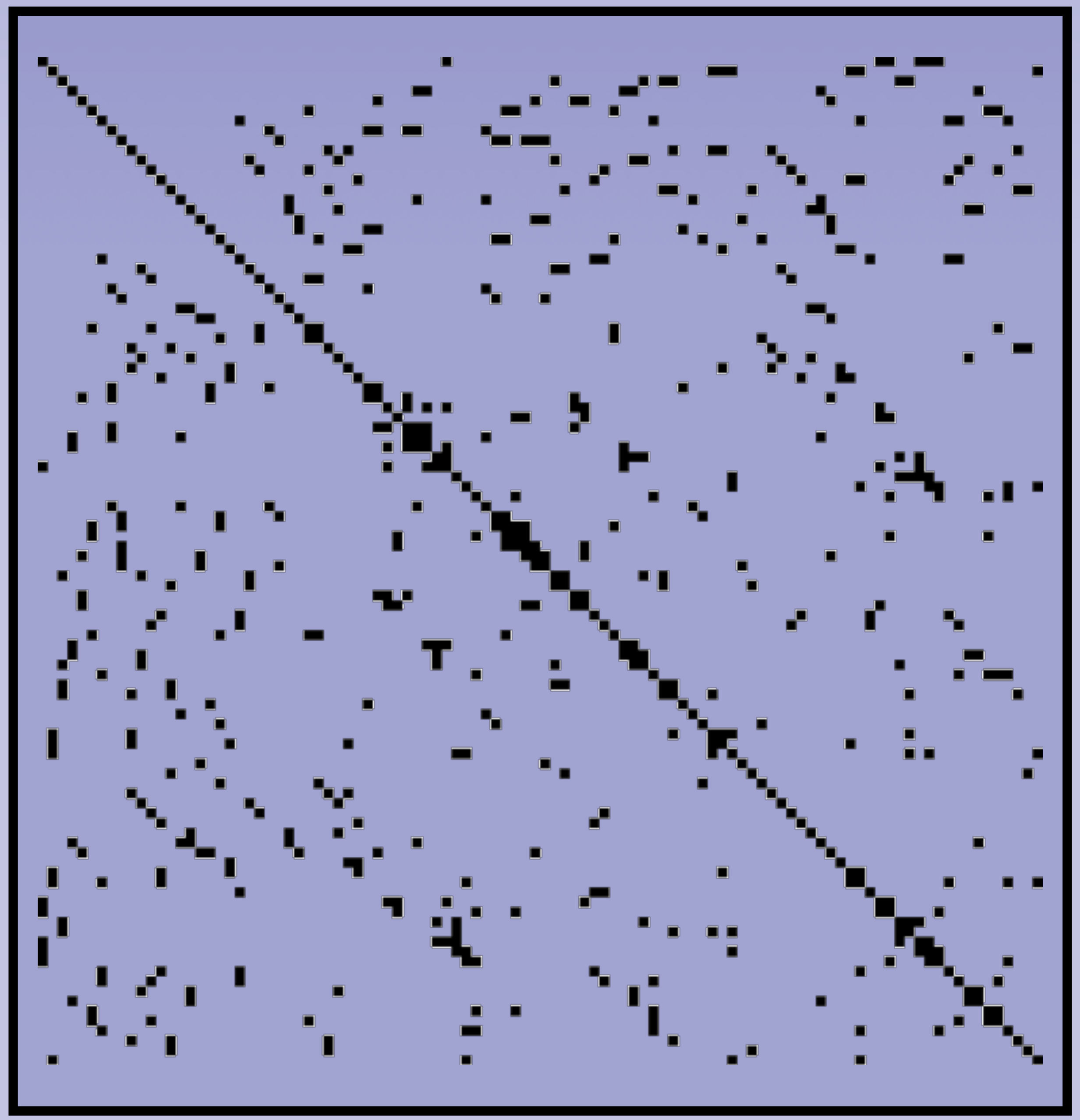
"Discretized" Conformal Maps?

- Ok, that's the *"discrete"* definition…
- …What about *"discretized"* notions of conformal maps?
  - these are much easier to come by
  - basically anything that converges under refinement
  - will see more of this as we discuss algorithms





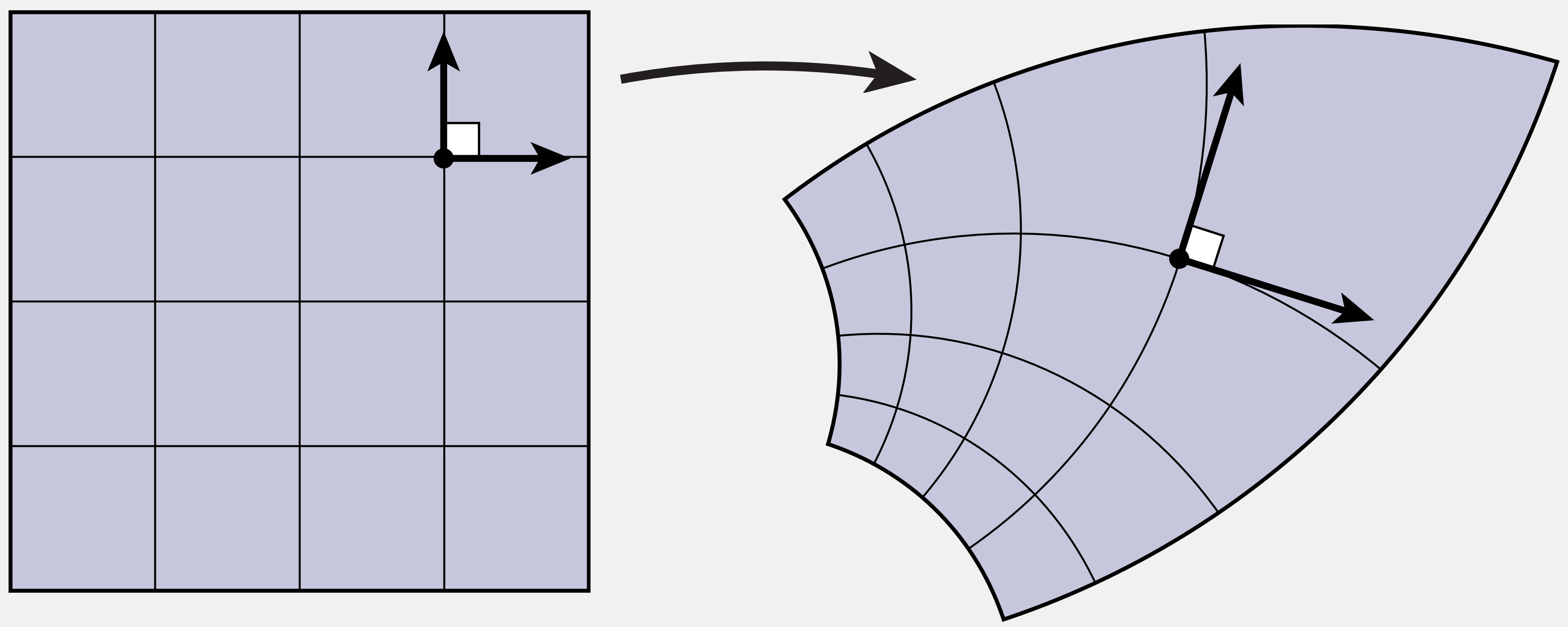
Part IV: Algorithms



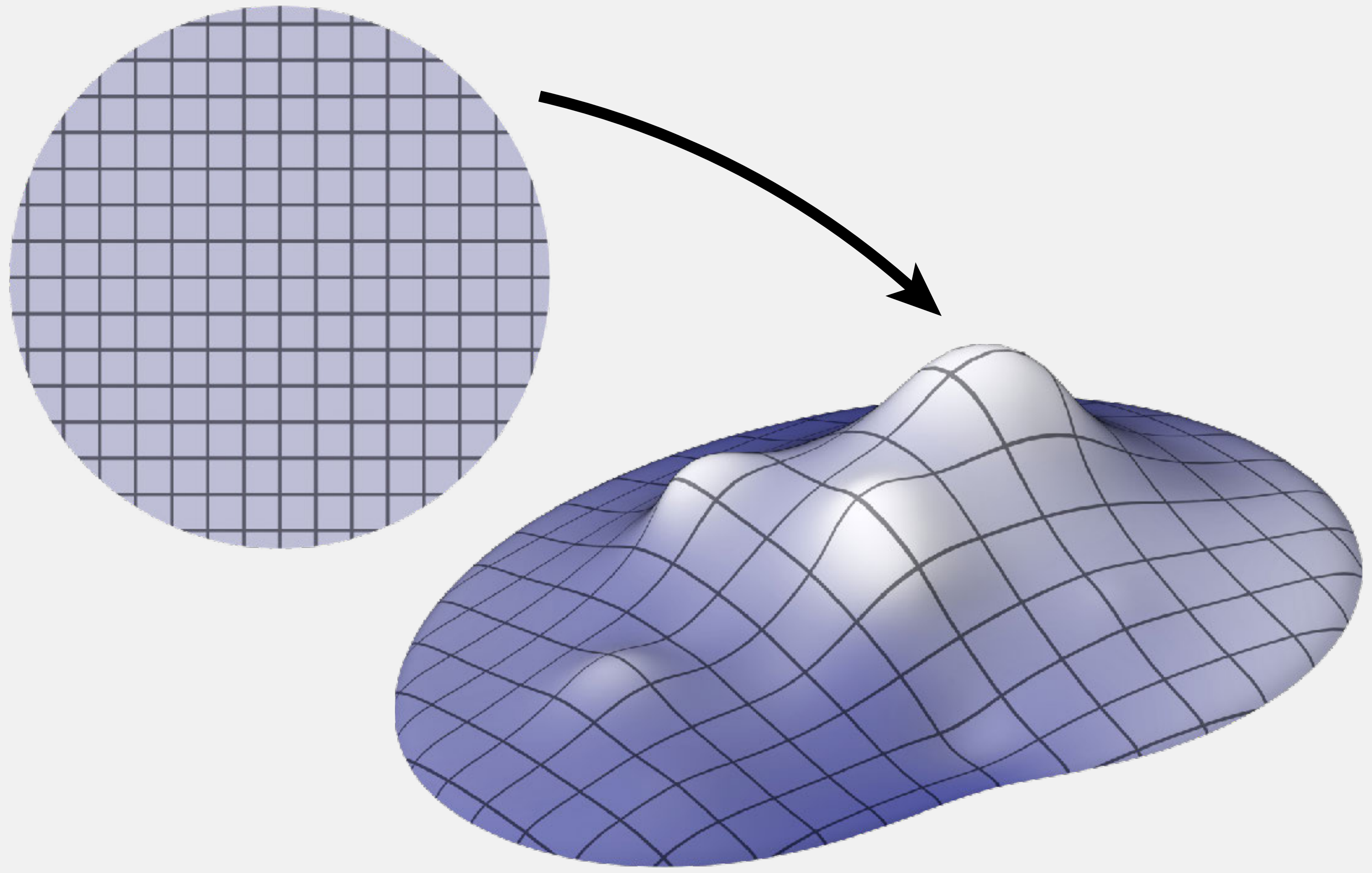
Conformal Geometry Processing

Keenan Crane • Symposium on Geometry Processing Grad School • Summer 2017

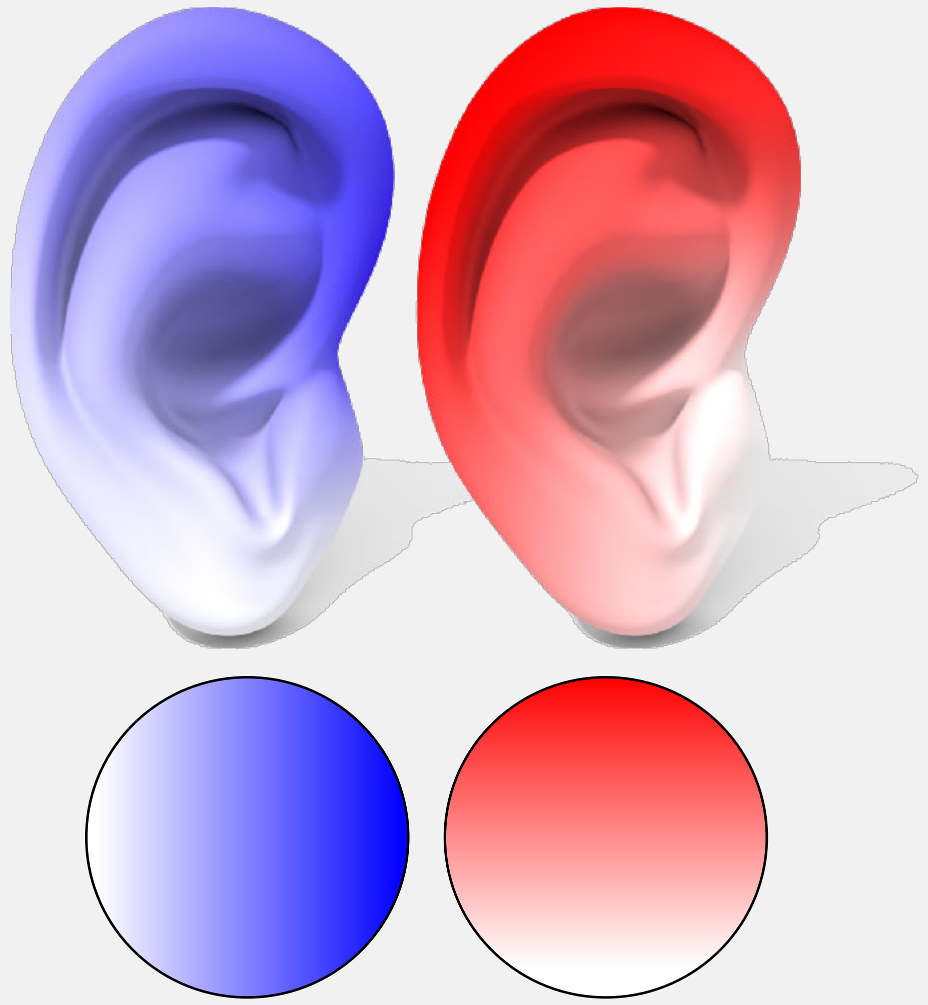
(Some) Characterizations of Conformal Maps



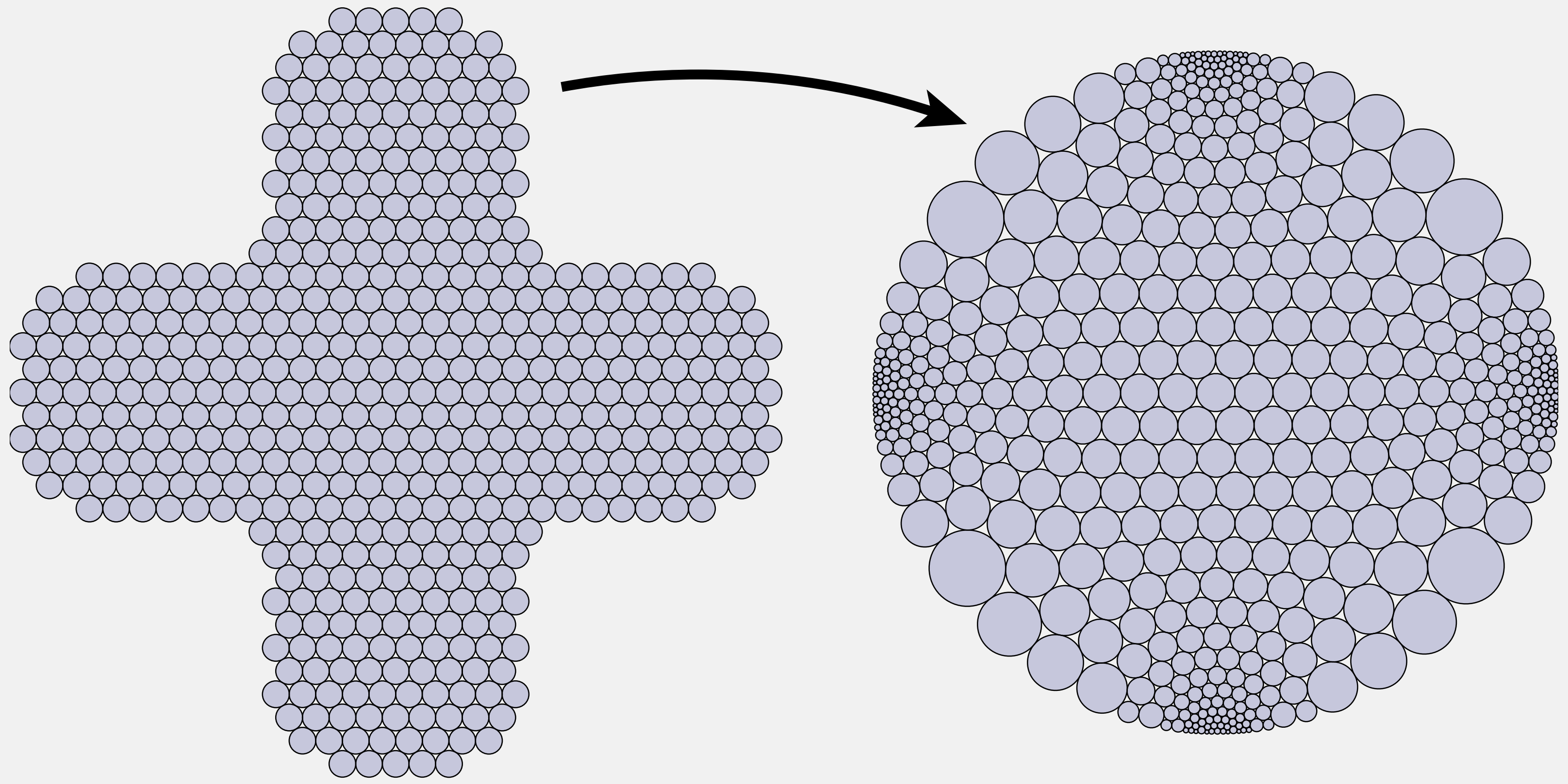
angle preservation



metric rescaling



conjugate harmonic functions



preservation of circles

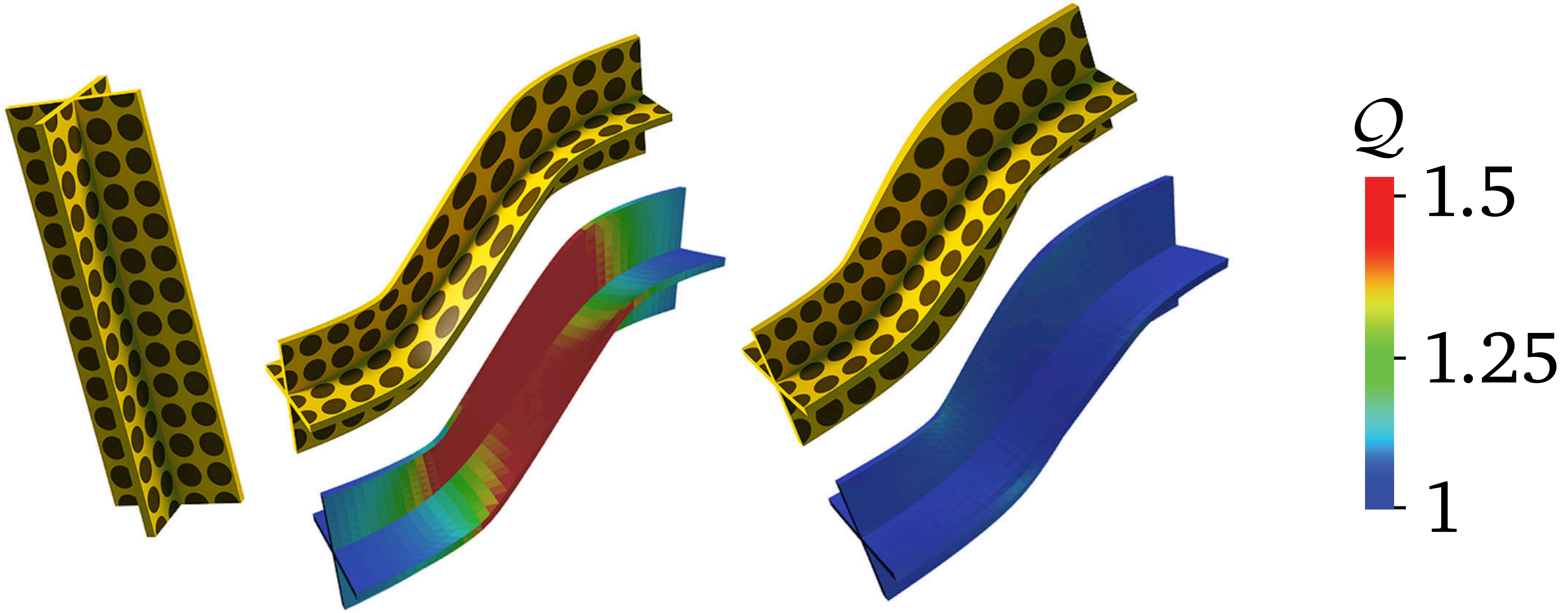
critical points of Dirichlet energy

(Some) Conformal Geometry Algorithms

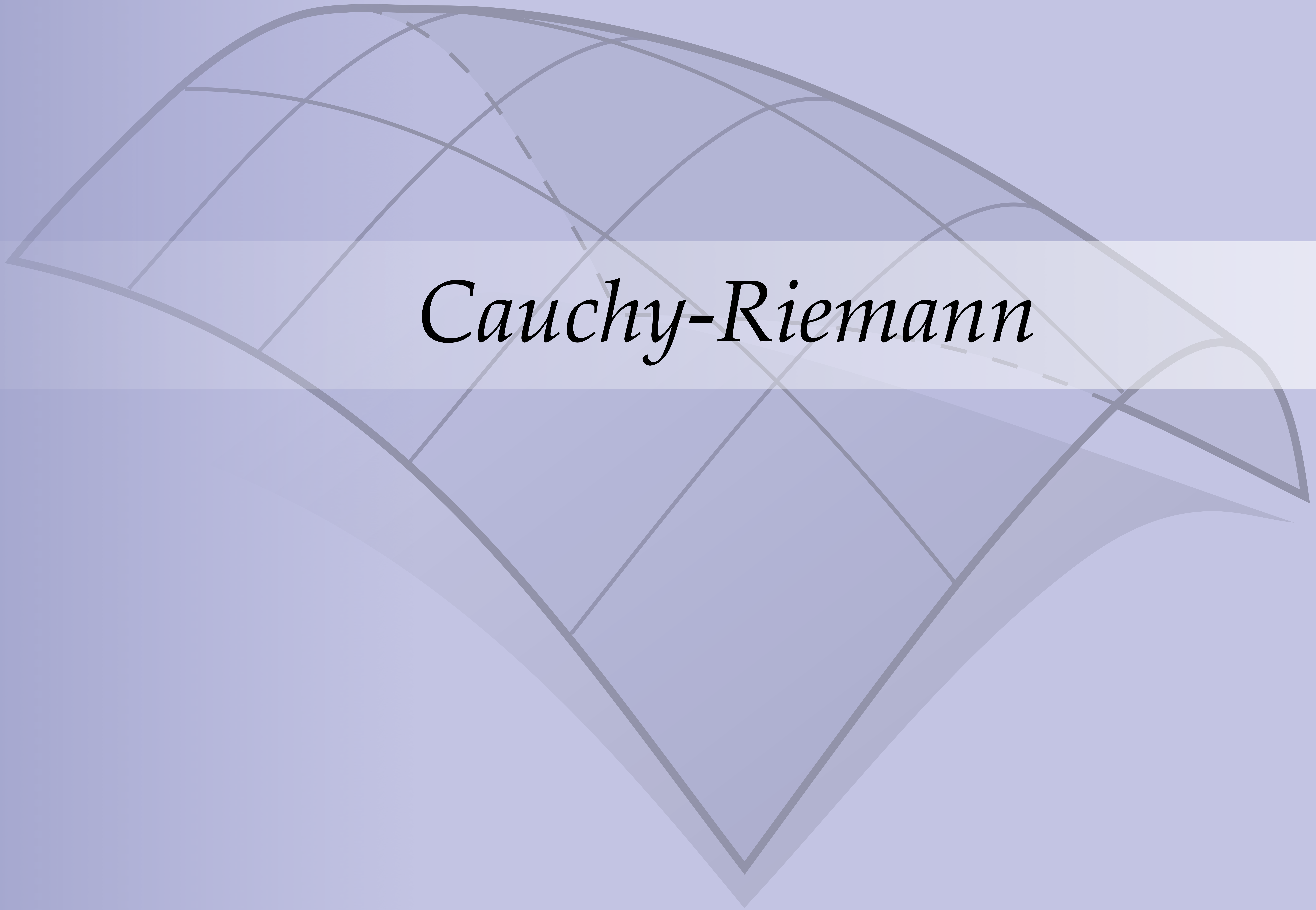
CHARACTERIZATION	ALGORITHMS
Cauchy-Riemann	*least square conformal maps (LSCM)*
Dirichlet energy	*discrete conformal parameterization (DCP)* *genus zero surface conformal mapping (GZ)*
angle preservation	*angle based flattening (ABF)*
circle preservation	*circle packing* *circle patterns (CP)*
metric rescaling	*conformal prescription with metric scaling (CPMS)* *conformal equivalence of triangle meshes (CETM)*
conjugate harmonic	*boundary first flattening (BFF)*

Quasiconformal Distortion

- Only conformal map from triangle to triangle is *similarity* (rigid + scale)
- *Quasiconformal distortion* (***Q***) is ratio of singular values in each triangle
- Measures "how conformal" (want $Q = 1$ everywhere)

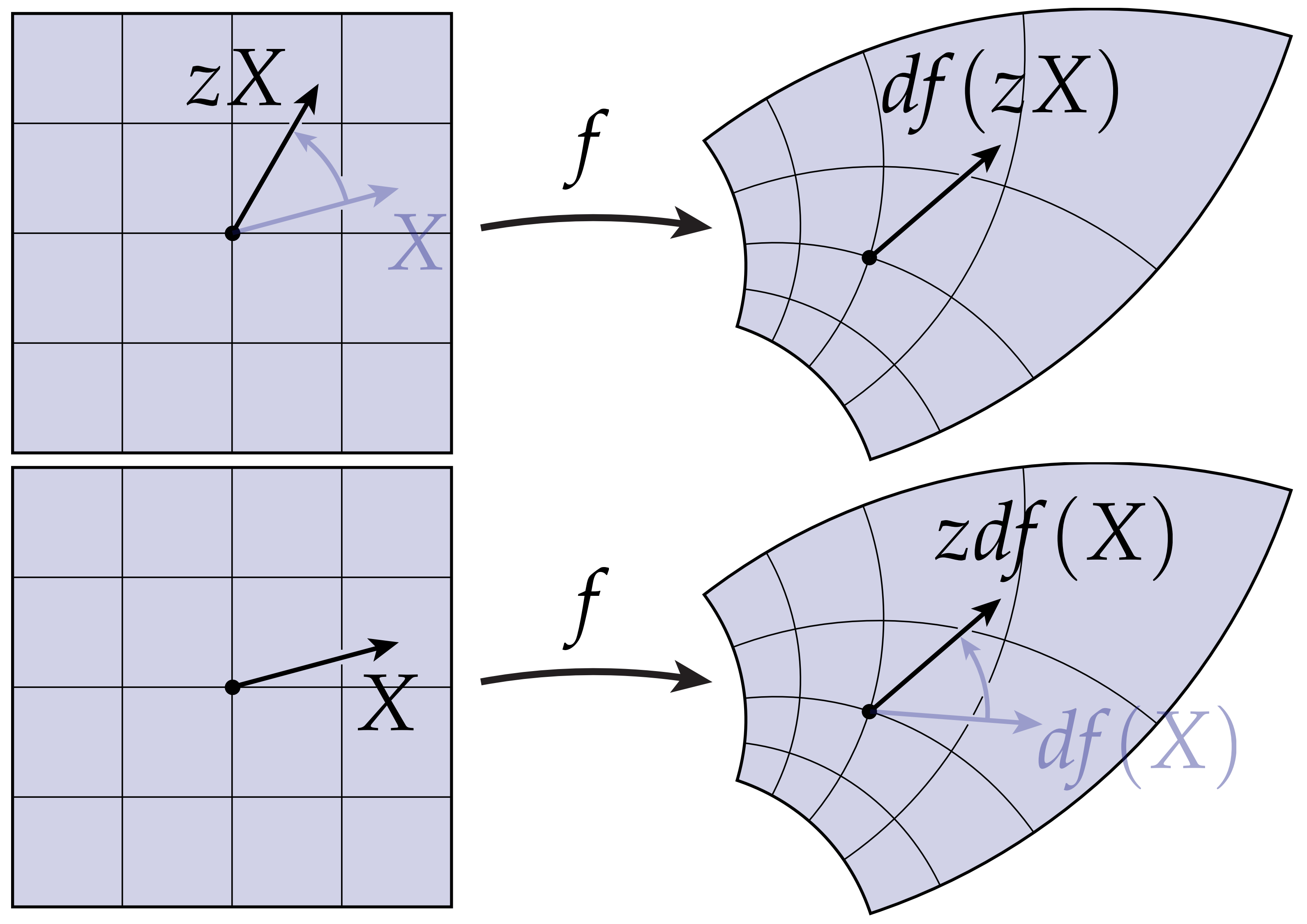


Cauchy-Riemann



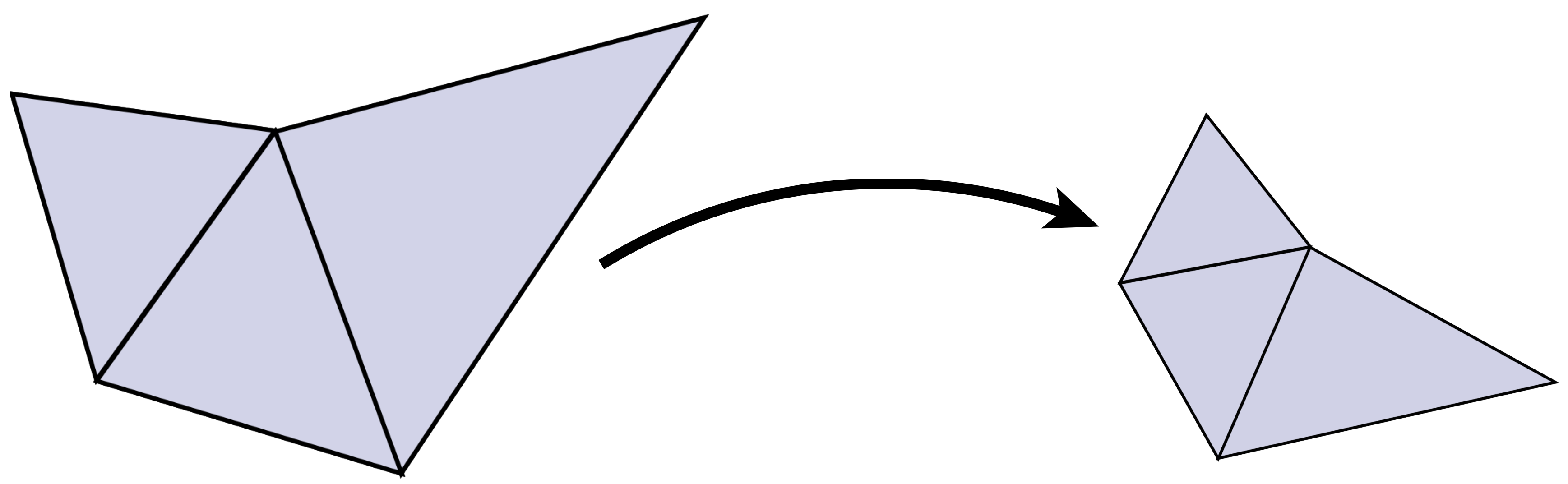
From Cauchy-Riemann to Algorithms

- Natural starting point: solve Cauchy-Riemann equation
- Already know that there will be no *exact* solutions for a triangle mesh
- Instead, find solution that minimizes residual
- Leads to *least squares conformal map (LSCM)*
- Very popular; in *Maya*, *Blender*, `libigl`, …
- Fully automatic; *no control over target shape*

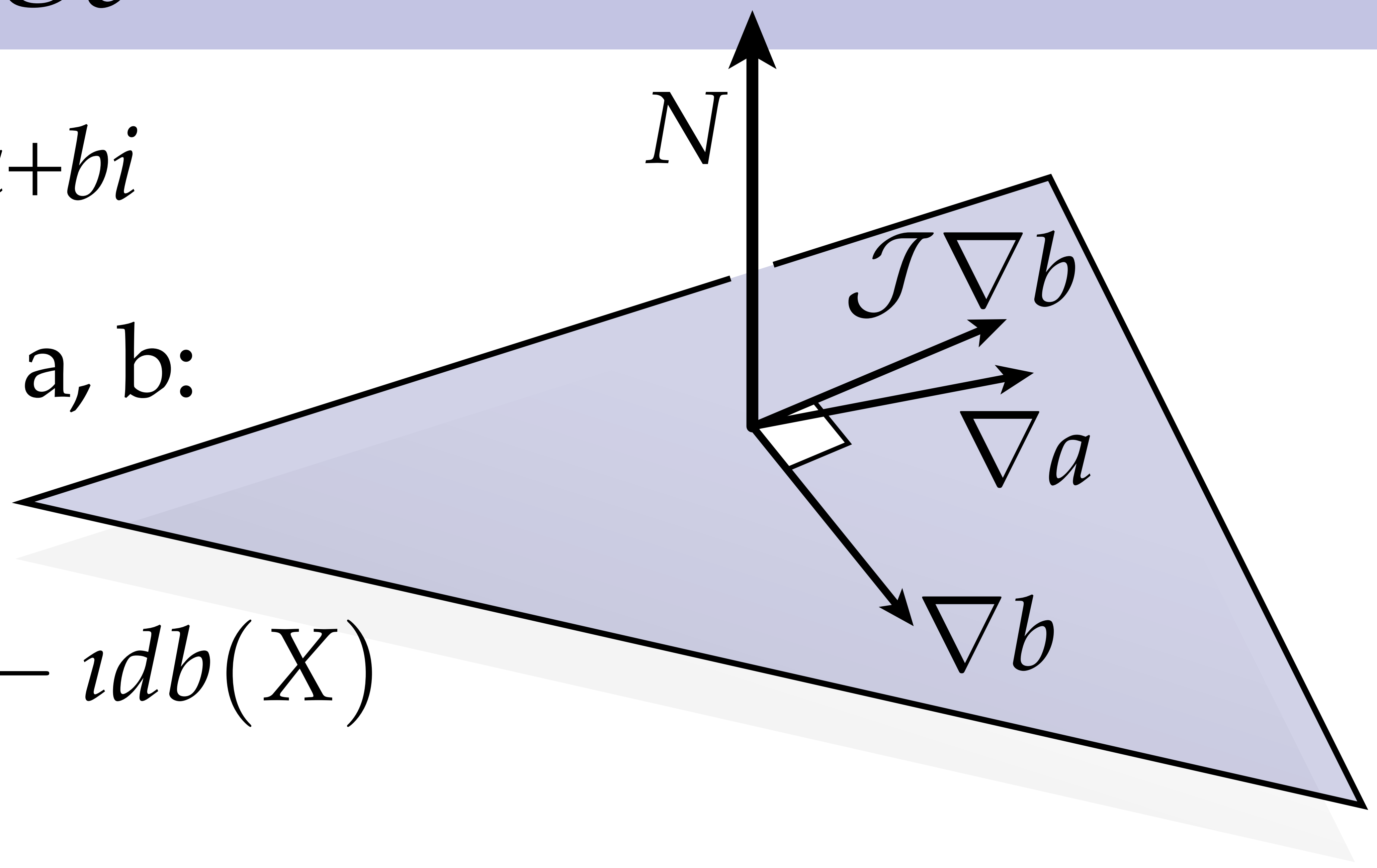


$$\boxed{df(zX) = zdf(X)}$$

Cauchy-Riemann



Least Square Conformal Energy



- Write map as pair of real coordinates: $f = a+bi$
- Express Cauchy-Riemann as condition on a, b:

$$\begin{array}{rrcl} & df(\mathcal{J}X) & = & \imath df(X) \\ \Longleftrightarrow & da(\mathcal{J}X) + \imath db(\mathcal{J}X) & = & \imath da(X) - \imath db(X) \\ \Longleftrightarrow & \nabla a & = & -\mathcal{J}\nabla b \end{array}$$

- Sum failure of this relationship to hold over all triangles:

$$E_{\text{LSCM}}(a,b) := \sum_{ijk\in F} \mathcal{A}_{ijk} \left((\nabla a)_{ijk} - N_{ijk} \times (\nabla b)_{ijk} \right)^2$$

- Resulting energy is *convex* and *quadratic* (*i.e.*, "easy"!)

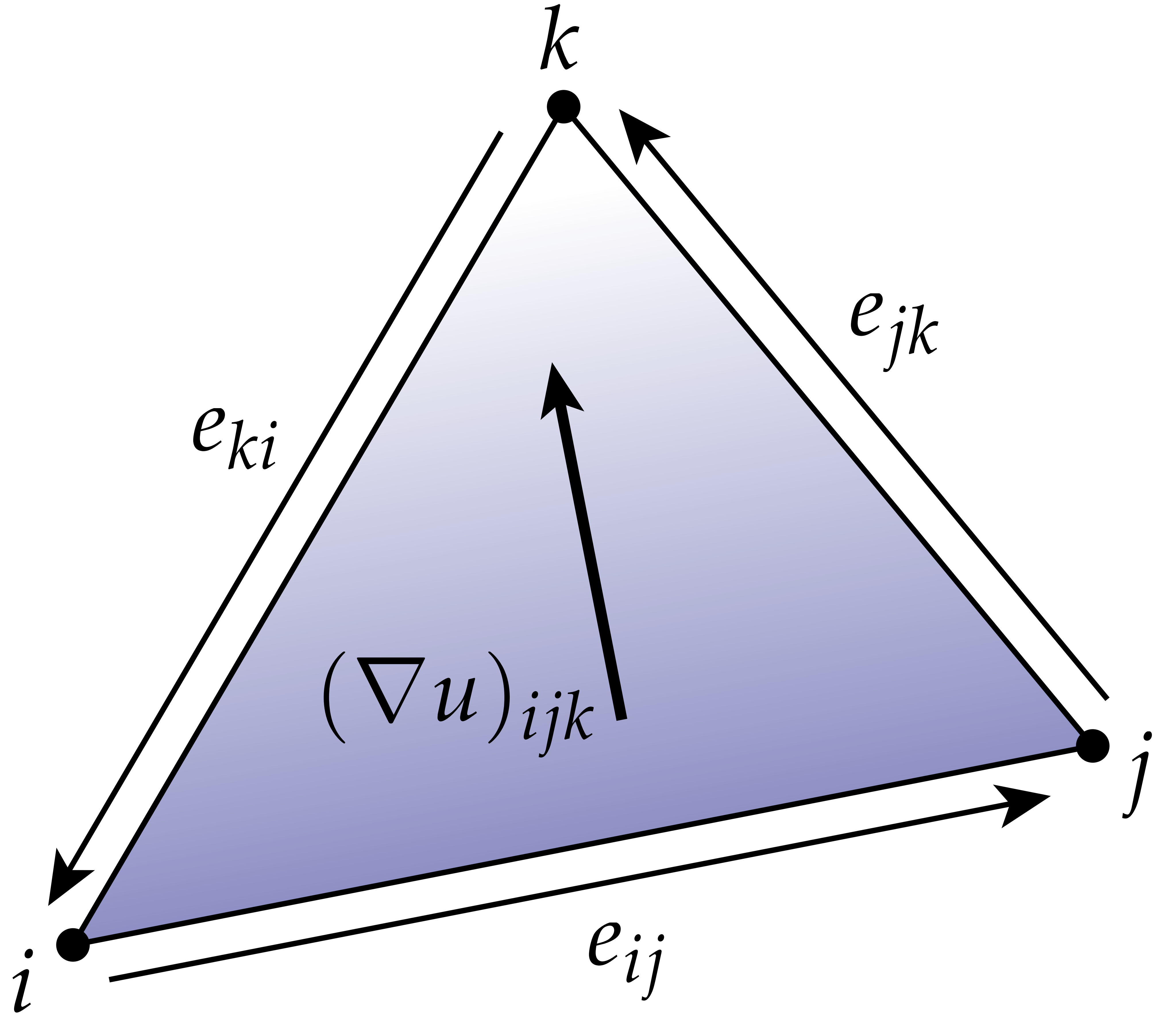
Gradient of a Piecewise Linear Function

- Many geometry processing algorithms need *gradient* of a function (i.e., direction of "steepest increase")

- Easy formula on a triangle mesh:

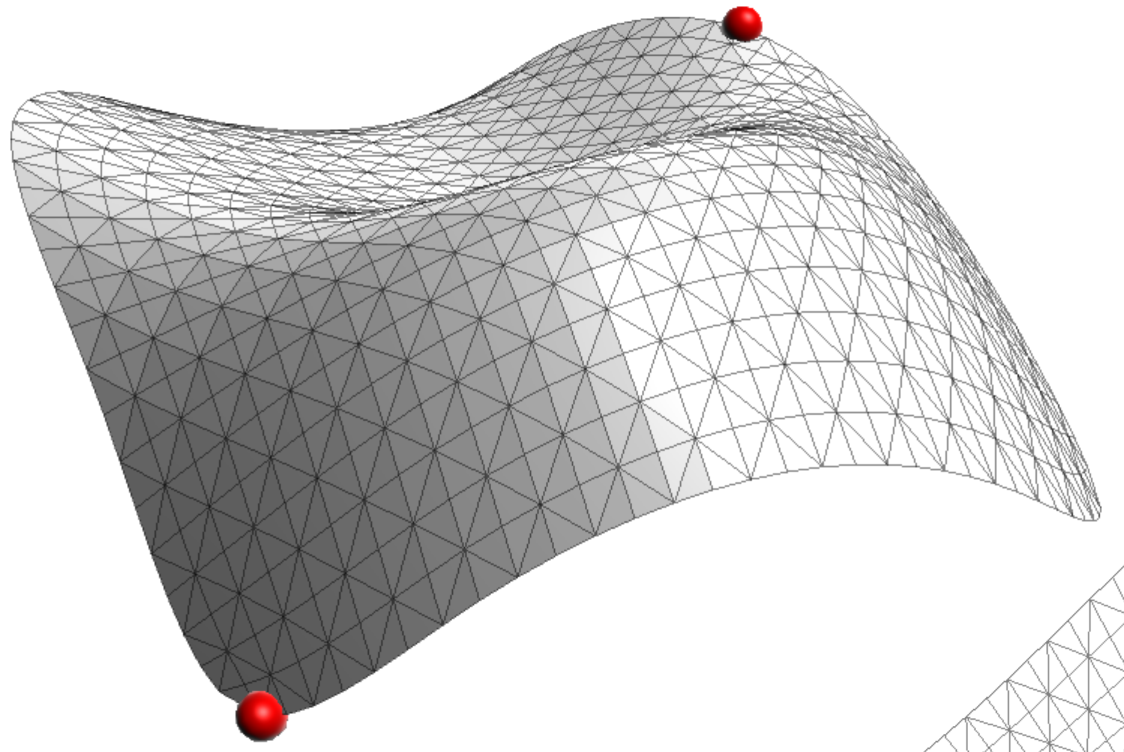
$$\boxed{(\nabla u)_{ijk} = \frac{1}{2\mathcal{A}_{ijk}}(u_i e_{jk} + u_j e_{ki} + u_k e_{ij})}$$

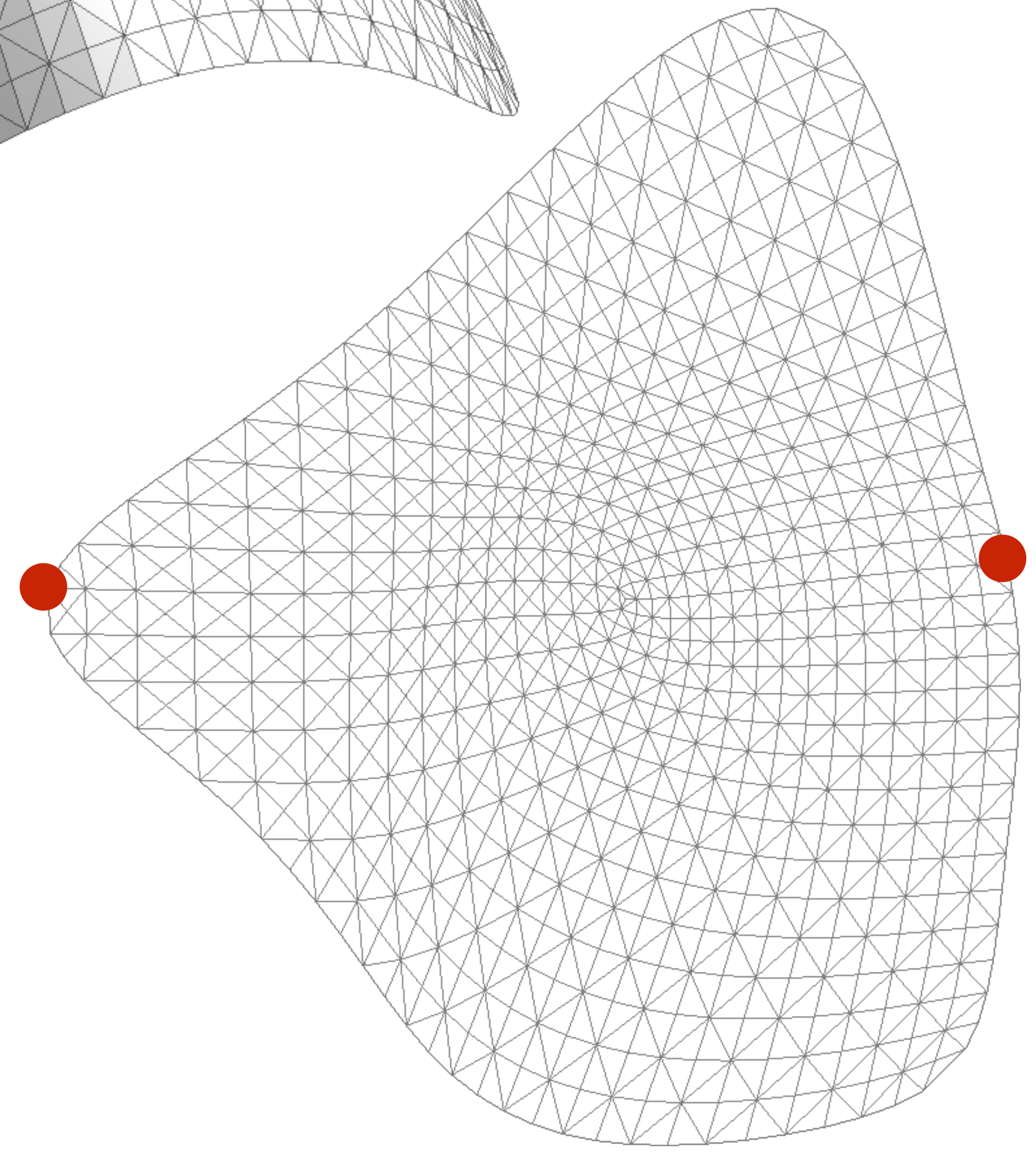
- Since function is *linear*, gradient is *constant* across each triangle.



Least Square Conformal Maps (LSCM)

- Coordinate functions (*a*,*b*) that minimize E_{LSCM} give the “best” map
- **Problem:** *constant* functions have zero energy!
- **Solution***: *“pin”* two vertices to fixed locations
  - one vertex determines translation in plane
  - the other determines rotation & scale
- *Will see later that this solution is still not quite right…
- To minimize, set gradient to zero and solve for (*a*,*b*)
- Numerical problem is sparse linear system (very easy to solve)





Least Square Conformal Maps (LSCM)

- Coordinate functions (a,b) that minimize E_{LSCM} give the "best" map
- Can encode energy as a quadratic form:

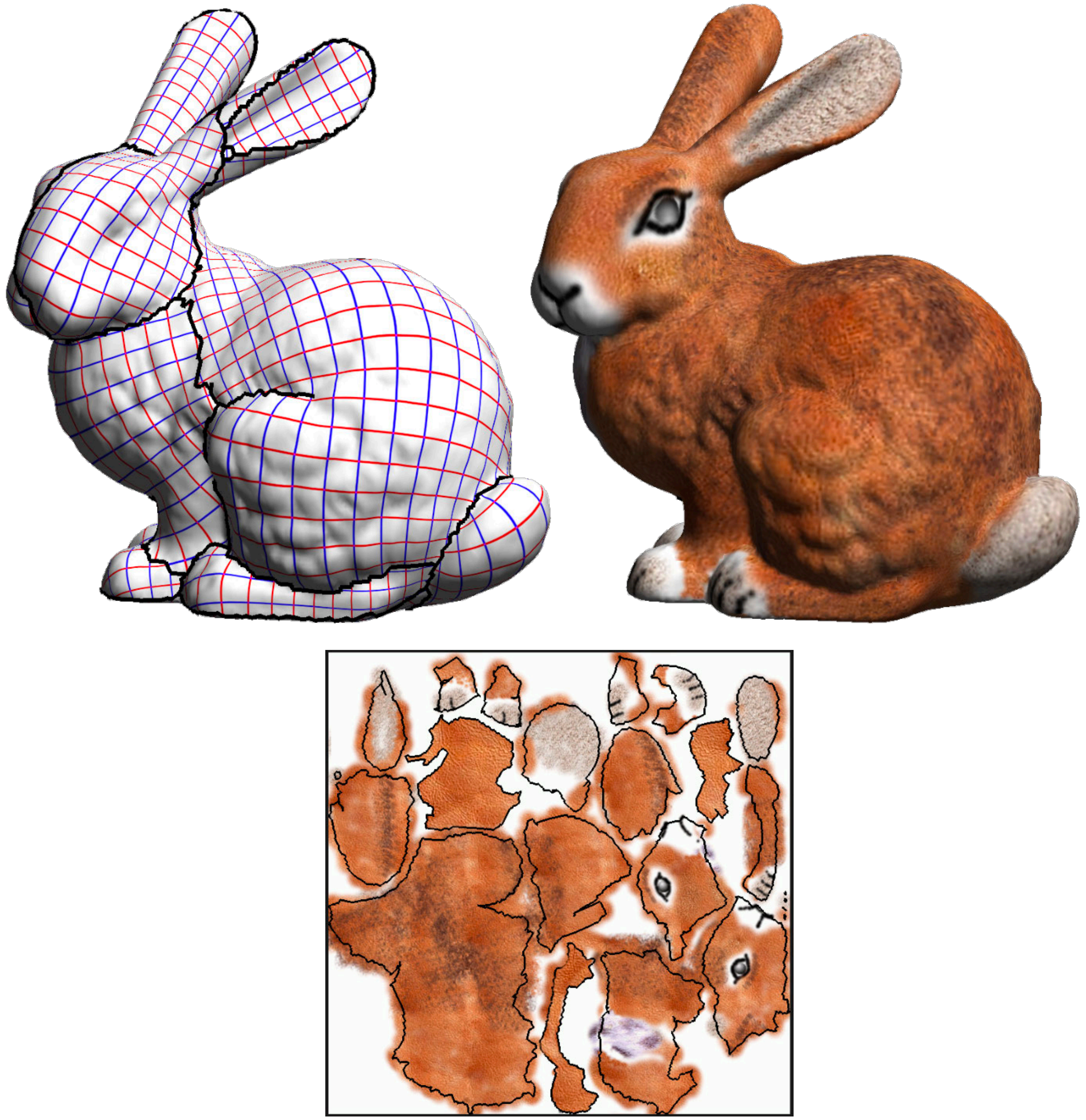
$$\mathsf{x} := \left[\begin{array}{ccccc} a_1 & b_1 & \cdots & a_n & b_n \end{array}\right]^{\mathsf{T}}$$

$$E_{\mathrm{LSCM}}(a,b) = \tfrac{1}{2}\,\mathsf{x}^{\mathsf{T}}\mathsf{A}\,\mathsf{x}, \quad \mathsf{A} \in \mathbb{R}^{2n\times 2n}$$

- Minimize by setting gradient equal to zero:

$$\mathsf{A}\mathsf{x} = 0$$

- Just need to solve a linear system
- **Problem:** has trivial solution $\mathsf{x} = 0$!



LÉVY, PETITJEAN, RAY, MAILLOT, *"Least Squares Conformal Maps for Automatic Texture Atlas Generation"* (2011)

LSCM—Nontrivial Solution via "Pinning"

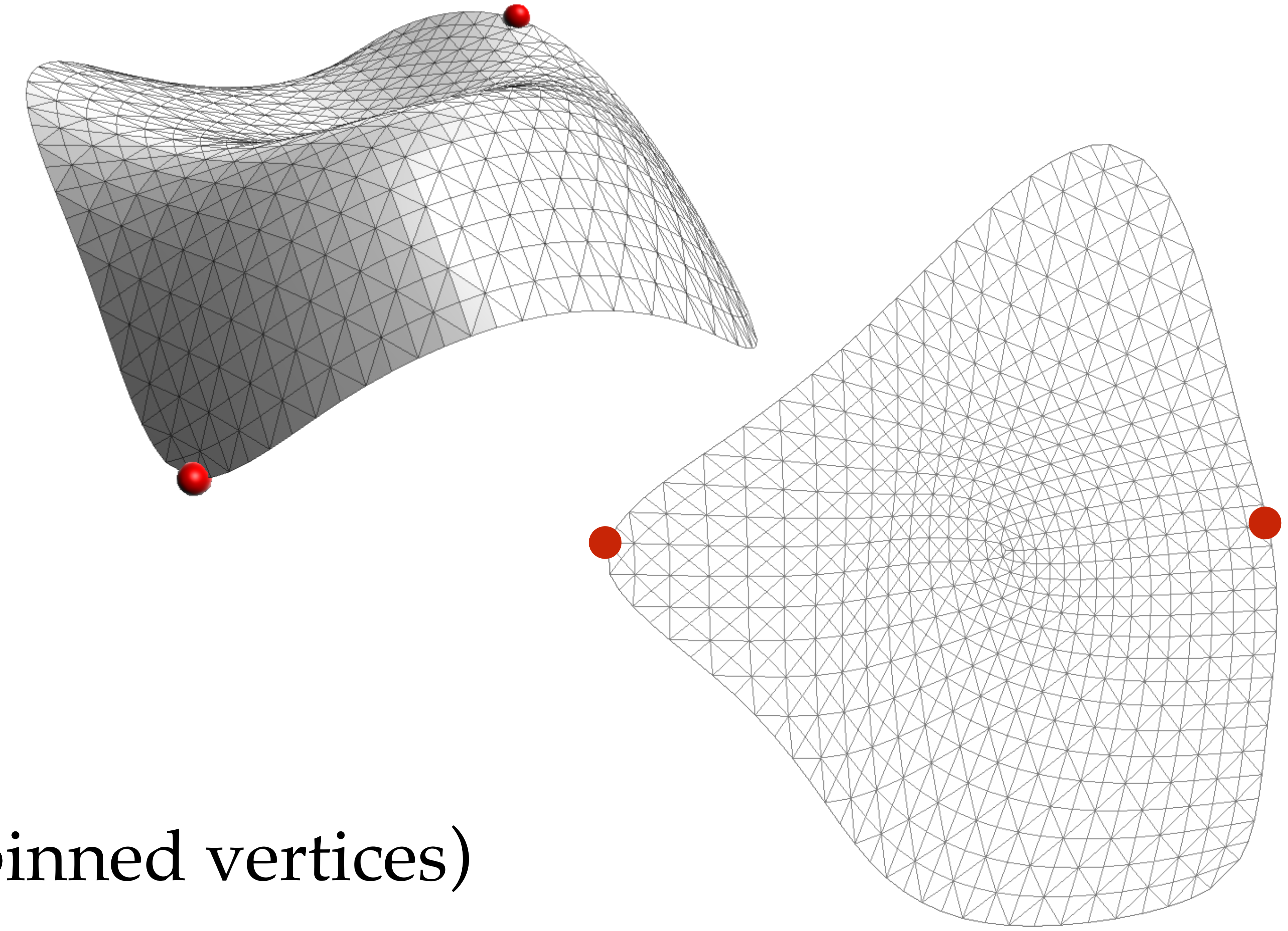
- In fact, any *constant* map will have zero energy, since gradient is zero:

$$E_{\mathrm{LSCM}}(a,b) := \sum_{ijk\in F} \mathcal{A}_{ijk}\left((\nabla a)_{ijk} - N_{ijk}\times(\nabla b)_{ijk}\right)^2$$

- Idea: "pin" any two vertices to arbitrary locations
  - one vertex determines global translation
  - another vertex determines scale/rotation
- Linear system now has nonzero RHS:

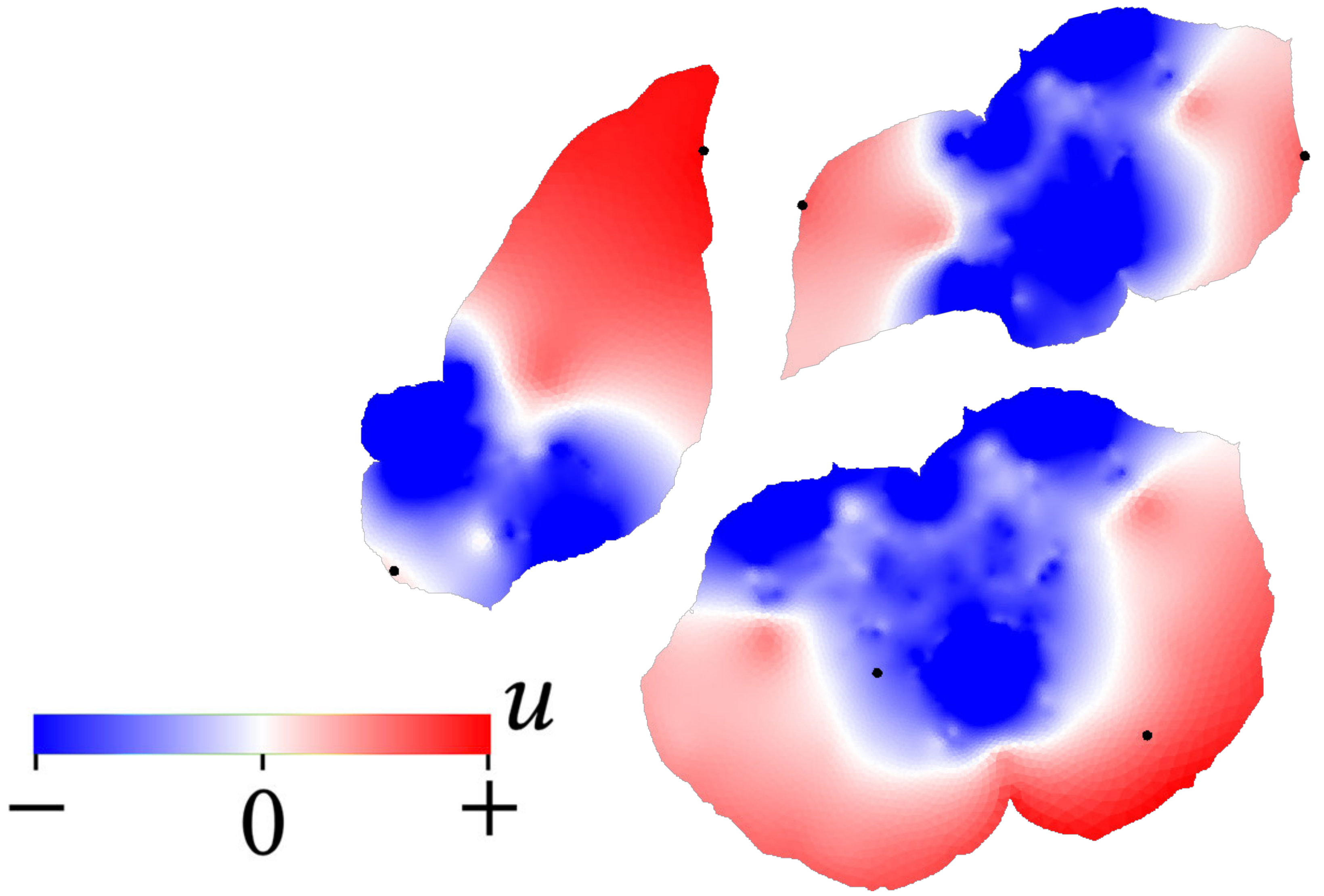
$$\hat{\mathsf{A}}\hat{\mathsf{x}} = \mathsf{b}$$

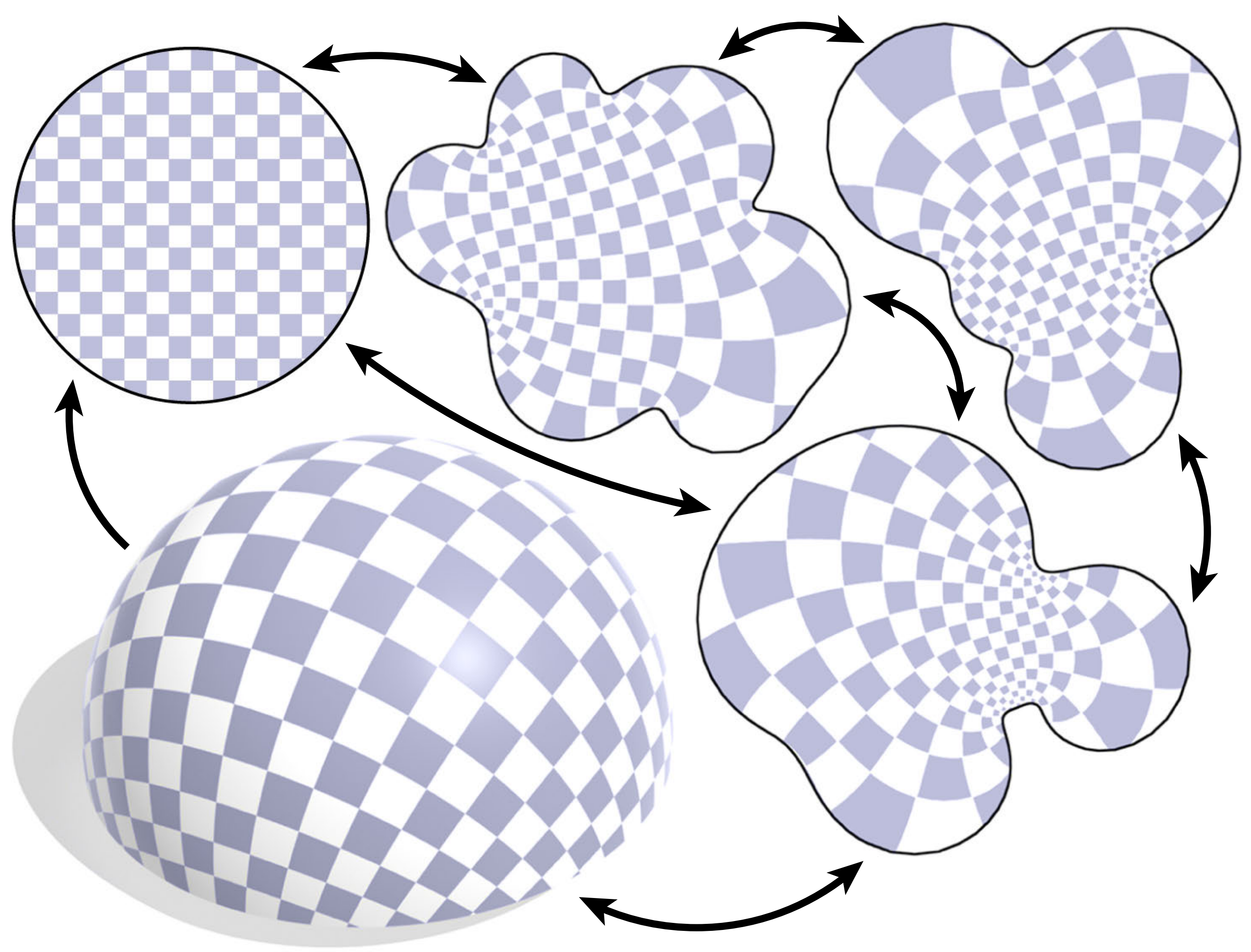
("hat" indicates removed rows/columns, corresponding to pinned vertices)



Problems with Pinning

- To get a unique solution we "pinned down" two vertices
- Two problems with this approach:
  1. map can be unpredictable, distorted depending on choice of vertices
  2. we should have ***way*** more choice about what target shape looks like!

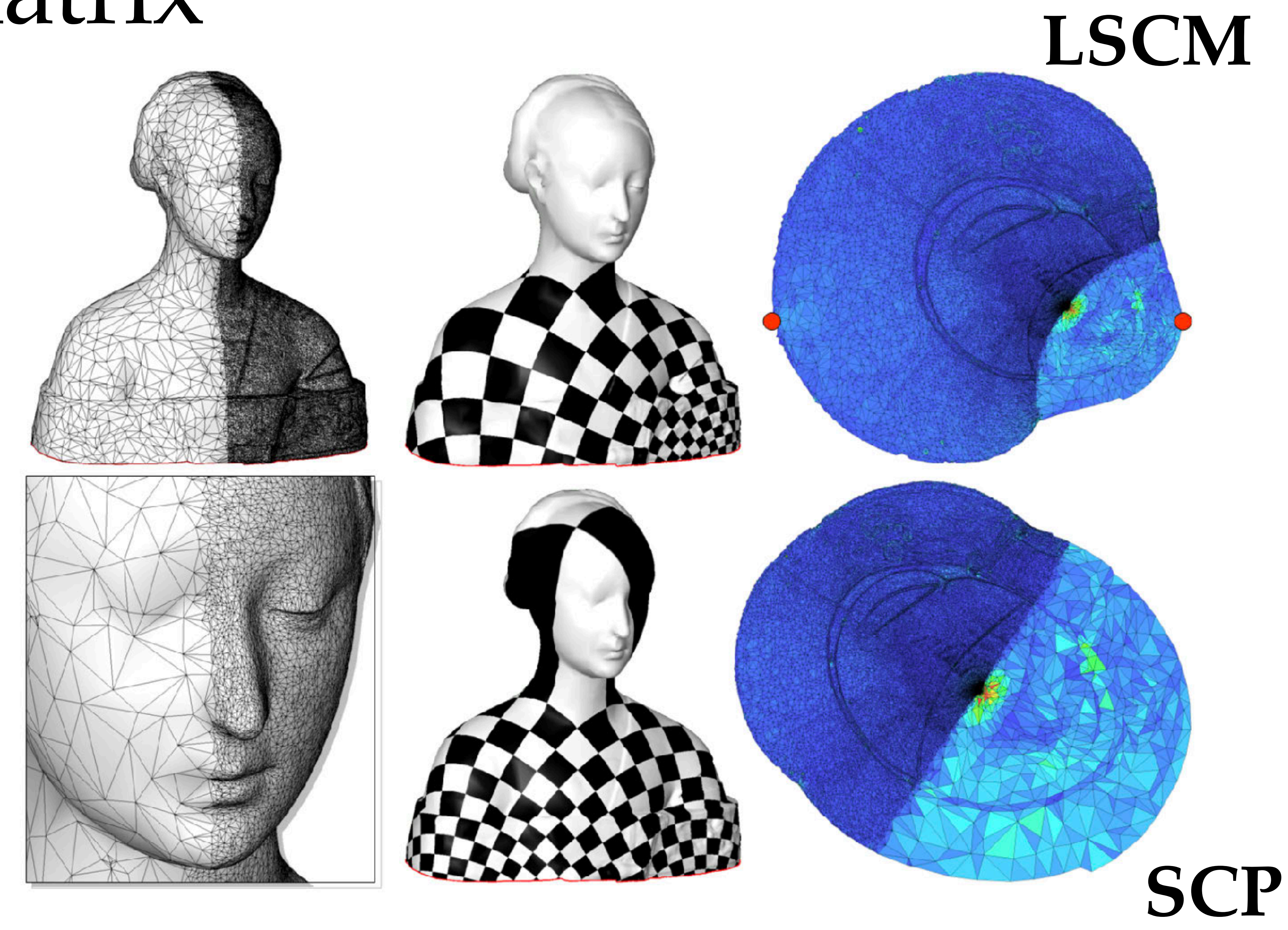




Will address the first issue first…

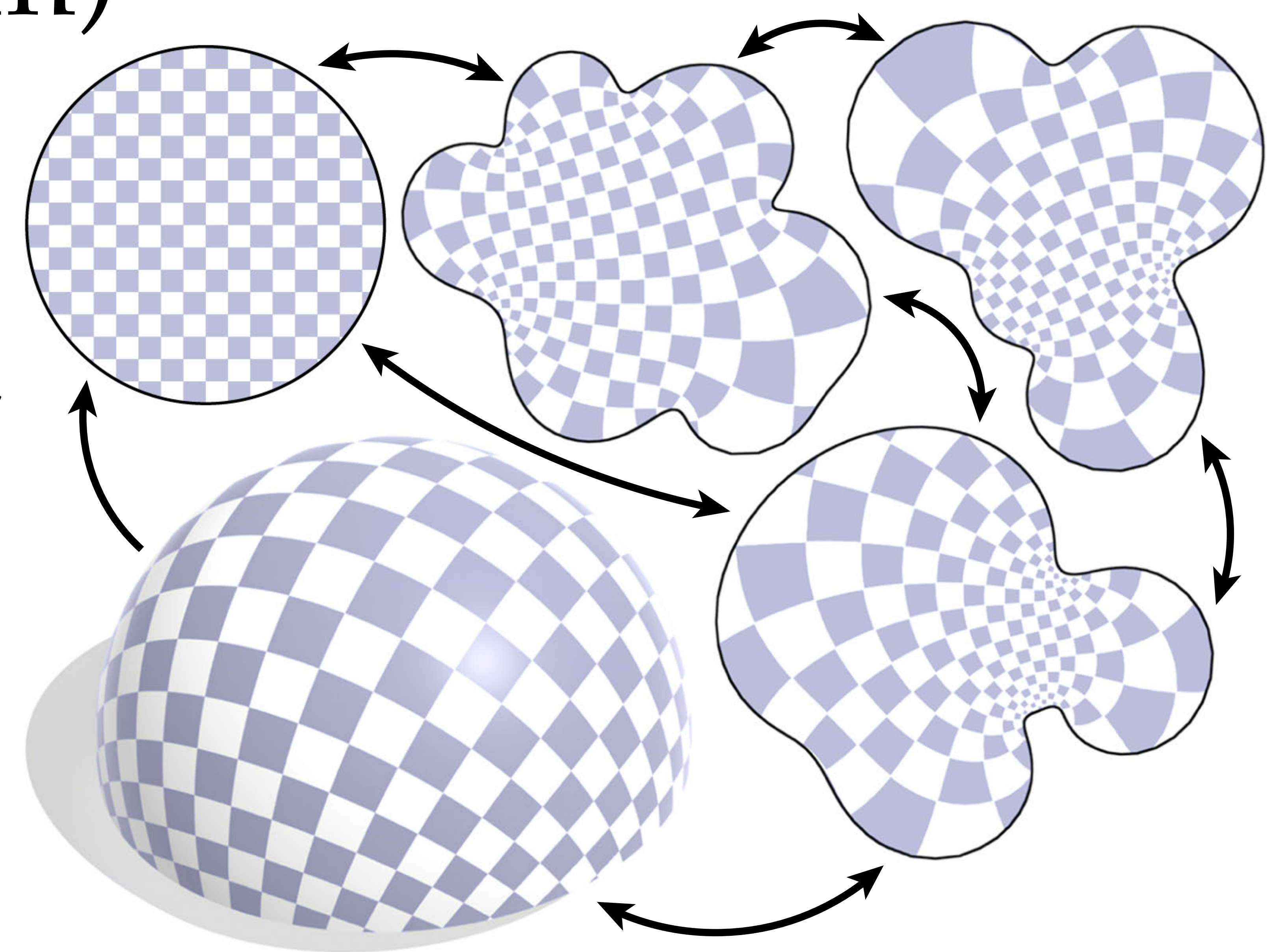
Spectral Conformal Parameterization (SCP)

- "Pinning" was used to prevent degenerate (constant) solution
- Alternatively, can ask for smallest energy among all *unit-norm* solutions
- Compute principal eigenvector of energy matrix
- **Q:** Why does this work better?
  - *identical* from perspective of linear algebra
  - (much) better accuracy in floating-point



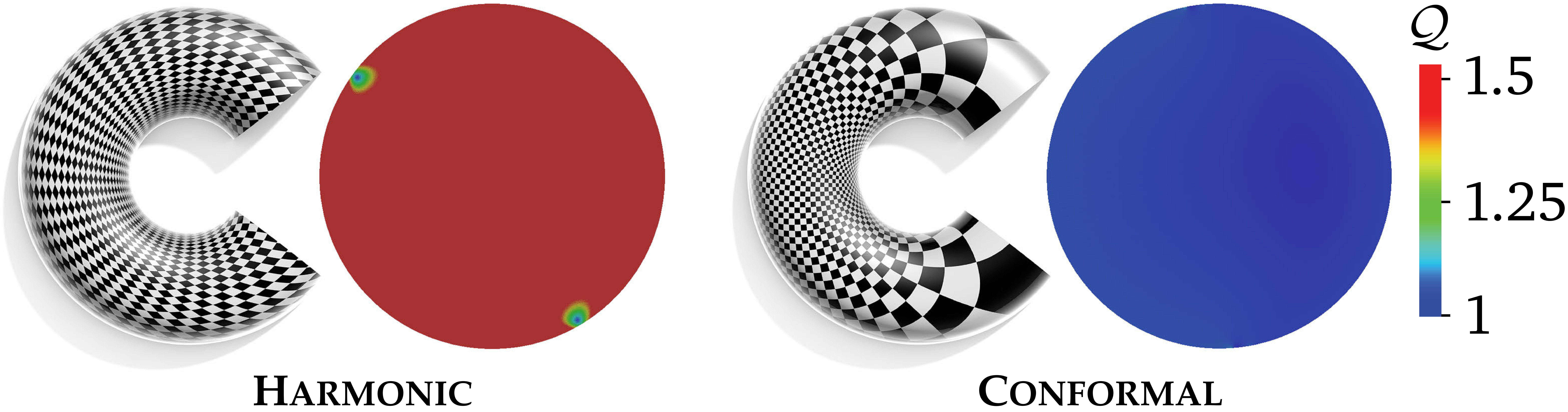
Conformal Maps—Boundary Conditions?

- *Something is still wrong!*
  - In the discrete setting, specified just two points on boundary (just rigid motion & scaling in the plane)
  - In the smooth setting, there are **far** more ways to conformally flatten (Riemann Mapping Theorem)
- What happened here?
  - Among *piecewise linear* maps, "most conformal" solution is unique (up to rigid motion).
  - But what if we want to *control* target shape?

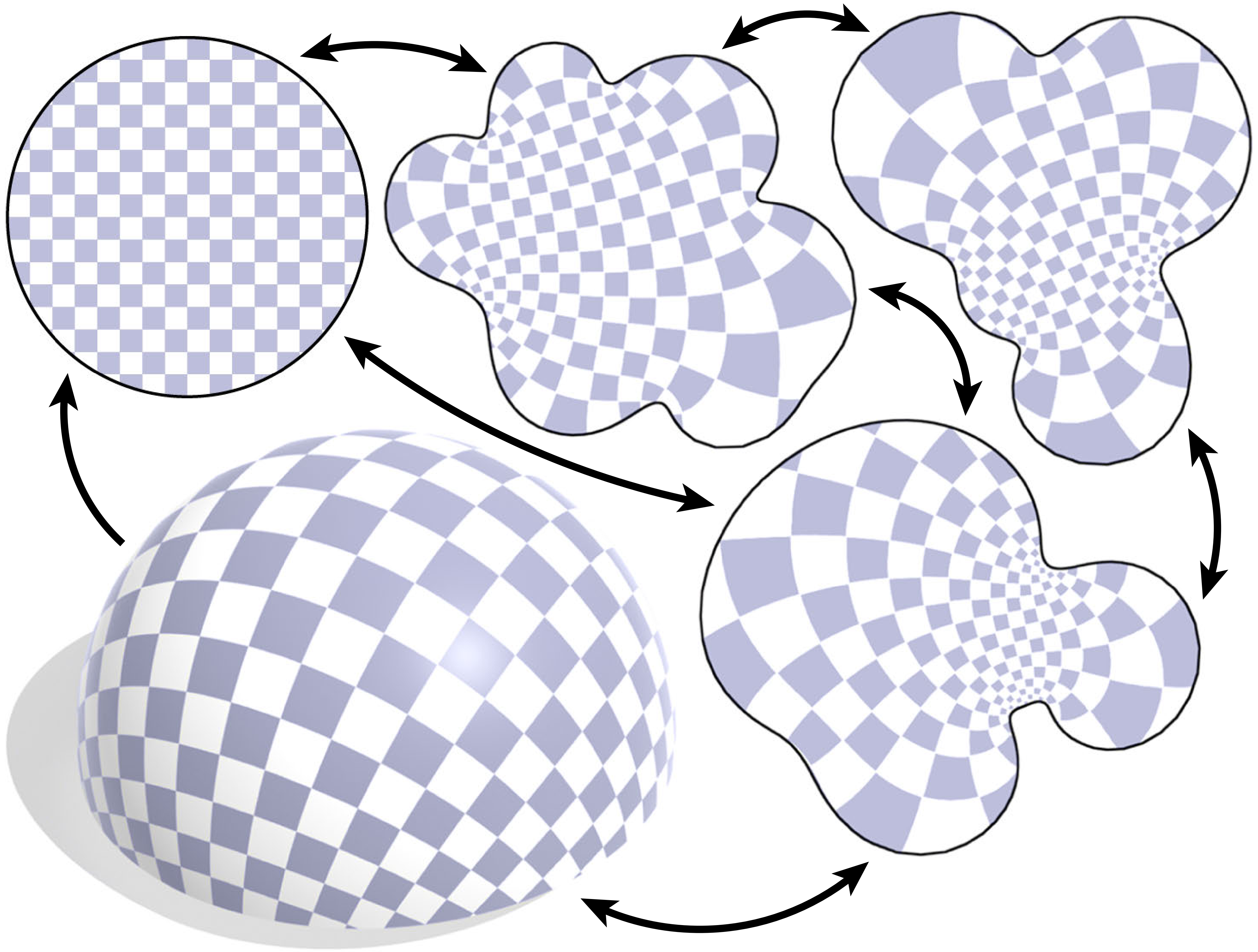


Prescribing the Entire Boundary Doesn't Work

- **First attempt:** pin *all* boundary points to desired target shape
- **Problem:** In general there is no conformal map compatible with a given map along the boundary
- Least-squares yields *harmonic* map with severe angle distortion:

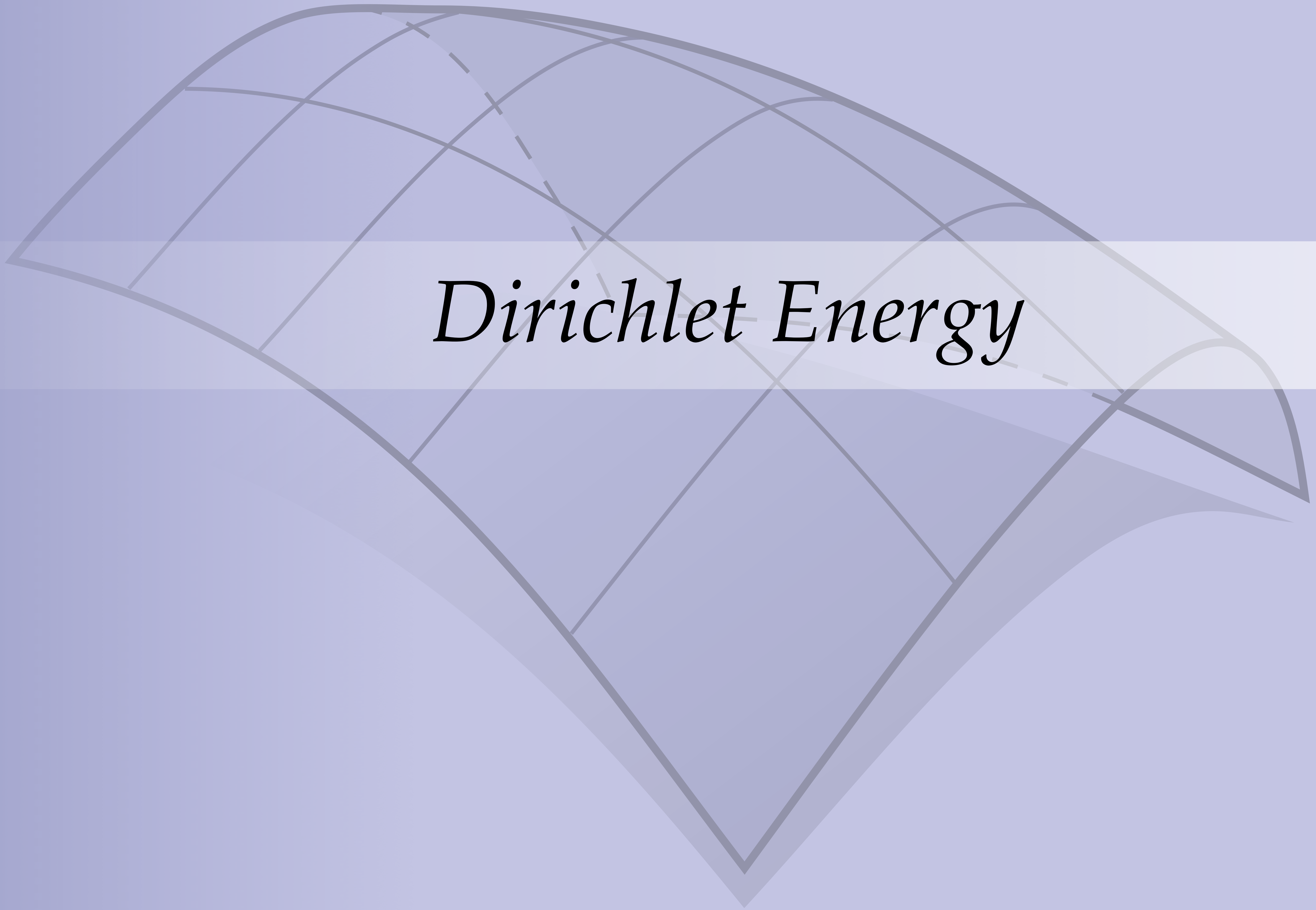


...So what if we want to control target shape?



Will revisit this question later—when we have more tools at our disposal!

Dirichlet Energy



Dirichlet Energy

- Different characterization of conformal maps: critical points of so-called *Dirichlet energy*
  - Physical analogy: elastic membrane that wants to have *zero* area
  - When this energy is minimized, we get a conformal map…
  - …under very special assumptions on the domain / boundary conditions!
- Alternative route to LSCM (a.k.a DCP) & other algorithms

Smooth Dirichlet Energy

- Consider any map f between manifolds M and N
- *Dirichlet energy* is given by:

$$E_D(f) := \int_M |df|^2$$

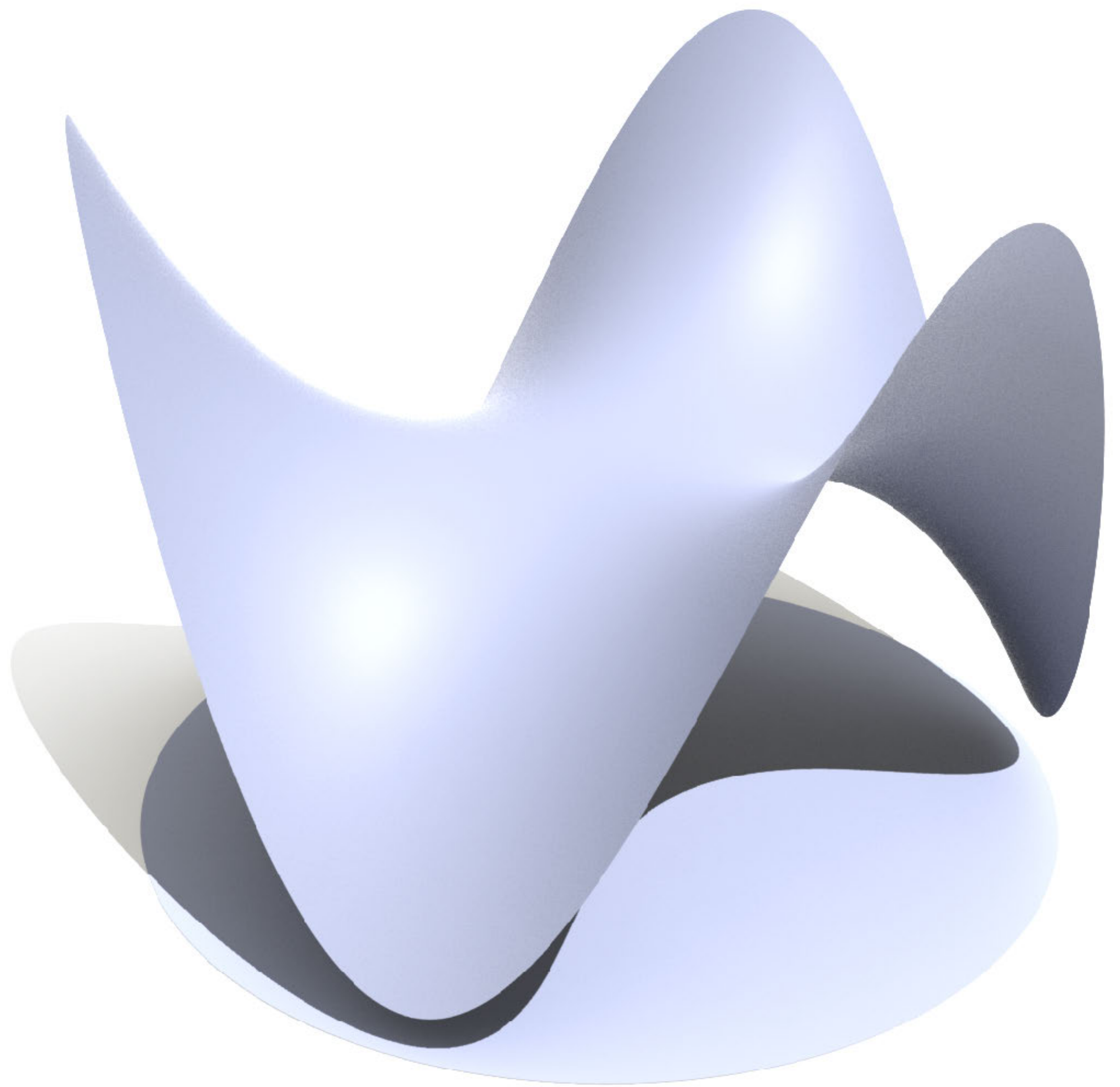
- Any critical point (*e.g.*, local minimum) is called a *harmonic map*.
- Perhaps most common case in geometry processing:
  - M is a surface
  - N is just the real line

Real Harmonic Functions

- Intuitively, a *harmonic* function is the “smoothest” function that interpolates given values on the boundary; looks “saddle-like”
- A function is *harmonic* if applying the Laplacian yields zero
- E.g., in 2D:

$$f(x,y) = x^3 - 3x^2y - 3xy^2 + y^3$$

$$\begin{aligned}\Delta f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \\ &= (6x - 6y) + (-6x + 6y) \\ &= 0\end{aligned}$$



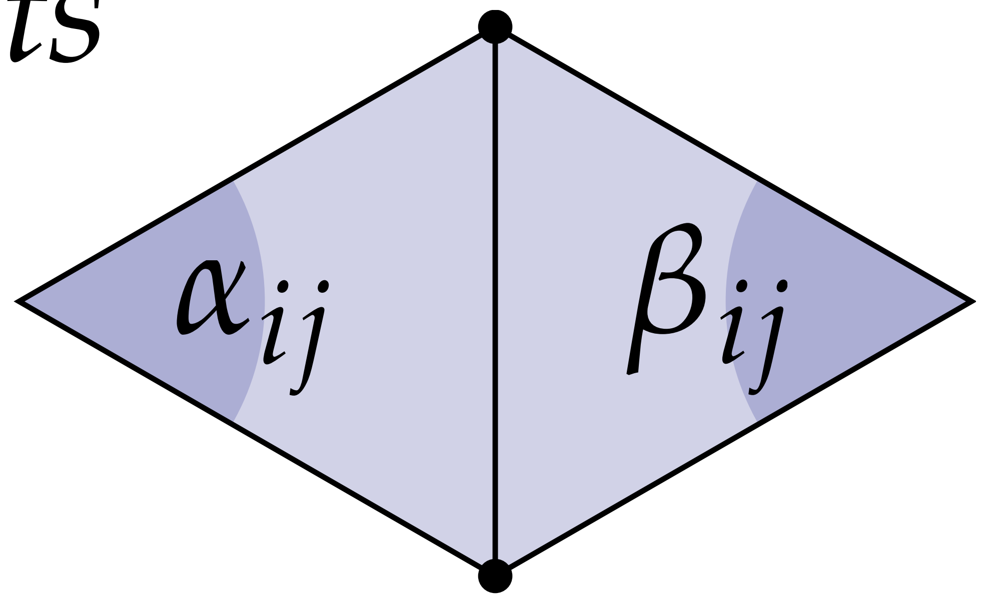
Discrete Harmonic Functions

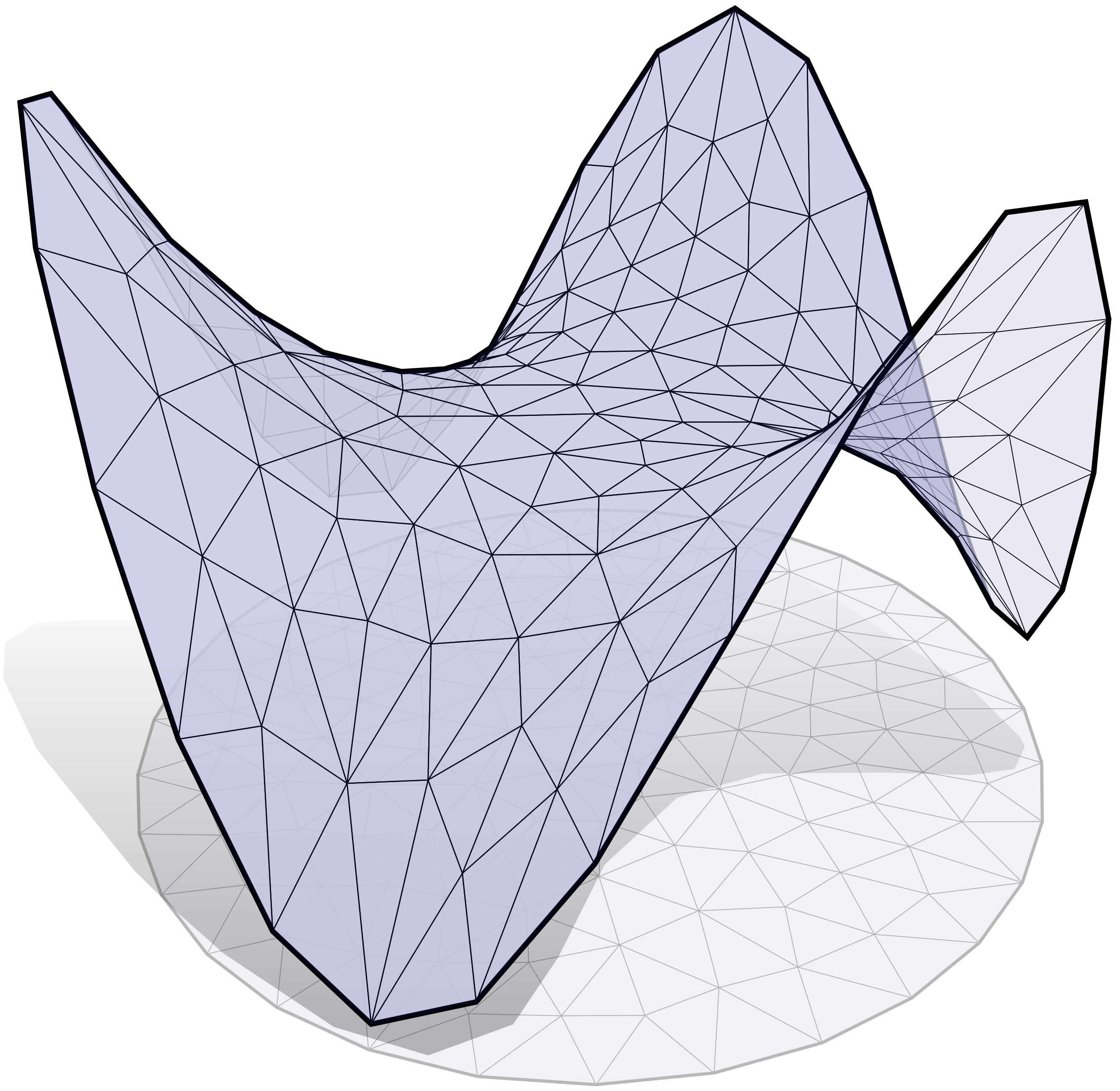
- Harmonic functions are easy to compute on a triangle mesh.
- Roughly speaking: every value is (weighted) average of its neighbors.
- More precisely, at every vertex i we want

$$f_i = \sum_{ij} w_{ij} f_j \Big/ \sum_{ij} w_{ij}$$

- Typical choice for w are *cotan weights*

$$w_{ij} = \tfrac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})$$

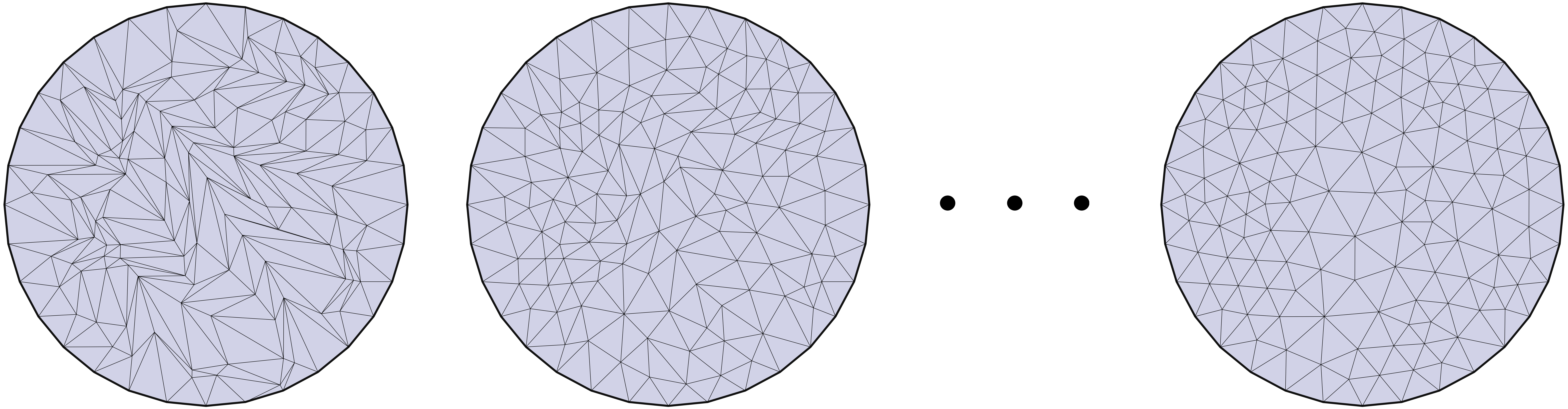




- Boundary values f_i are fixed
- Sparse linear system; many (fast!) ways to solve

Discrete Harmonic Map—Neanderthal method

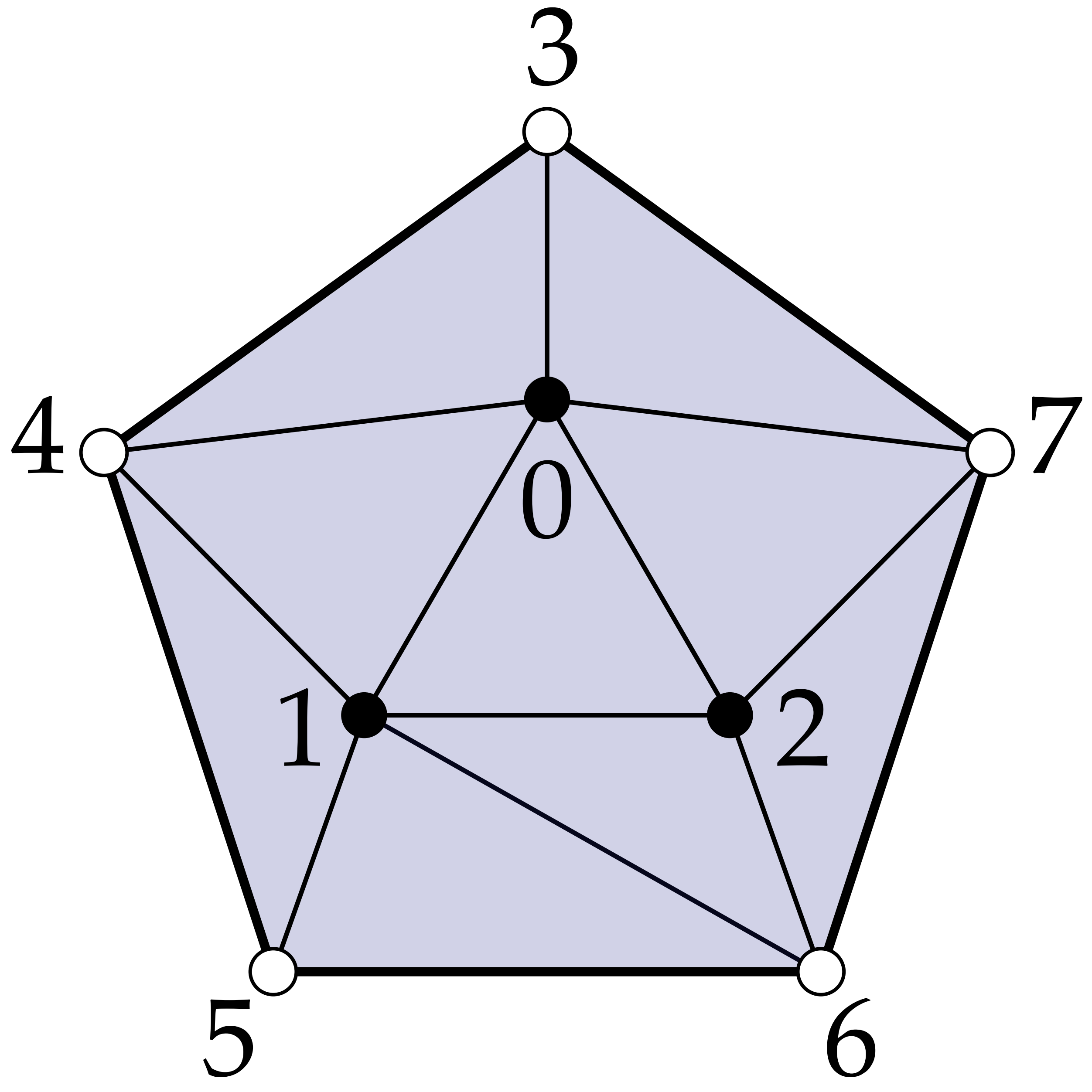
- How can we actually *compute* a harmonic map?
- Simple but stupid idea: repeatedly average with neighbors (*Jacobi)*
- Much better idea: express as linear system and solve with a fast solver.



input **iteration 1** **iteration 736** *(converged)*

Meshes & Matrices

- Common task in geometry processing: solve system of linear equations involving variables on vertices (or edges, or faces, …)
- Basic idea: give each mesh element a unique *index*; build a matrix encoding system of equations.
- E.g., find values u for **black** vertices that are average of neighbors:



$$\begin{aligned} u_0 &= (u_1 + u_2 + u_3 + u_4 + u_7)/5 \\ u_1 &= (u_0 + u_2 + u_4 + u_5 + u_6)/5 \\ u_2 &= (u_0 + u_1 + u_6 + u_7)/4 \end{aligned}$$

$$\Longleftrightarrow \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_3 + f_4 + f_7 \\ f_4 + f_5 + f_6 \\ f_6 + f_7 \end{bmatrix}$$

(Now solve with a *fast* linear solver.)

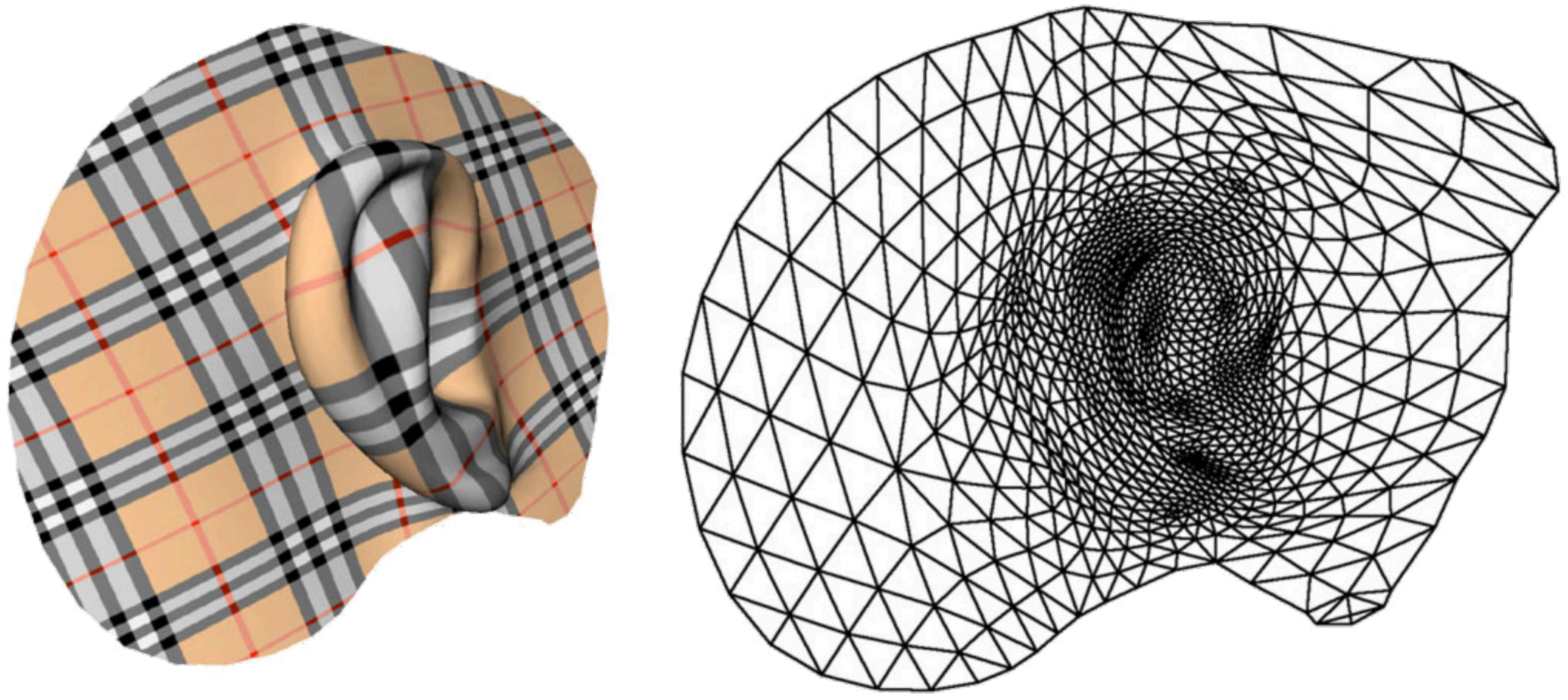
Dirichlet Energy and Harmonic Maps

- **Fact*:** the residual of Cauchy-Riemann equations can be expressed as difference of Dirichlet energy and (signed) target area:

$$||\star df - \imath df||^2 = E_D(f) - \mathcal{A}(f)$$

*For a derivation, see Crane et al, *"Digital Geometry Processing with Discrete Exterior Calculus"*, Section 7.4

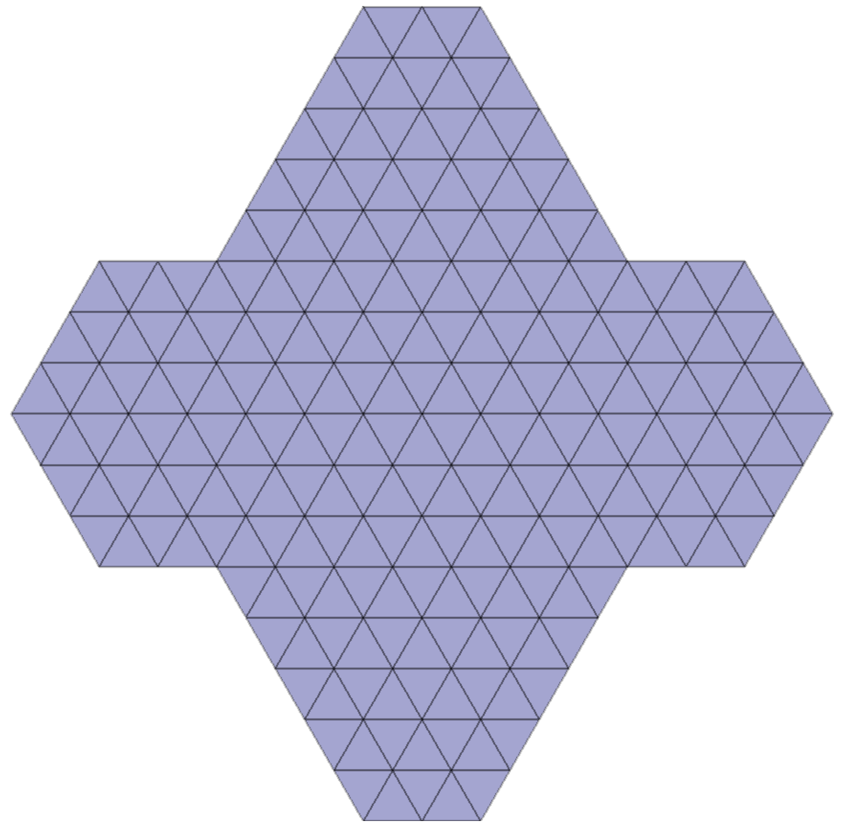
- Minimizing this energy turns out to be numerically equivalent to LSCM



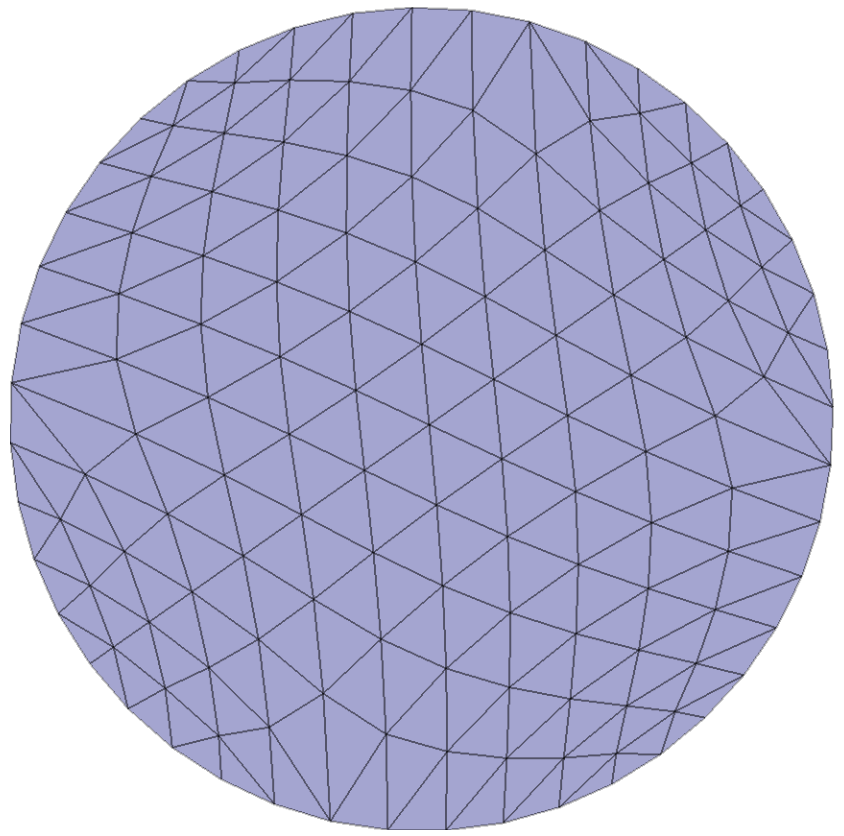
DESBRUN, MEYER, ALLIEZ, *"Intrinsic Parameterizations of Surface Meshes"* (2002)

Harmonic Map with Fixed Area

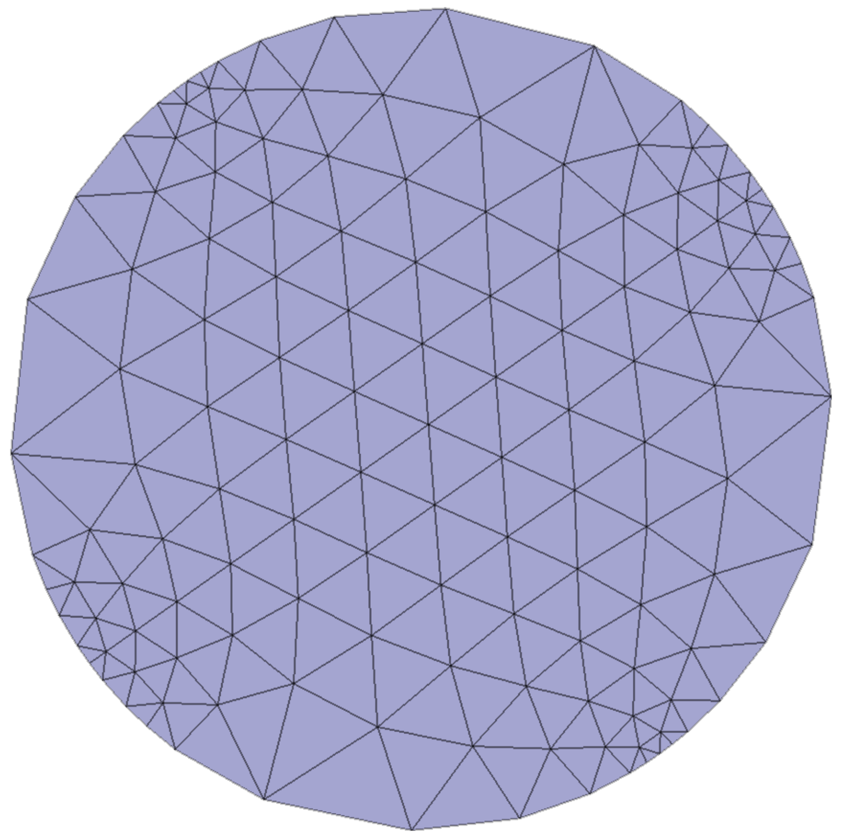
- **Special case:** if target area is fixed, one need only consider E_D
- *E.g.*, world's simplest algorithm for uniformization:
  - Iteratively average with neighbors
  - Project boundary vertices onto circle
- (Initialize by doing the same thing but with boundary fixed to circle)



domain



harmonic



conformal

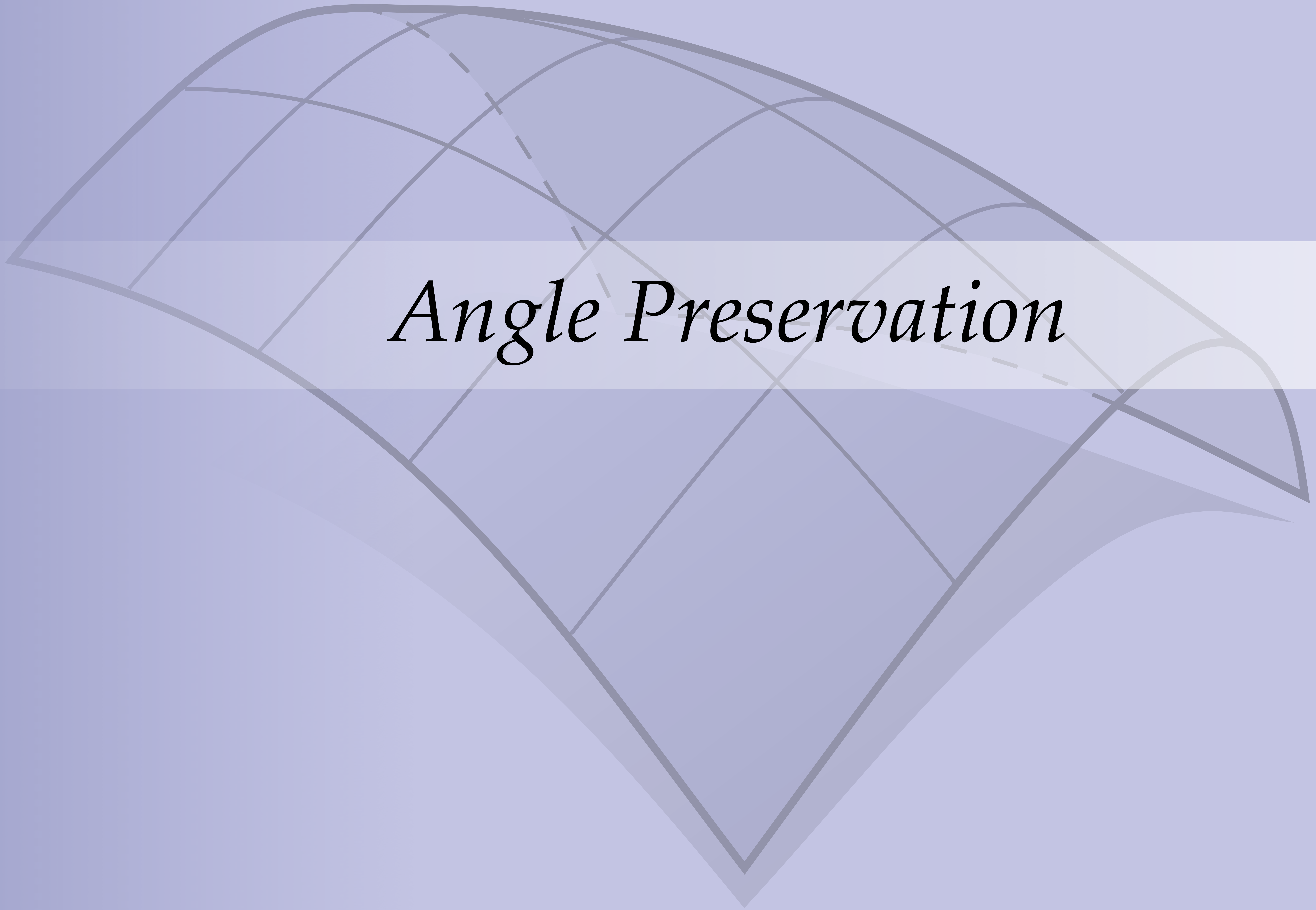
More sophisticated treatment: HUTCHINSON, *"Computing Conformal Maps and Minimal Surfaces"* (1991)

Aside: When is a Harmonic Map Conformal?

- When else can you play this "trick"? (*I.e.*, get a conformal map by just computing a harmonic map)
- Works for the sphere: just keep averaging w/neighbors, projecting
  - *Caveat*: may get stuck in a local minimum that is only holomorphic
  - As before, there are much more intelligent algorithms for the sphere!
- Full characterization given by Eells & Wood (1975):

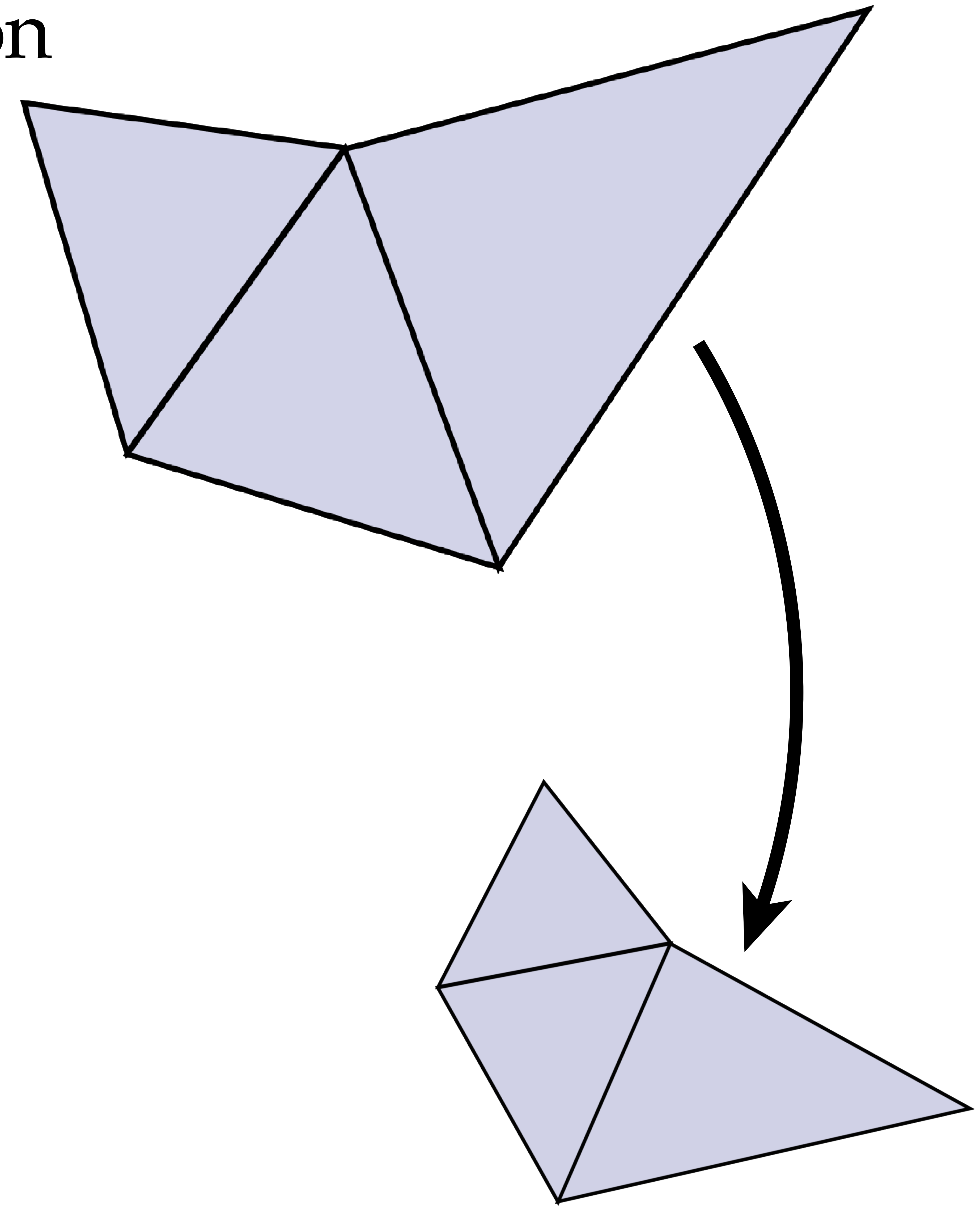
THEOREM. *If $\varphi : X \to Y$ is a harmonic map relative to Riemannian metrics g and h, and if $e(X) + |d_\varphi e(Y)| > 0$, then φ is $\pm$ holomorphic relative to the complex structures determined by g and h.*

Angle Preservation



Angle Preservation

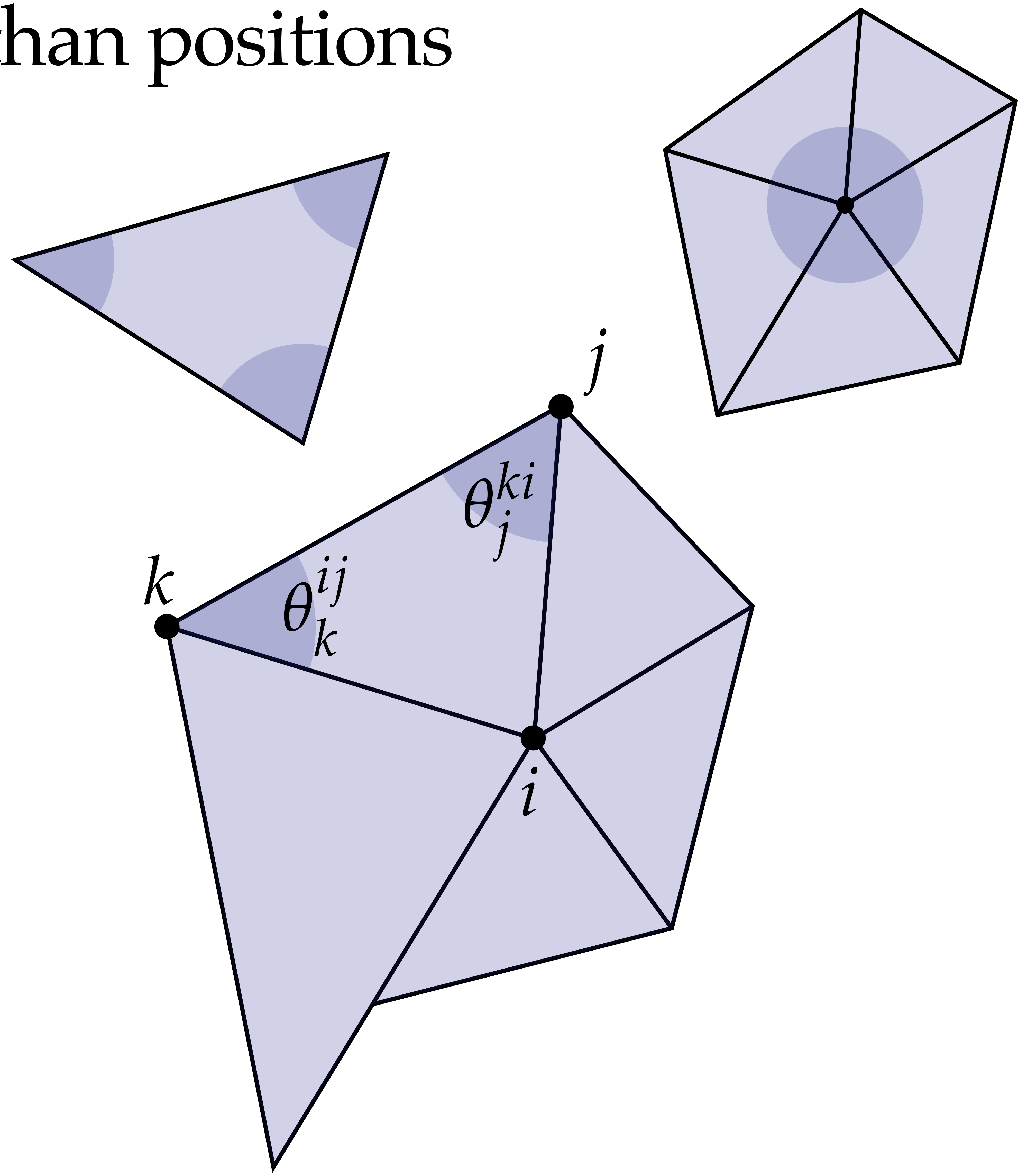
- As discussed earlier, exact angle preservation is *too rigid* (most meshes can't be flattened)
- But, can still continue down this path:
  - Find a collection of angles that describe a flat mesh
  - Approximate original angles "as well as possible"
  - Still provides good approximation of conformal map as we refine (*"discretized"*)



Compatibility of Angles

- Encode flat mesh by *interior angles* rather than positions
- Must satisfy three conditions:
  1. Angles sum to π in each triangle
  2. Sum to 2π around interior vertices
  3. Compatible lengths around vertices:

$$\prod_{ijk \in \mathrm{St}(i)} \frac{\sin \theta_j^{ki}}{\sin \theta_k^{ij}} = 1$$



Note: final condition is *nonlinear!*

Angle-Based Flattening

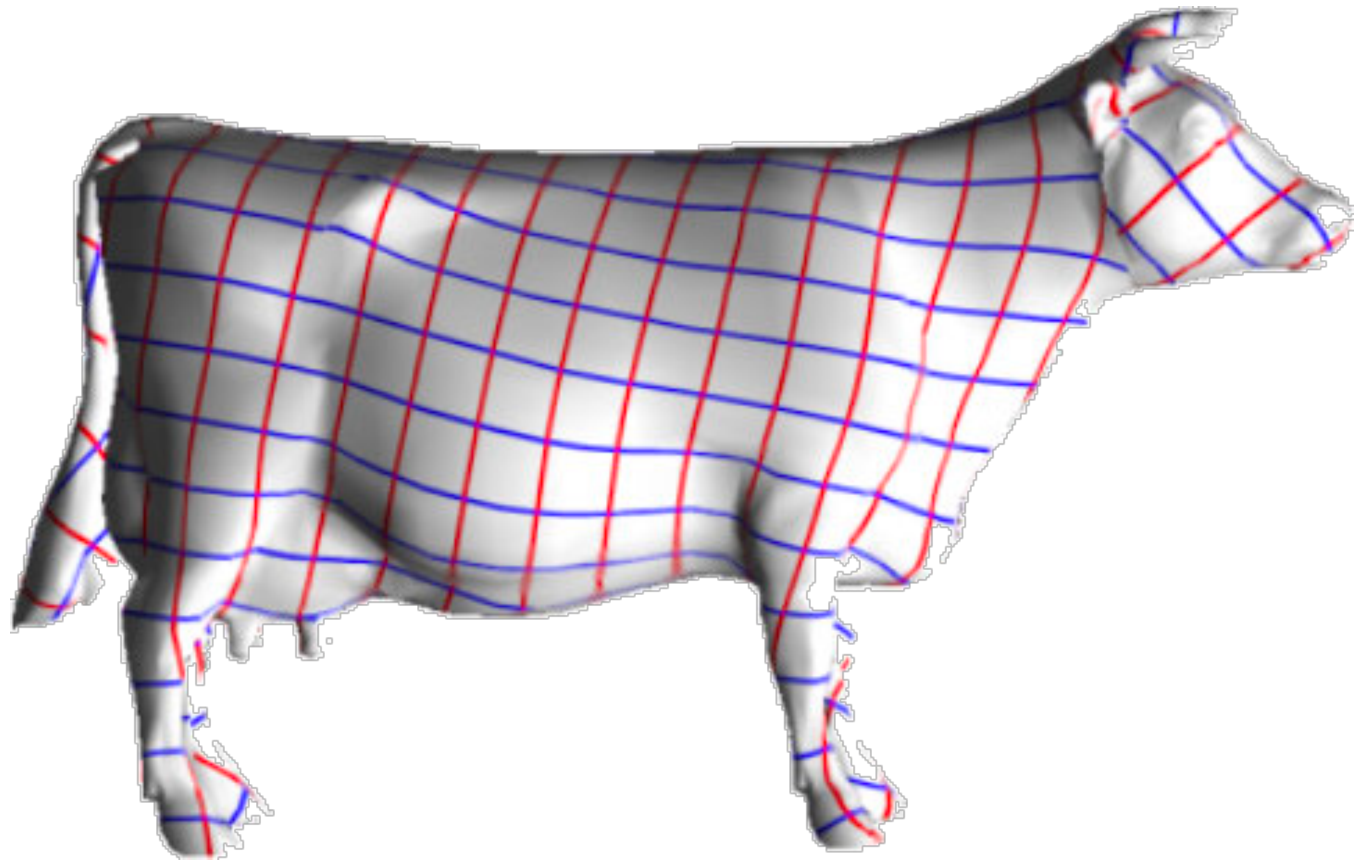
- **Given:** angles θ_0 for original mesh (usually from embedding in 3-space)
- **Find:** closest angles θ that describe a flat mesh
- Compute by solving nonconvex optimization problem:

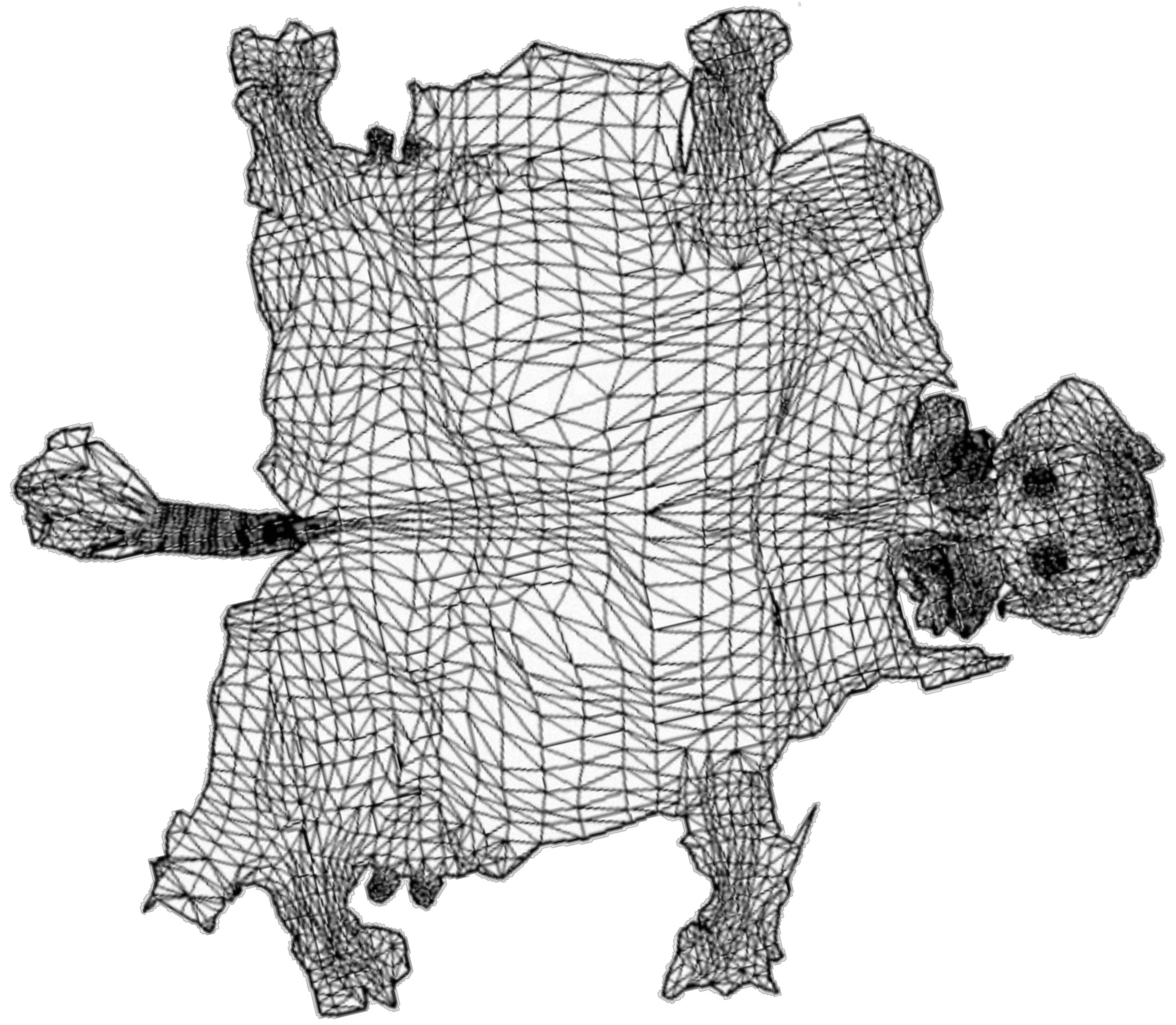
$$
\begin{array}{rll}
\min\limits_{\theta \in \mathbb{R}^{3|F|}} & \sum_i (\tilde{\theta}_i - \theta_i)^2 & \\
\text{s.t.} & \theta_i^{jk} + \theta_j^{ki} + \theta_k^{ij} = \pi, & \forall ijk \in F \\
& \sum_{ijk \in \mathrm{St}(i)} \theta_i^{jk} = \pi, & \forall i \in V \\
& \prod\limits_{ijk \in \mathrm{St}(i)} \sin\theta_j^{ki} / \sin\theta_k^{ij} = 1, & \forall i \in V
\end{array}
$$

SHEFFER, DE STURLER, *"Parameterization of Faceted Surfaces for Meshing using Angle Based Flattening"* (2001)

Linear Angle Based Flattening

- Original ABF problem is large, difficult to solve
- Approximate by a linear problem:
  - solve for *change* in angles that makes mesh flat
  - linearize nonlinear condition via log, Taylor series
- Results are nearly indistinguishable from original ABF

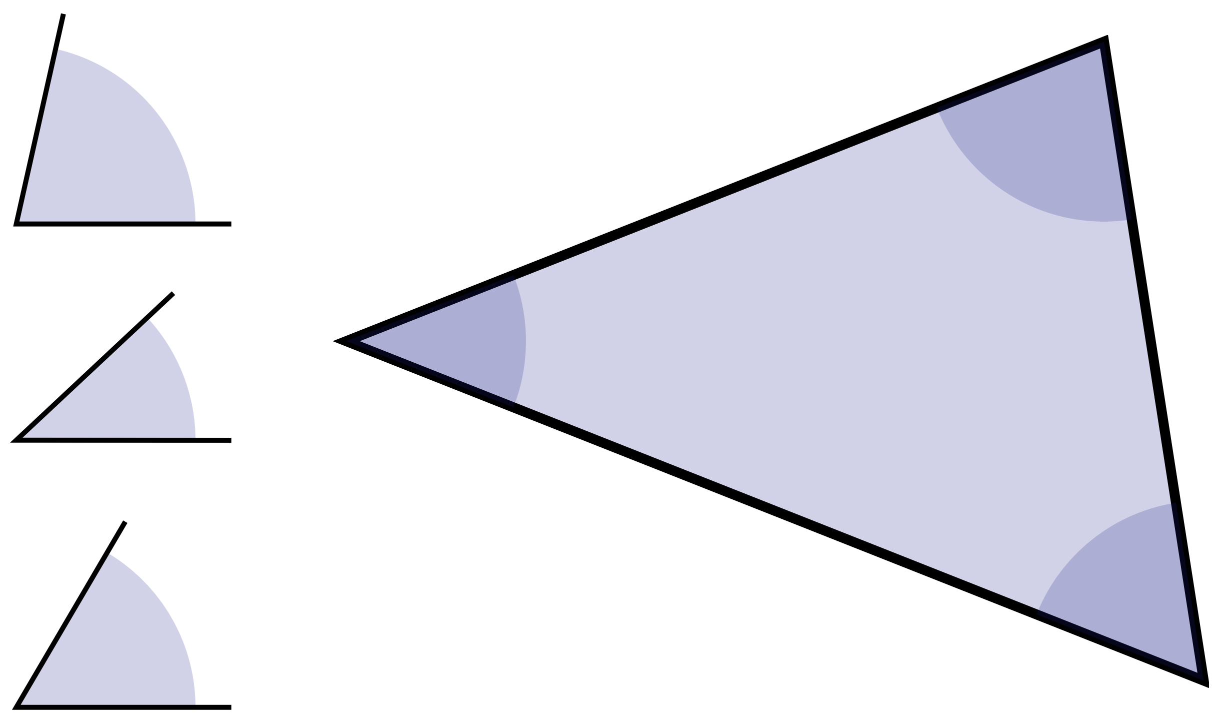


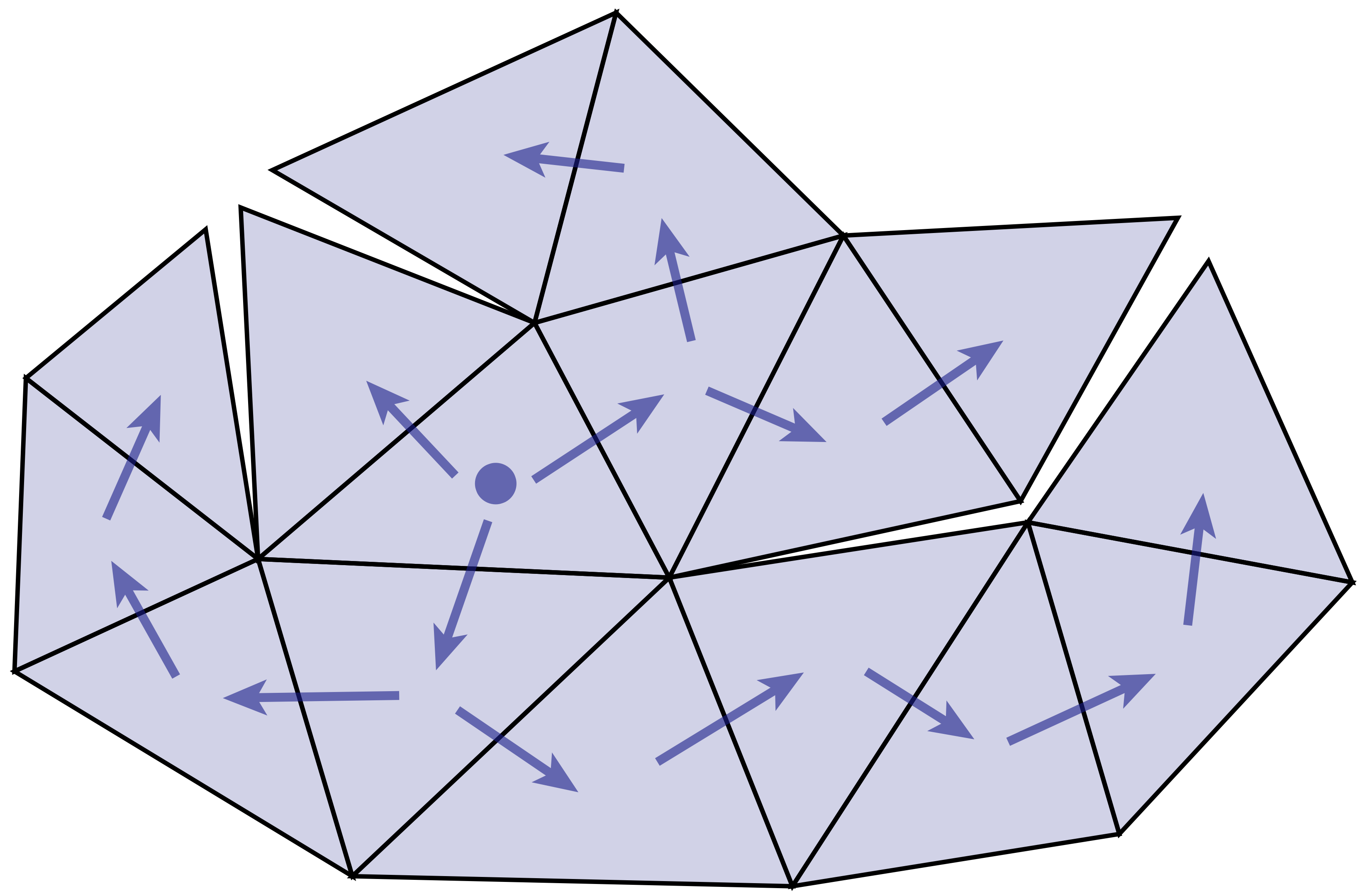


ZAYER, LÉVY, SEIDEL, *"Linear Angle Based Parameterization"* (2007)

Angle Layout Problem (Local Strategy)

- **Given:** Angles that describe a flat triangulation
- **Find:** Vertex positions that exhibit these angles
- *Local strategy*: start at any triangle and "grow out"
  - first triangle determined up to scale by three angles
  - Problem: accumulation of numerical error can cause cracks

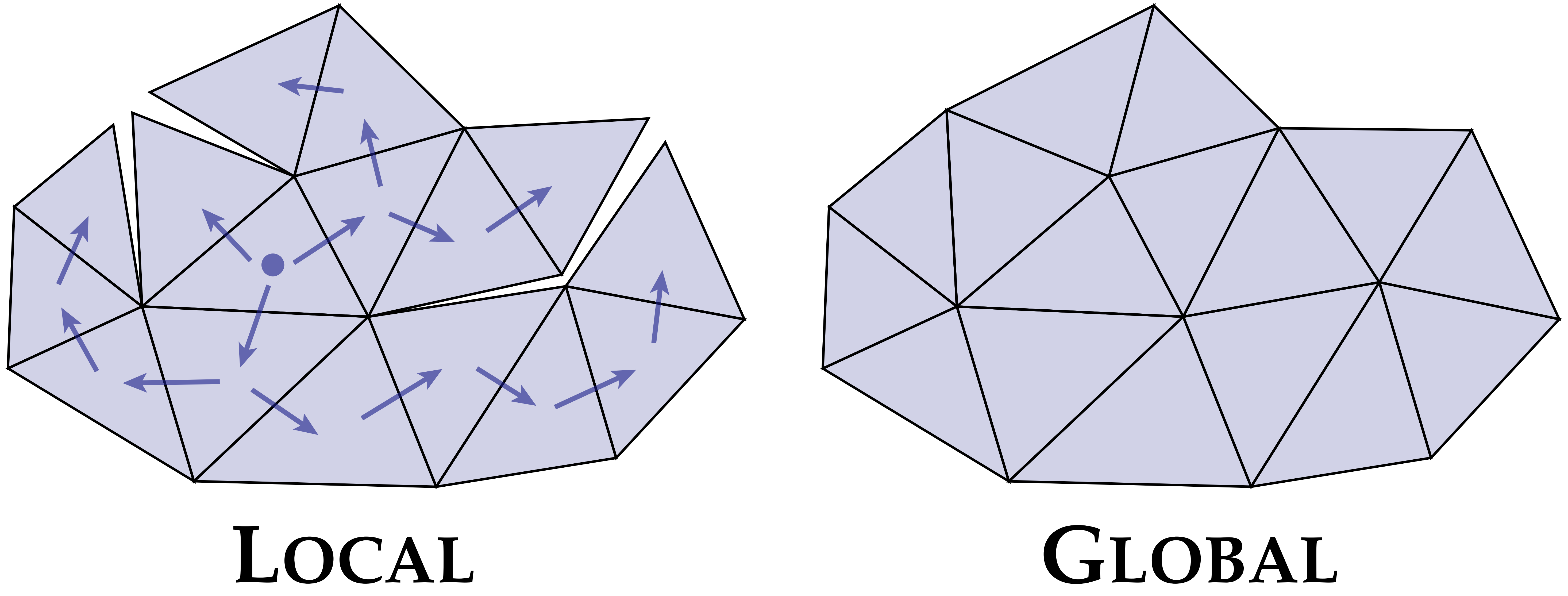




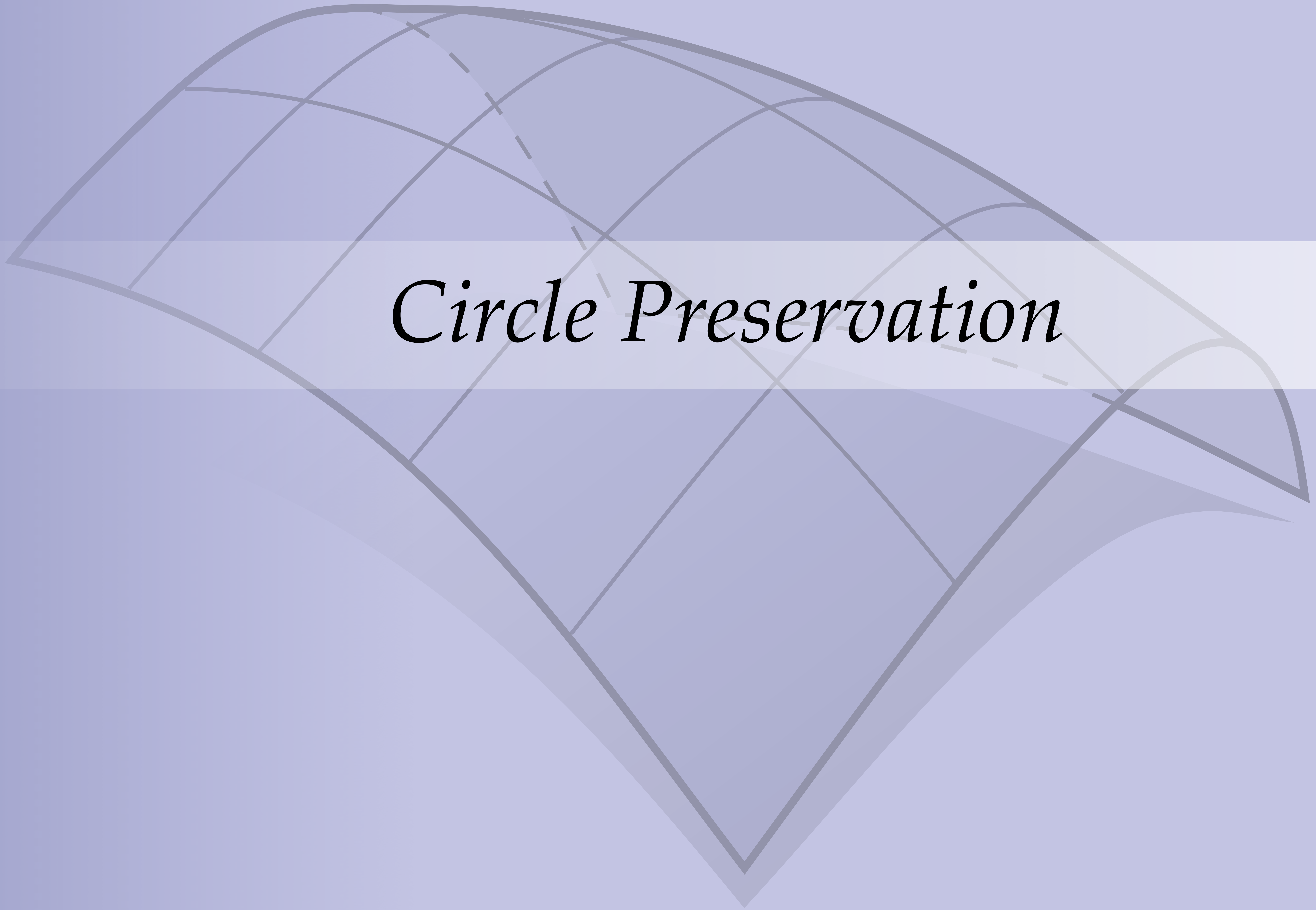
Angle Layout Problem (Global Strategy)

- *Global strategy*: solve large linear system for vertex positions that best match the given angles (see ABF++)
- **Observation:** linear system is equivalent to computing edge lengths from angles, running LSCM on new edge lengths.
  - *Interpretation*: ABF++ intrinsically "deforms" metric to something nearly flat; still needs LSCM to get final (extrinsic) map to the plane

(Will see this strategy again later…)

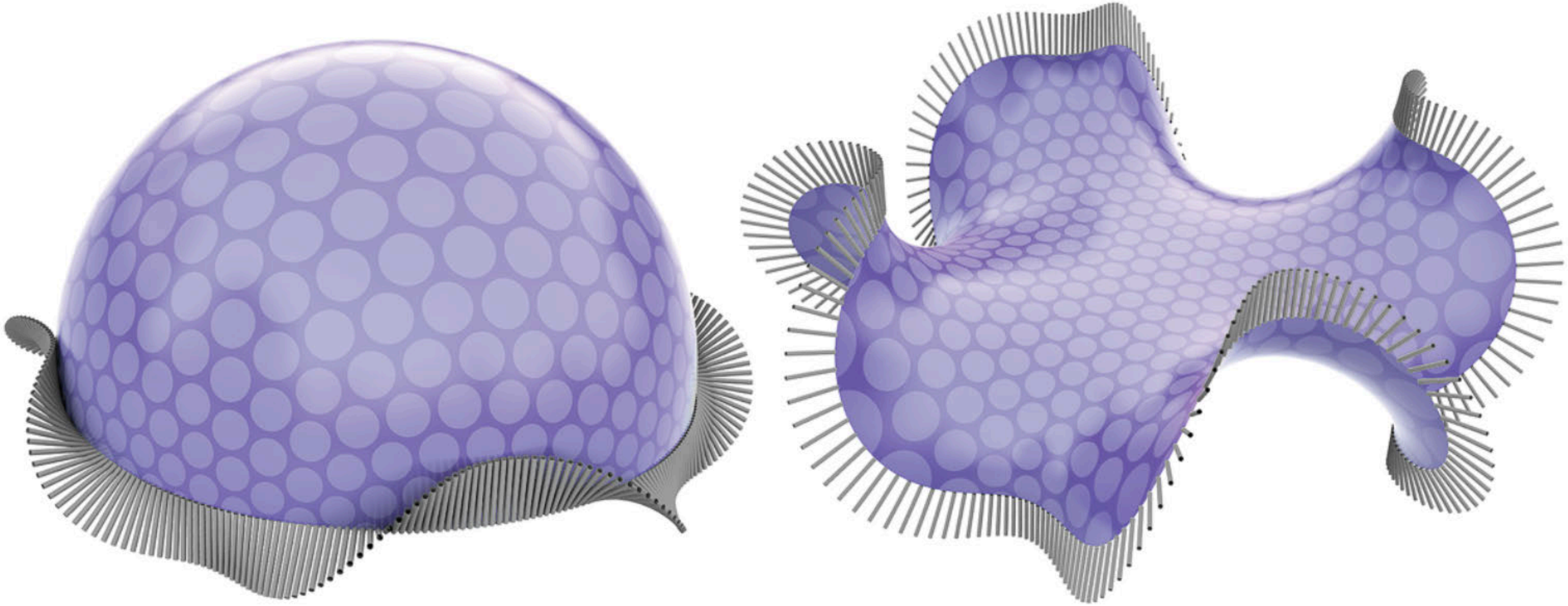


Circle Preservation



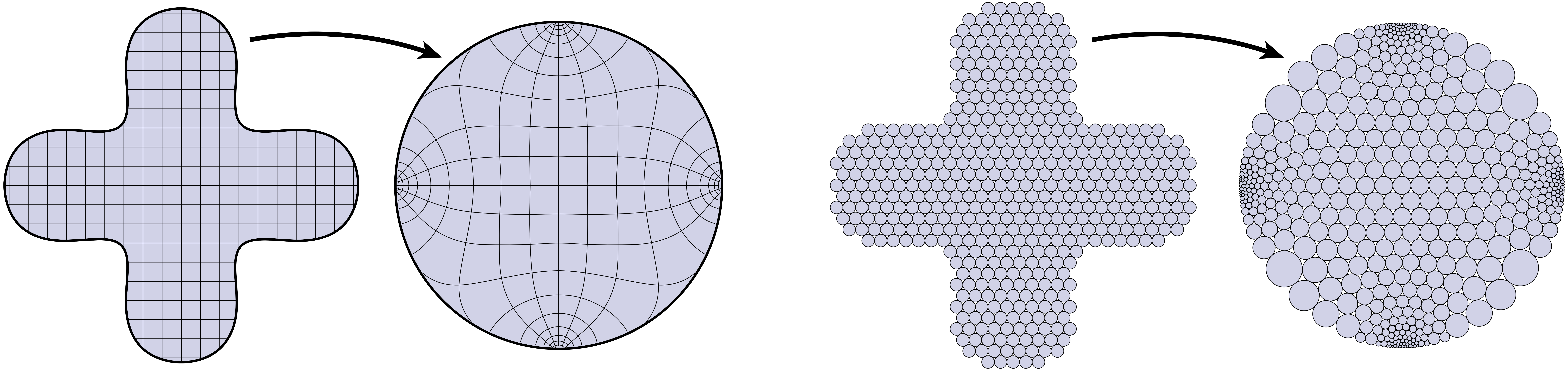
Circle Preservation

- **Smooth**: conformal maps preserve infinitesimal circles (why?)
- **Discrete**: try to preserve circles associated with mesh elements



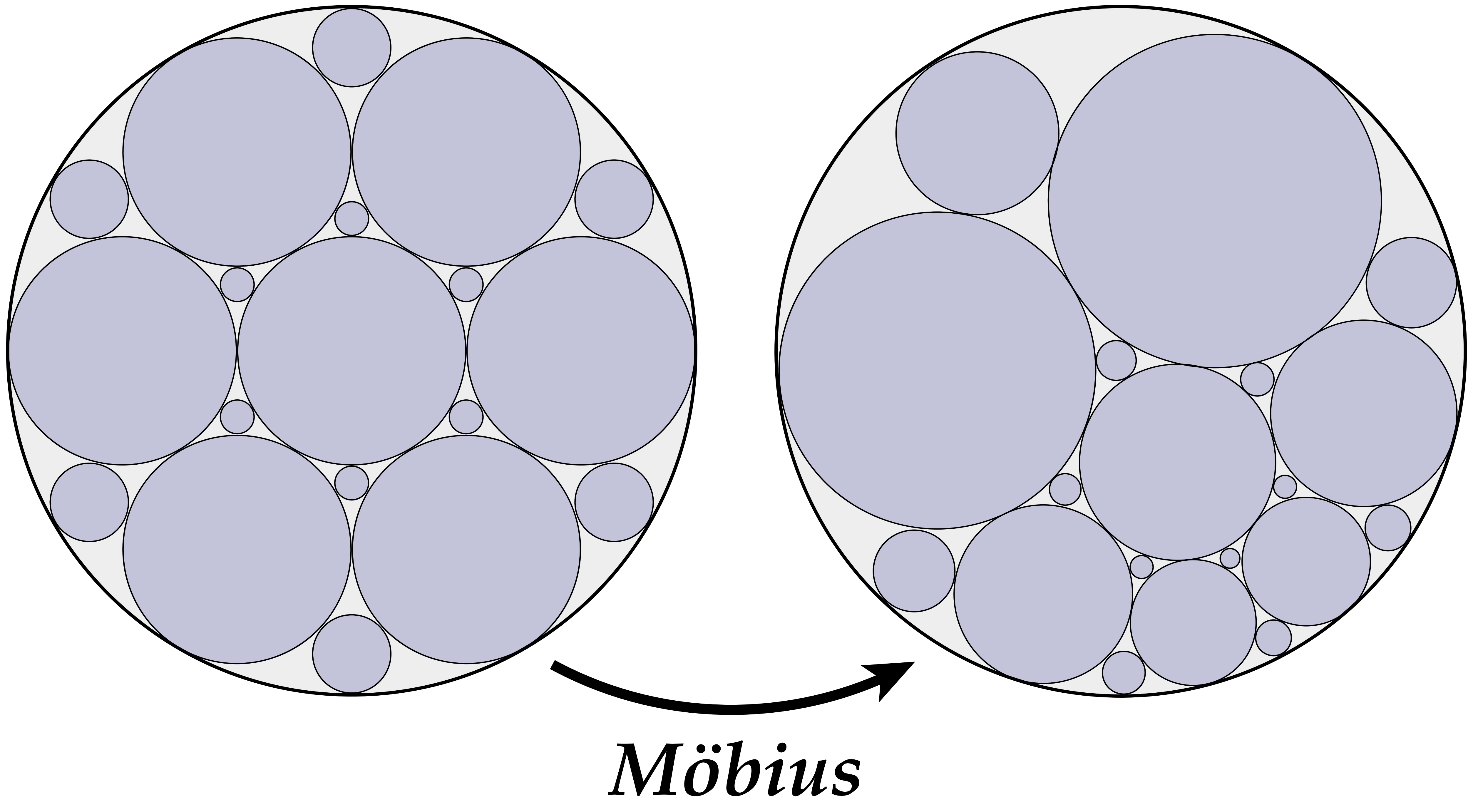
Circle Packing

- *Koebe*: every planar graph can be realized as collection of circles
  - one circle per vertex; two circles are tangent if they share an edge
- *Thurston*: cover planar region by regular tiling of circles; now make boundary circles tangent to unit circle. This "circle packing" approximates a smooth conformal map (Rodin-Sullivan).



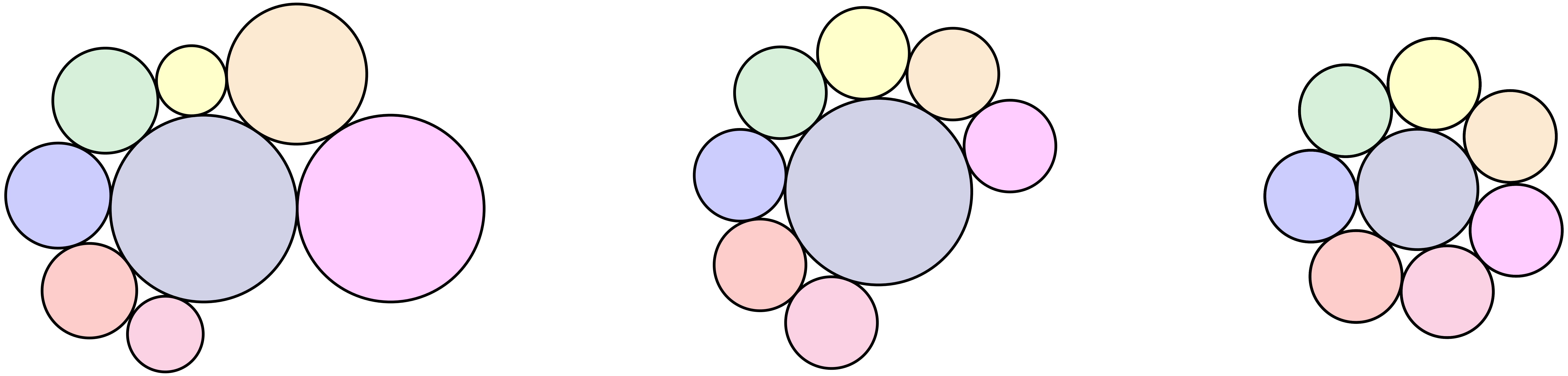
Circle Packing—Structure Preservation

- Theories based on circles naturally preserve certain properties of smooth conformal maps
- *E.g.*, since Möbius transformations take circles to circles, circle packing preserves dimension of solutions to Riemann mapping

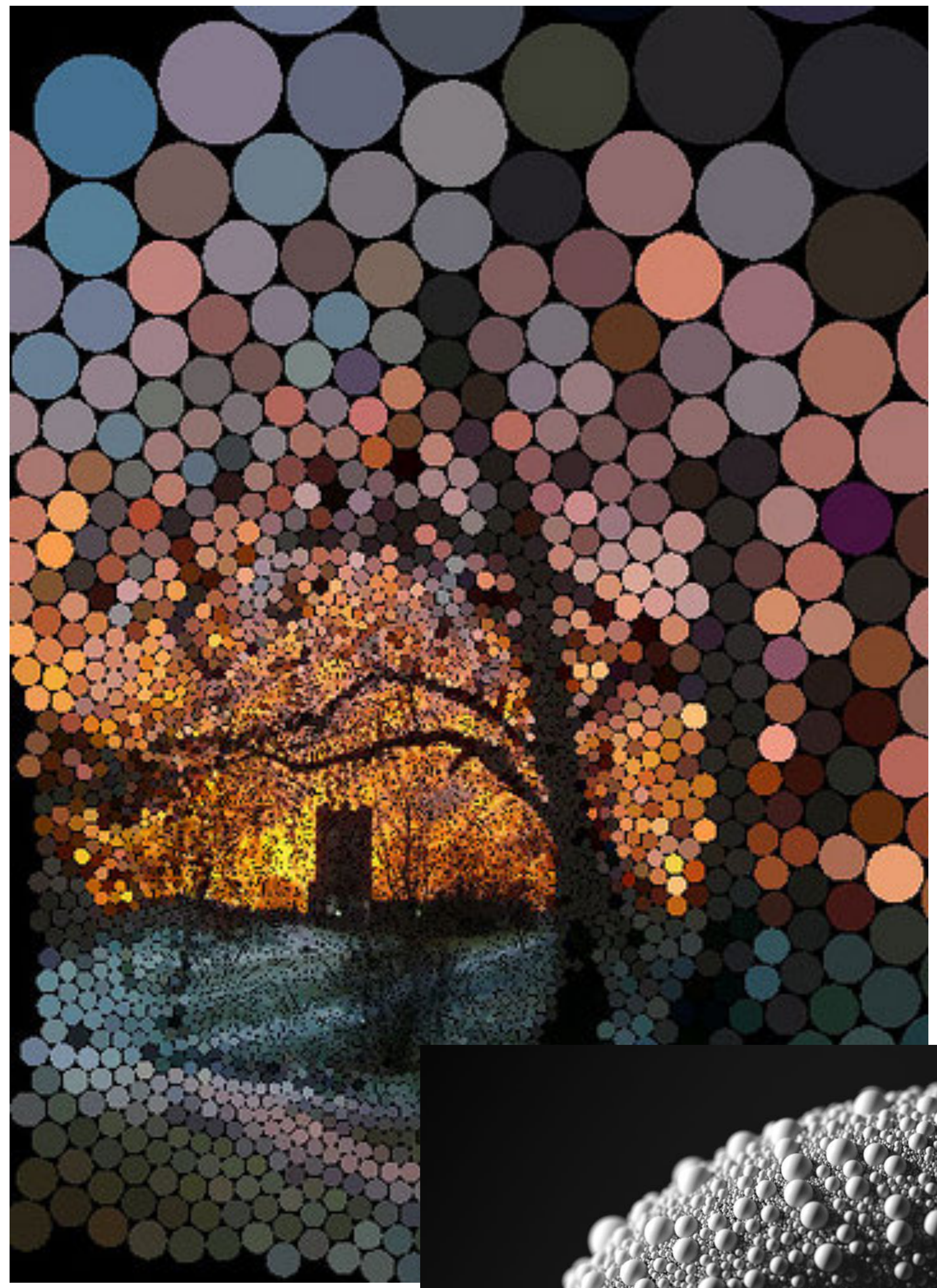


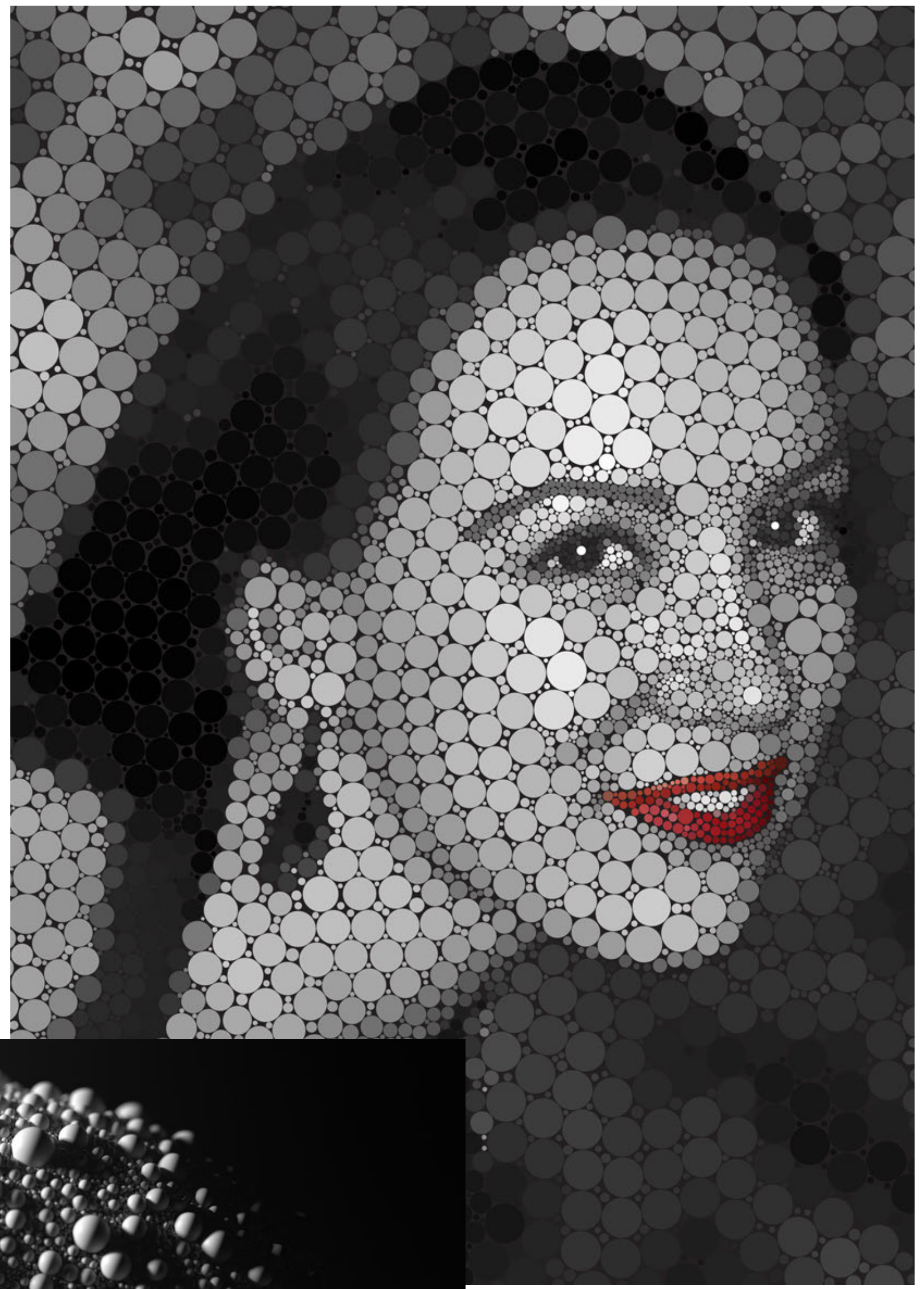
Circle Packing — Algorithm

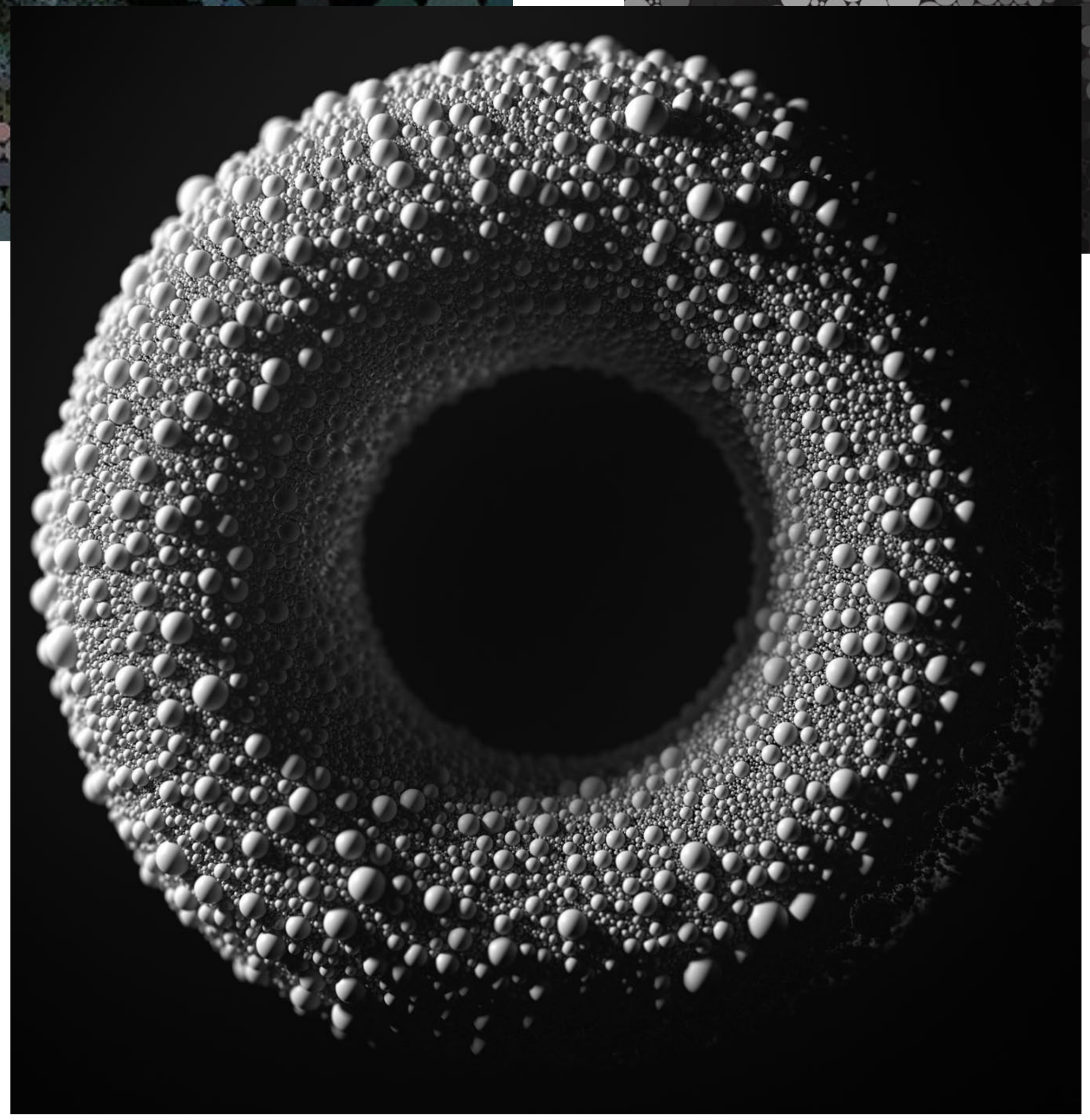
- Nonlinear problem, but simple iterative algorithm
- For each vertex i:
  - Let θ be total angle currently covered by k neighbors
  - Let r be radius such that k neighbors of radius r also cover θ
  - Set new radius of i such that k neighbors of radius r cover 2π
- *Repeat!*

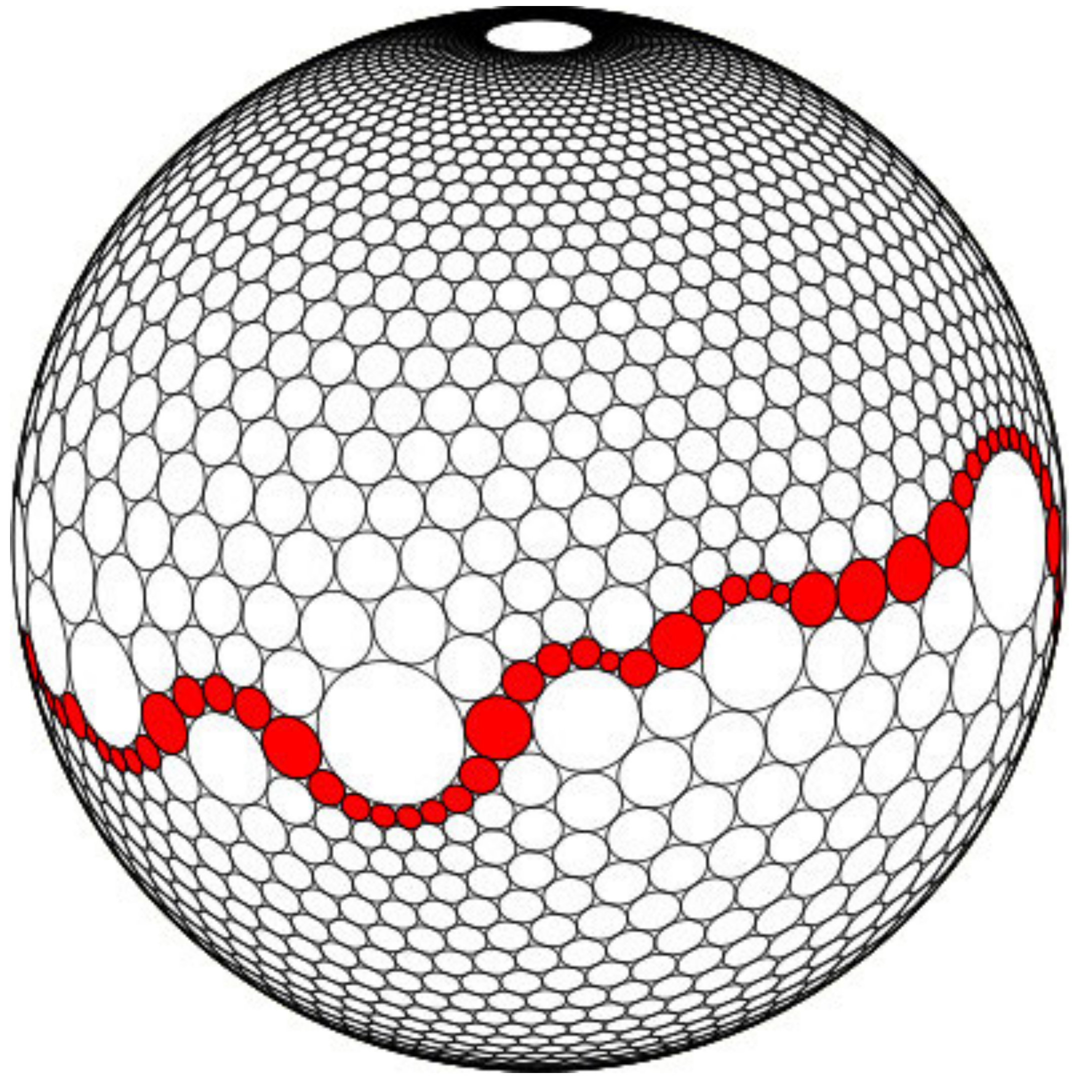


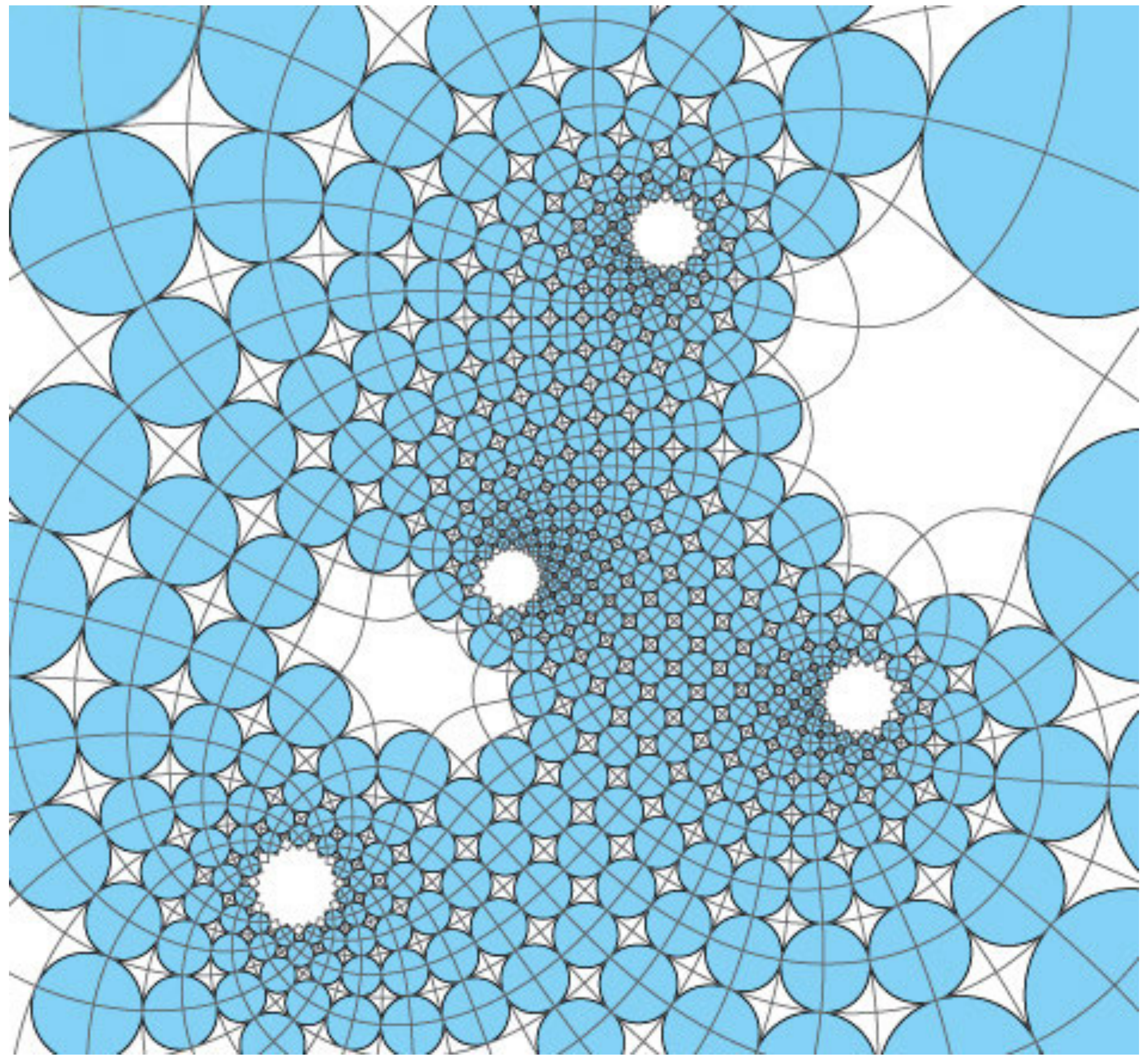
Circle Packing—Gallery

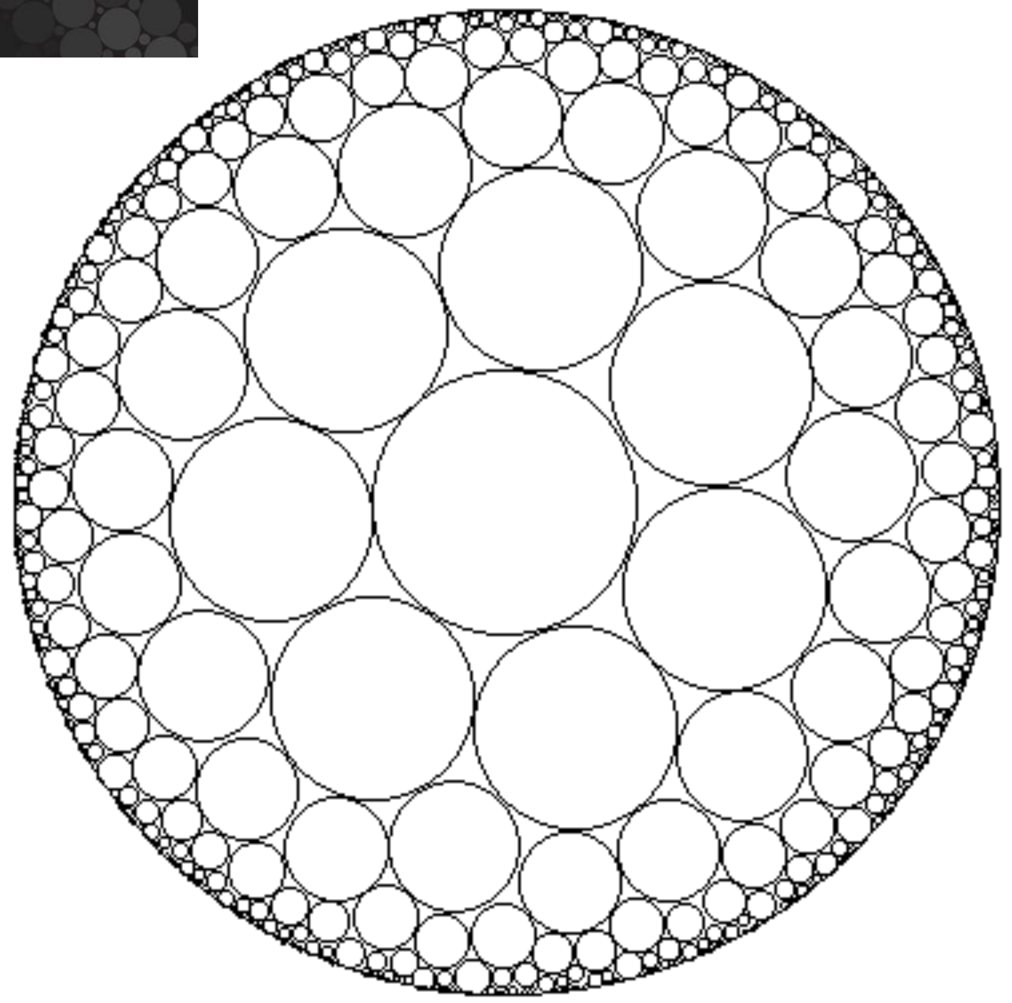


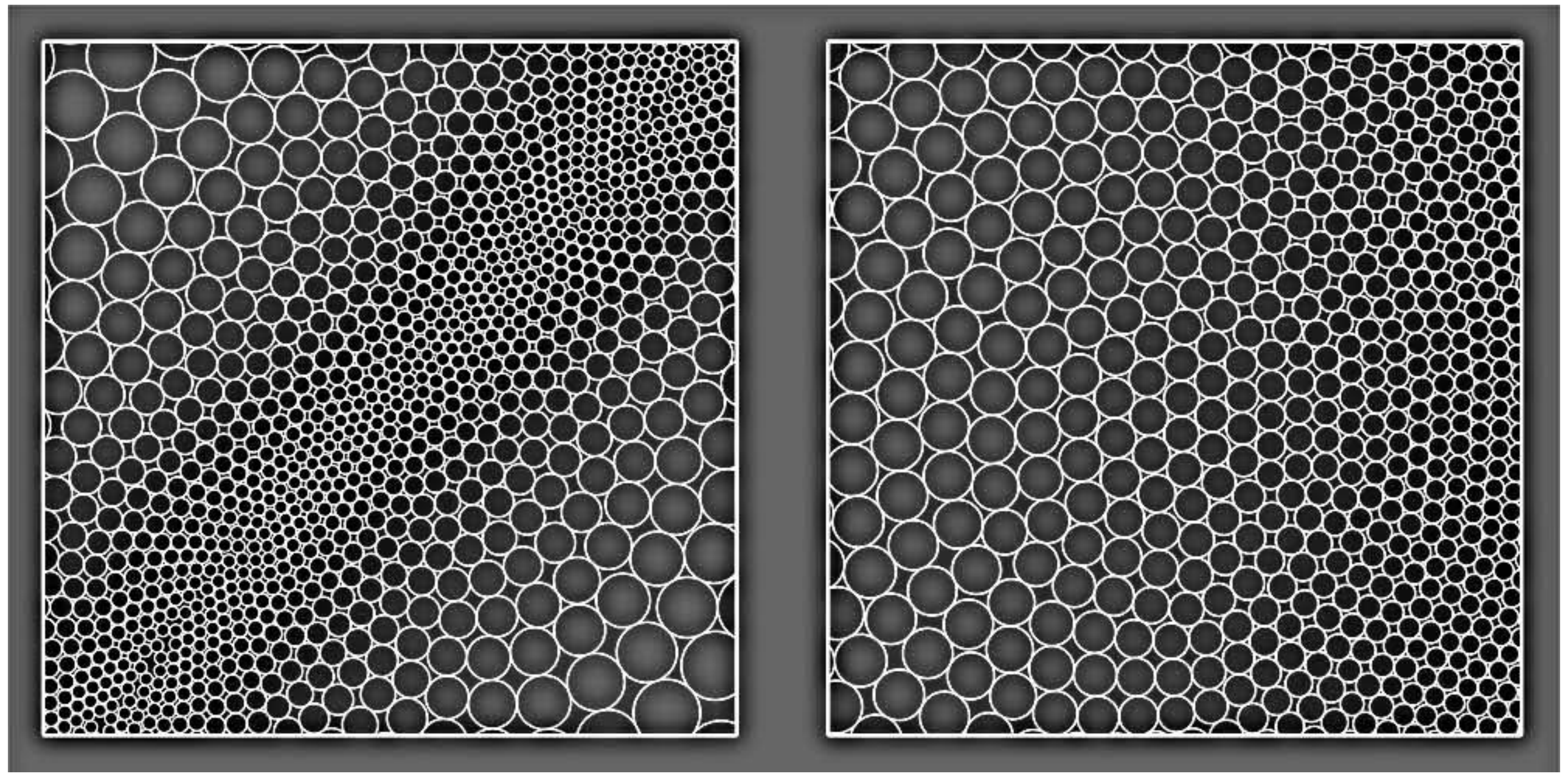






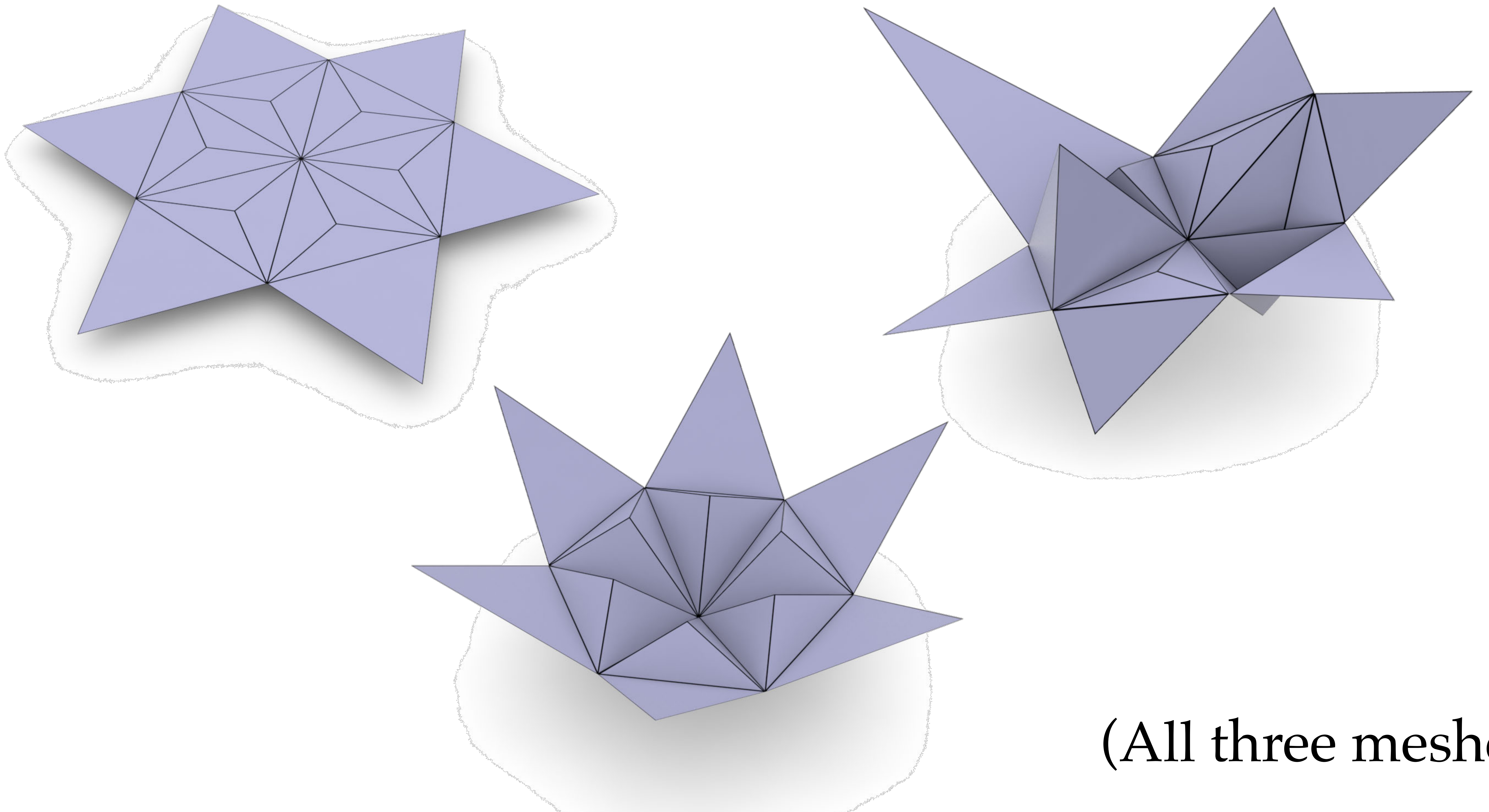


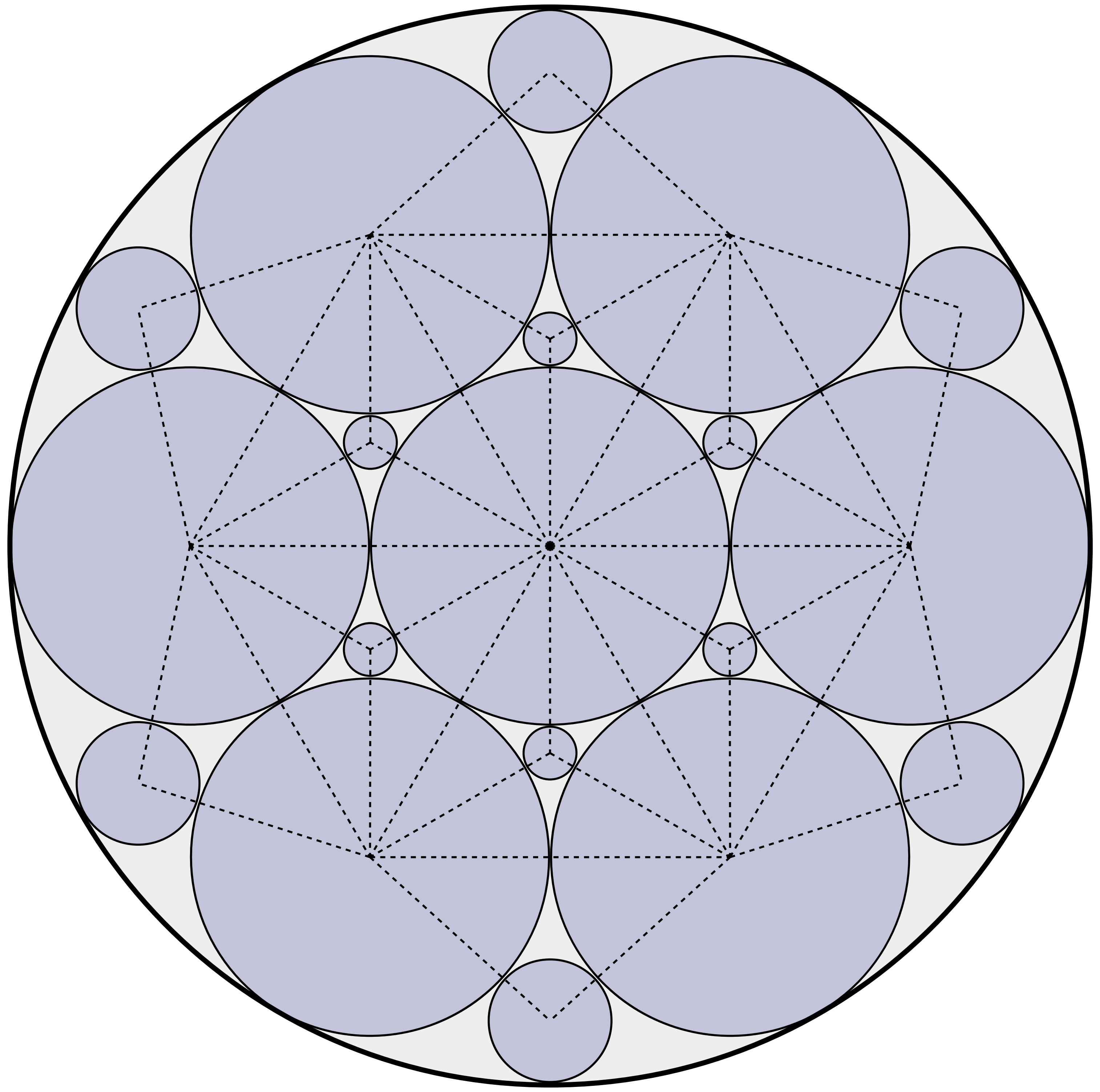




Circle Packings Ignore Geometry

- Circle packing is purely combinatorial (neighboring circles are tangent)
- For *geometry* processing, need definition that incorporates *geometry!*

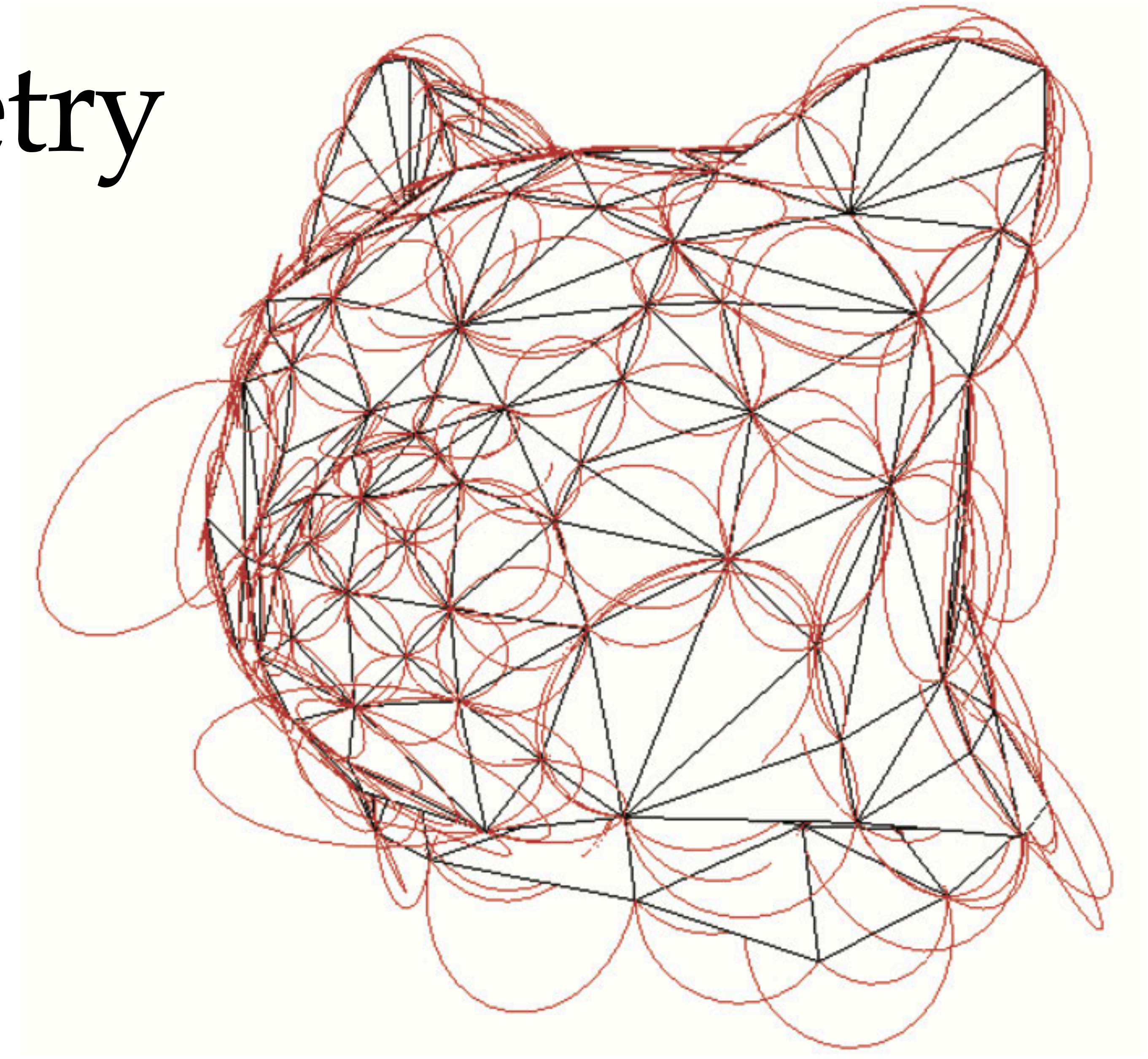


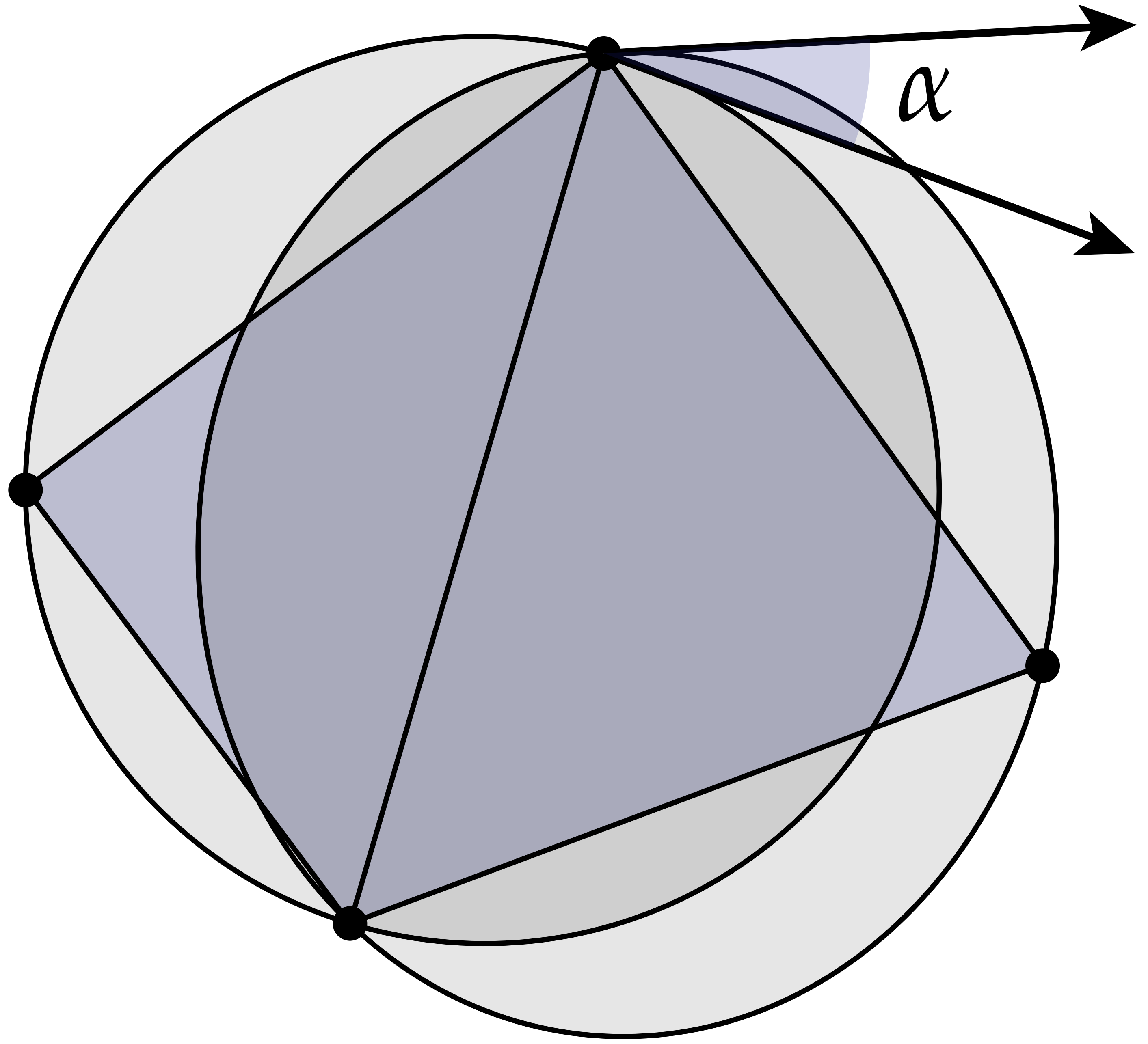


(All three meshes yield same circle packing.)

Circle Patterns

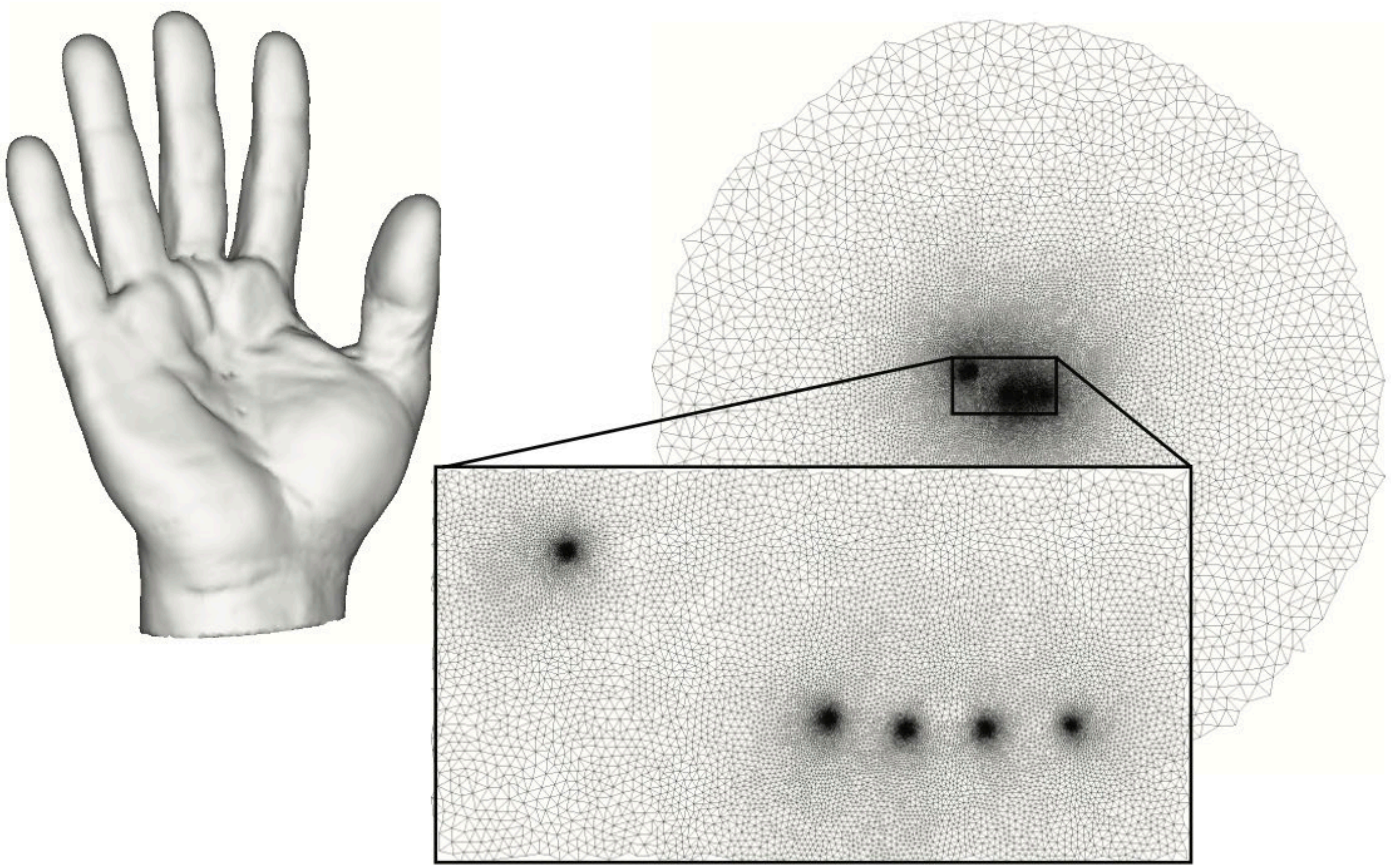
- Different idea: *circle patterns*
  - associate each face with its circumcircle (circle through three vertices)
  - consider "conformal" if circle intersection angles are preserved
- Nicely incorporates geometry
- Convex optimization
- *Still rigid!* (not obvious)

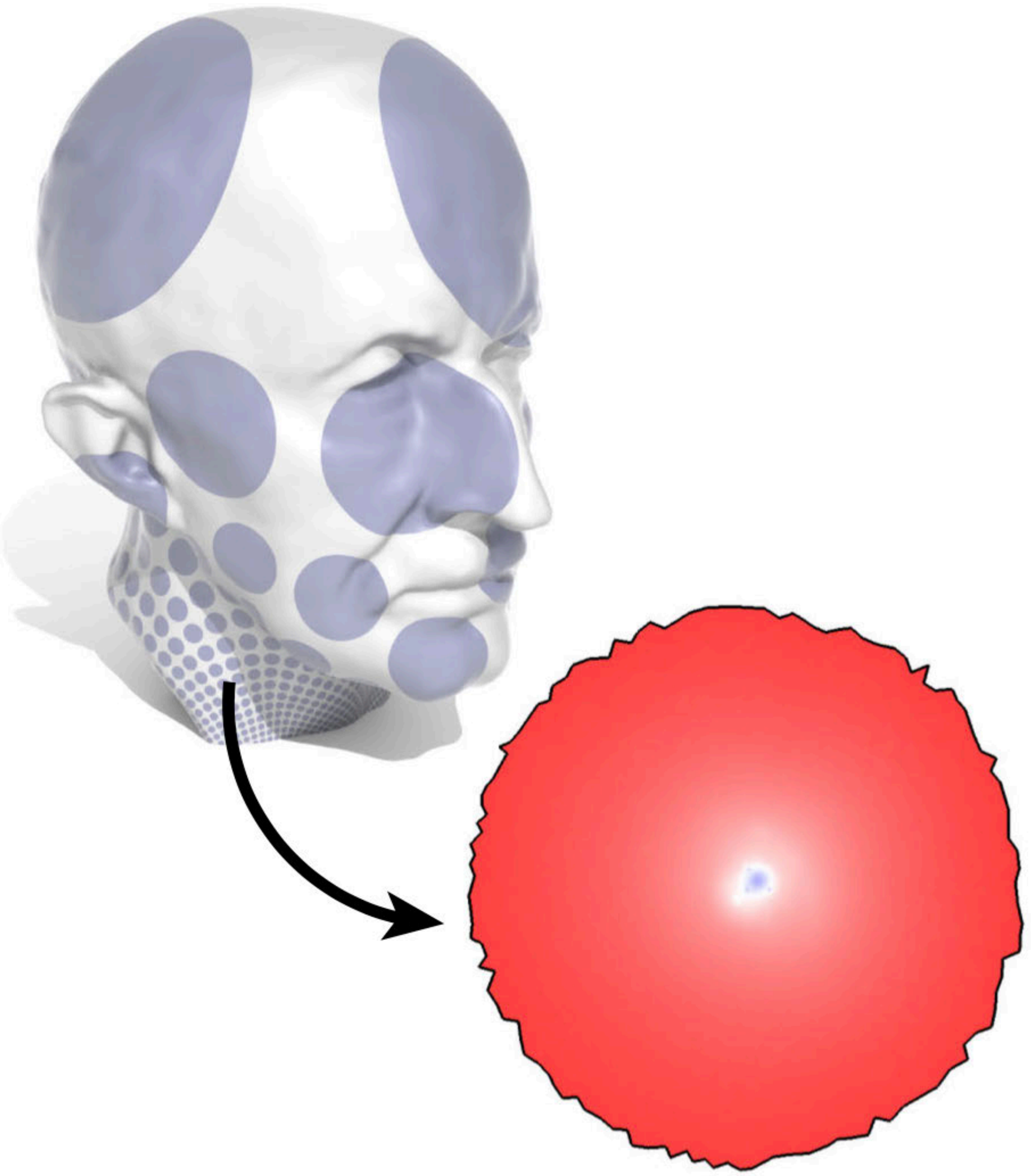




Cone Singularities—Motivation

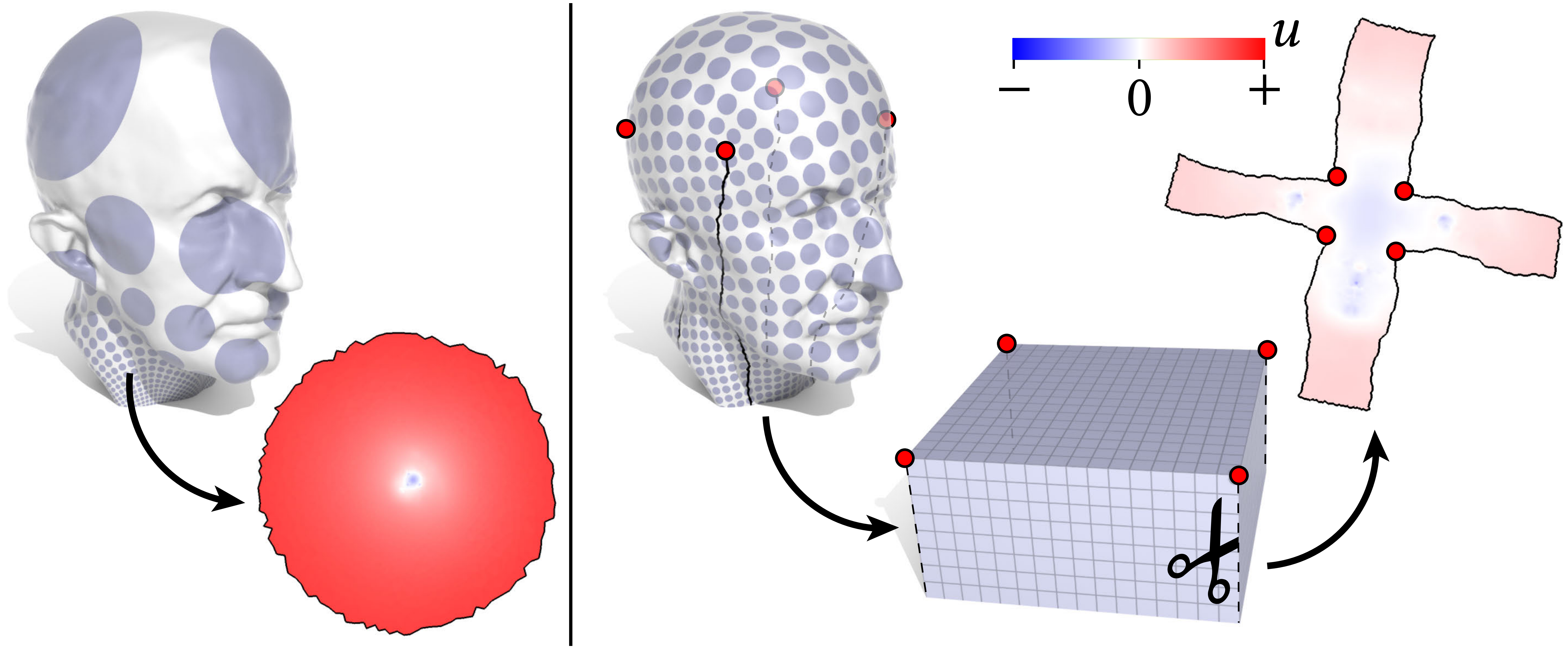
- Even in the best case, conformal flattening can exhibit significant area distortion:





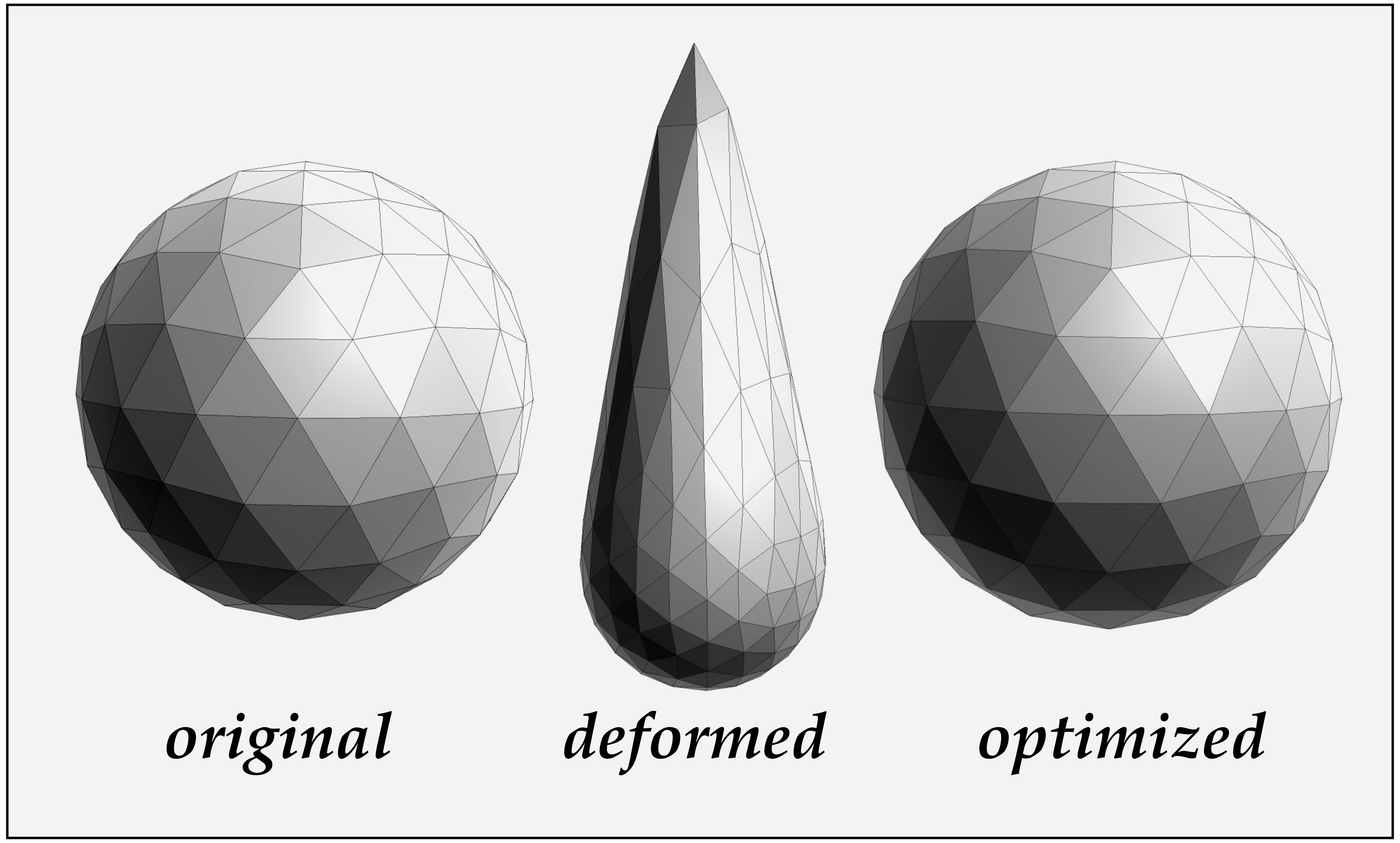
Cone Singularities

- **Idea:** *(Kharevych-Springborn-Schröder)*
  - first map to a surface that is flat except at a few “cone points”
  - then cut through cone points so that surface is flat everywhere
  - can now lay out in the plane with no additional stretching
- **Result:** lower overall area distortion (concentrated at cones)

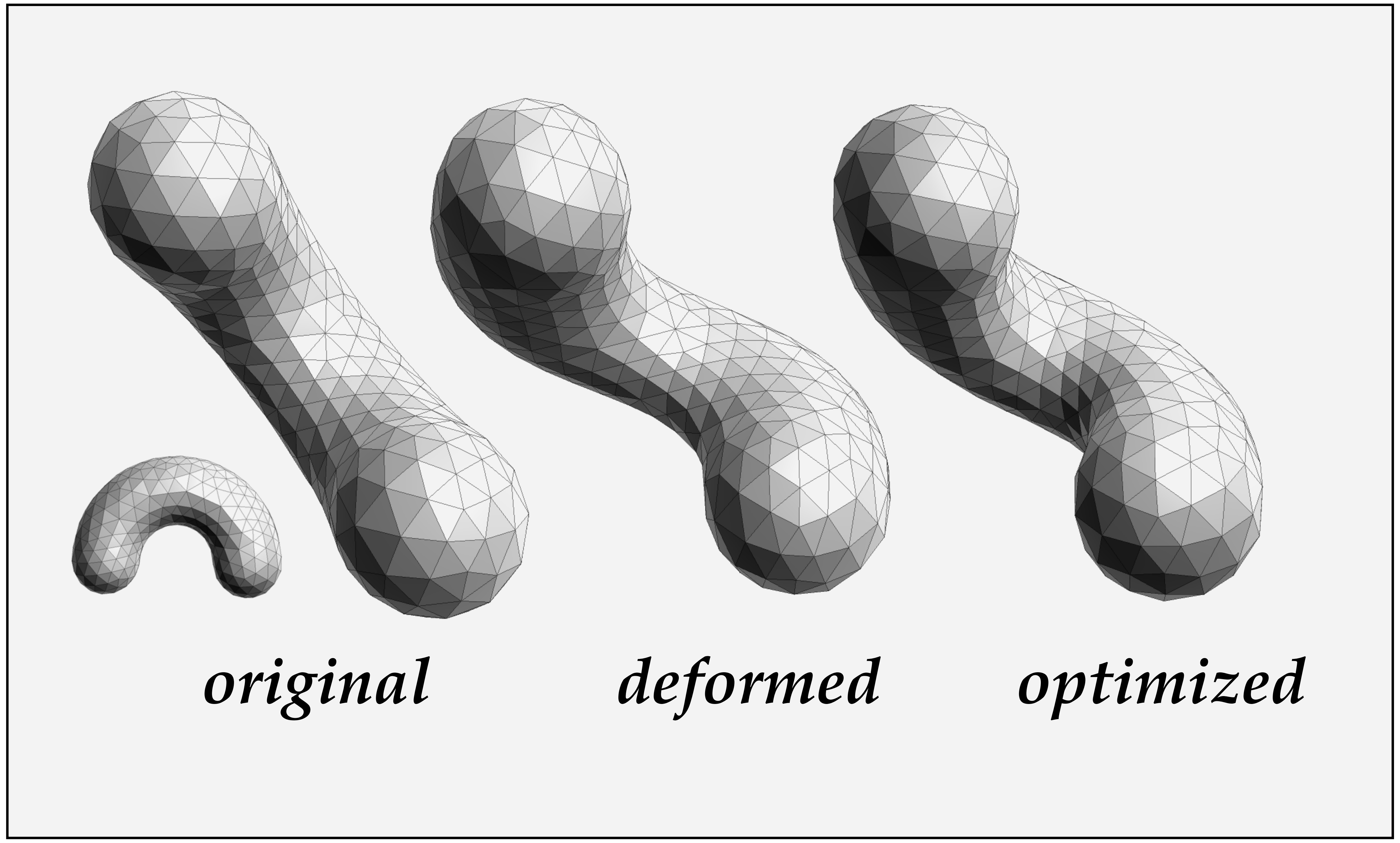


Rigidity of Circle Patterns

Experiment: deform mesh, then find (numerically) nearby mesh with same circle intersection angles as original mesh.



(CONVEX)

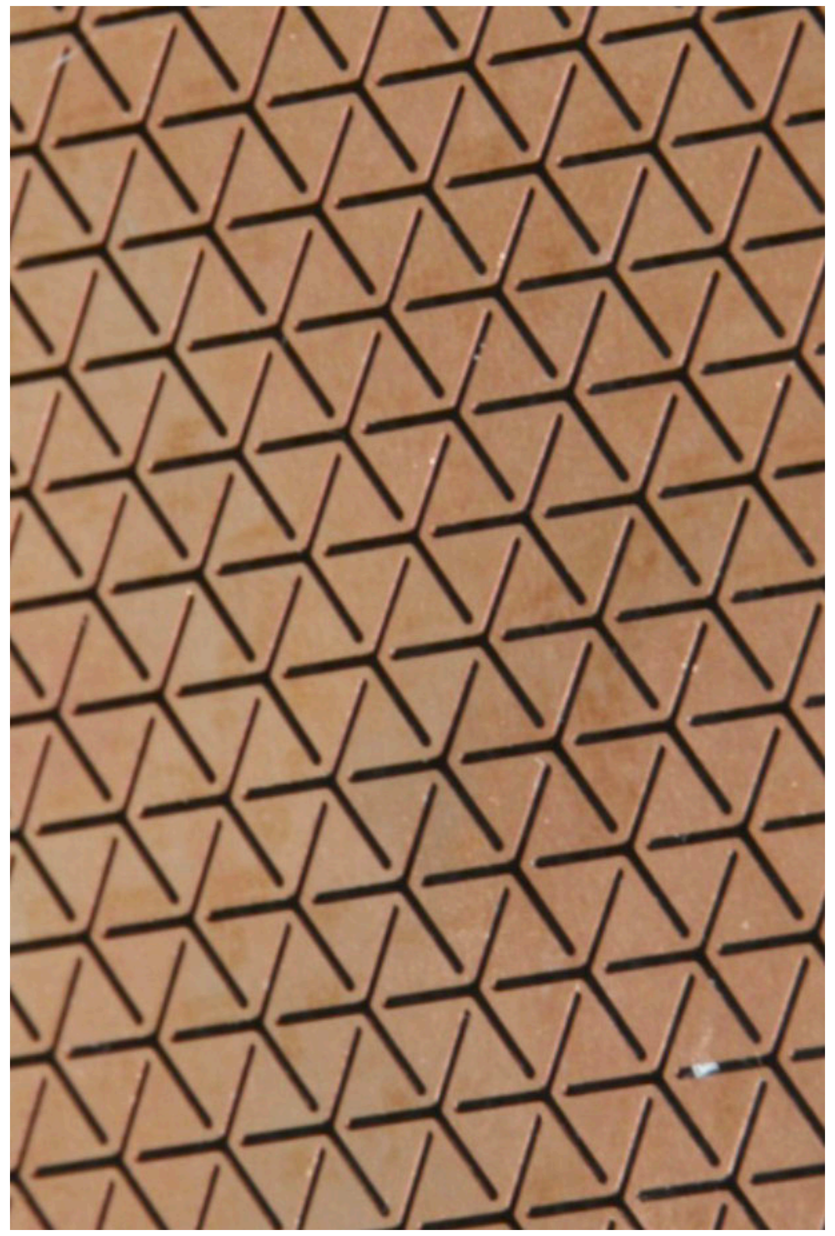


(NONCONVEX)

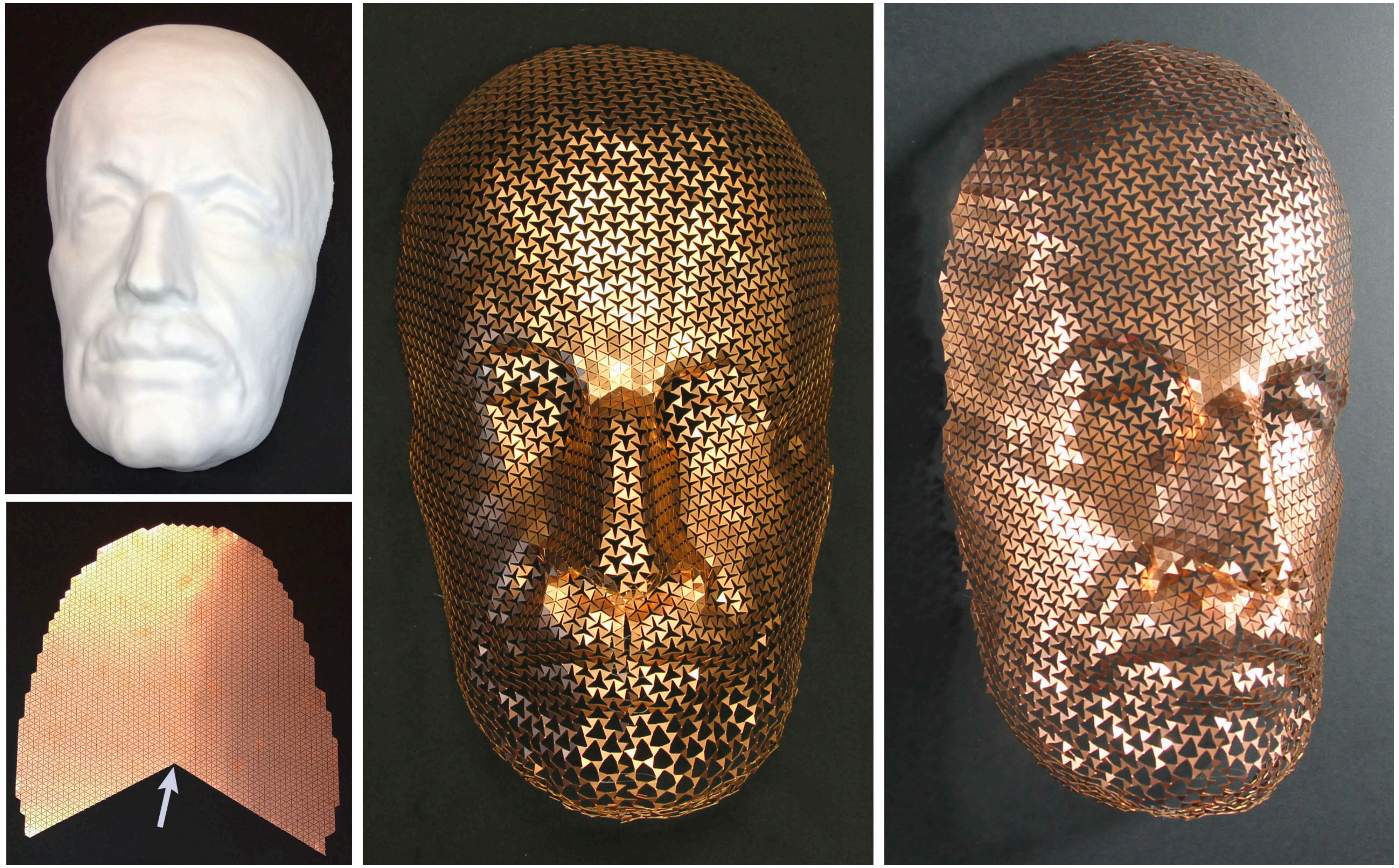
…More flexible than angle preservation, less flexible than smooth conformal maps…

Cone Singularities in Auxetic Design

- Useful for manufacturing from materials with limited ability to stretch:



(laser cut copper)



KONAKOVIC, CRANE, DENG, BOUAZIZ, PIKER, PAULY, "*… Computational Design … with Auxetic Materials*" (2016)

Metric Scaling

Discrete Conformal Flattening

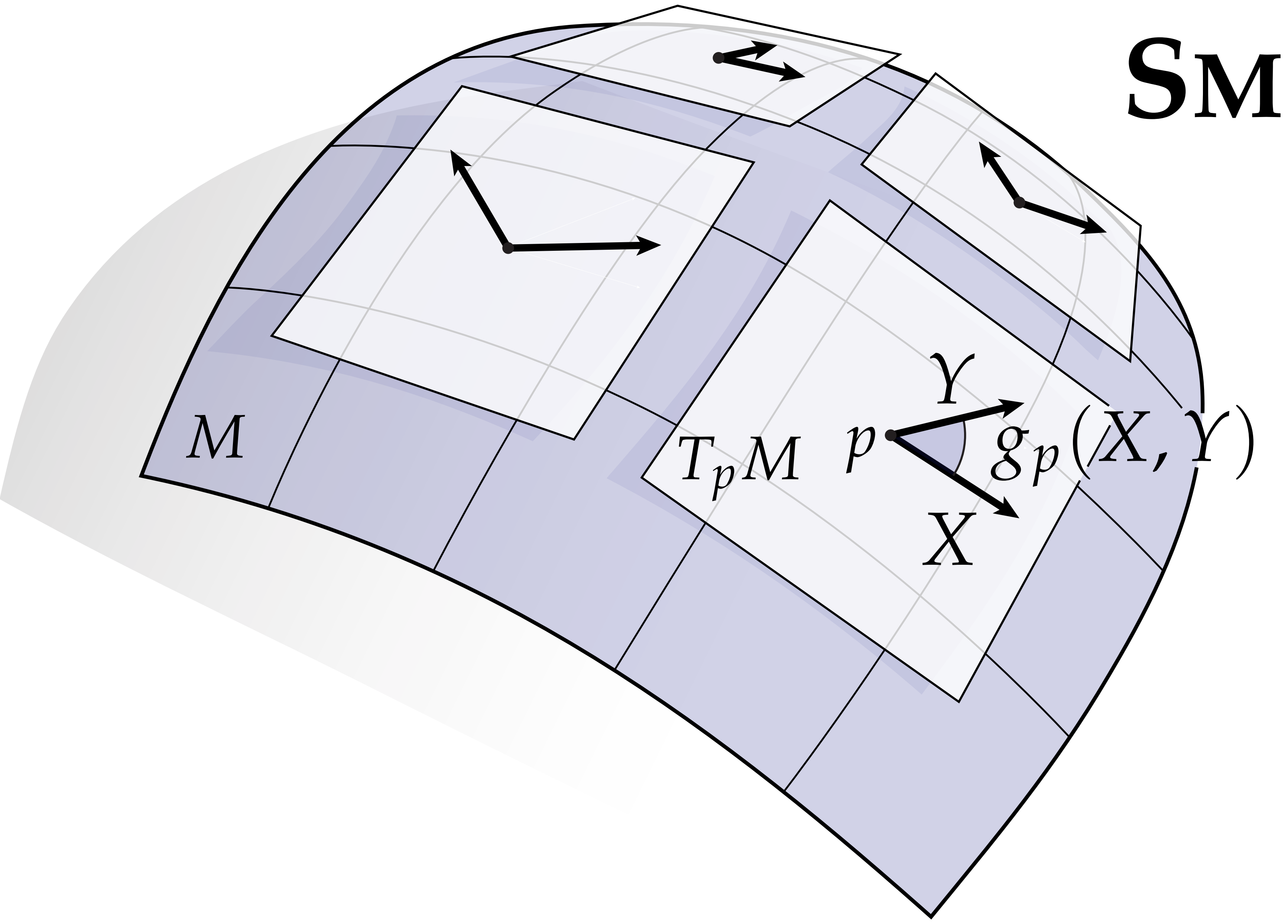
- Recall that two metrics are conformally equivalent if…

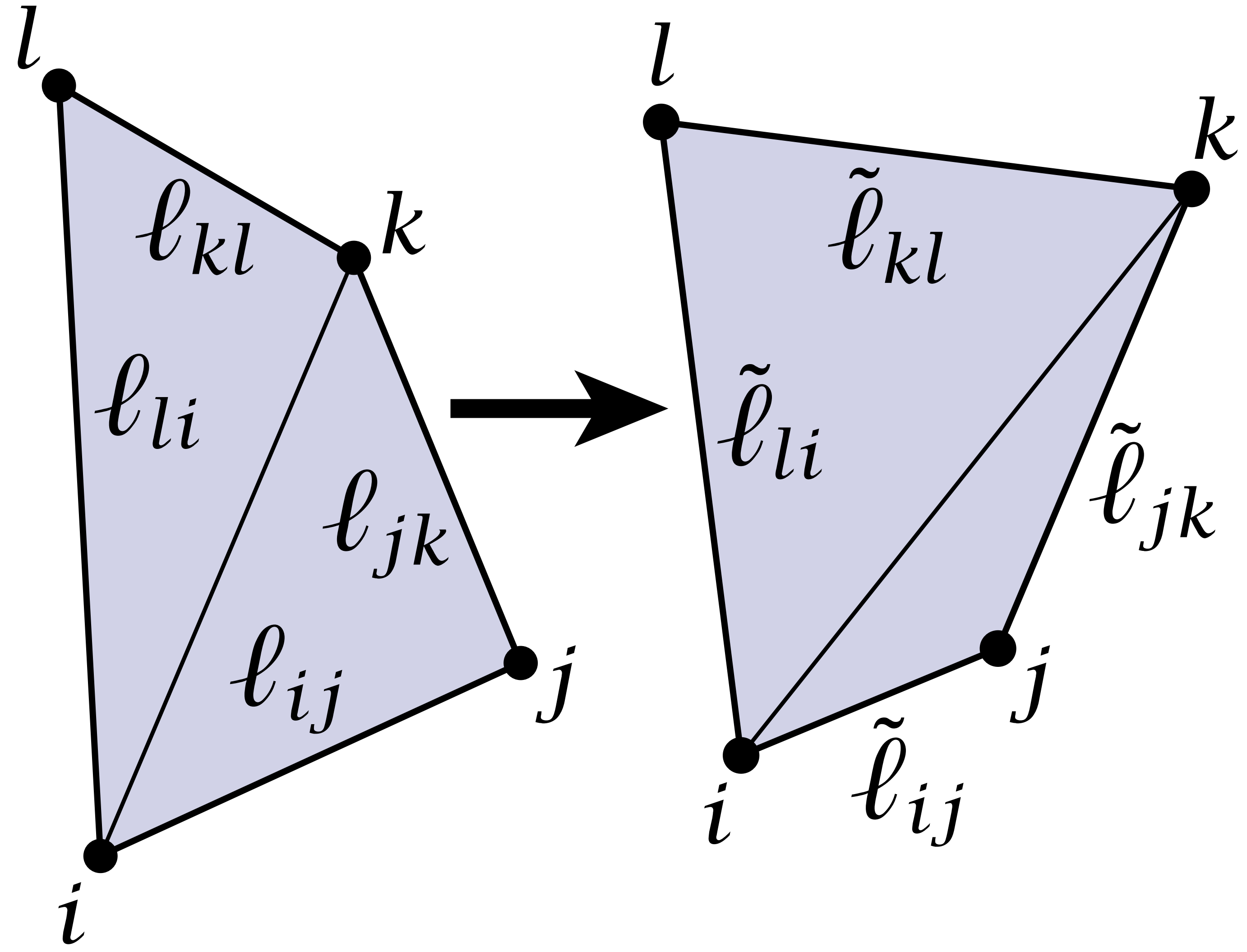
$$\boxed{\tilde{g} = e^{2u} g}$$

SMOOTH

$$\boxed{\tilde{\ell}_{ij} = e^{(u_i+u_j)/2} \ell_{ij}}$$

DISCRETE



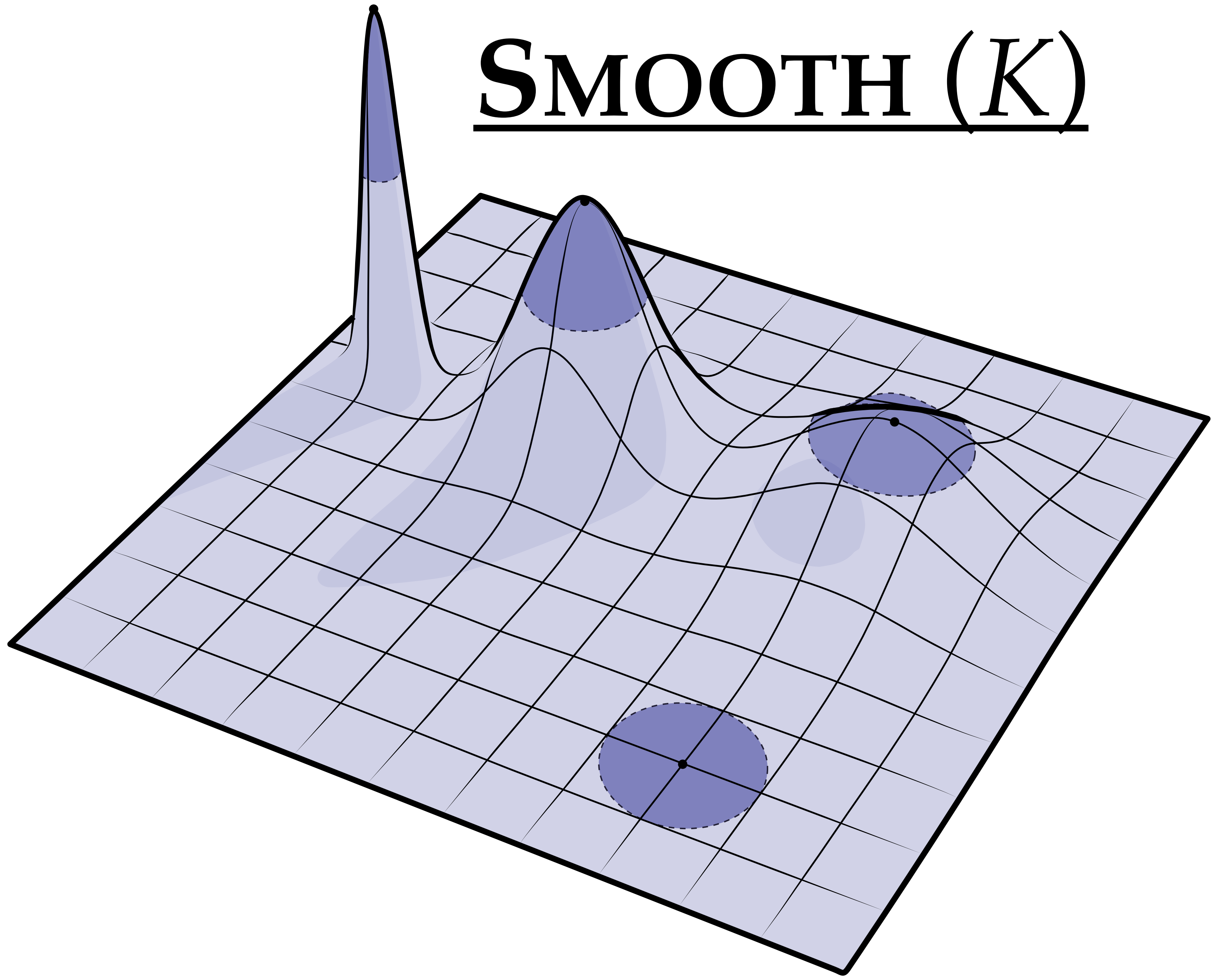


How do we *compute* a flattening that is conformally equivalent in this sense?

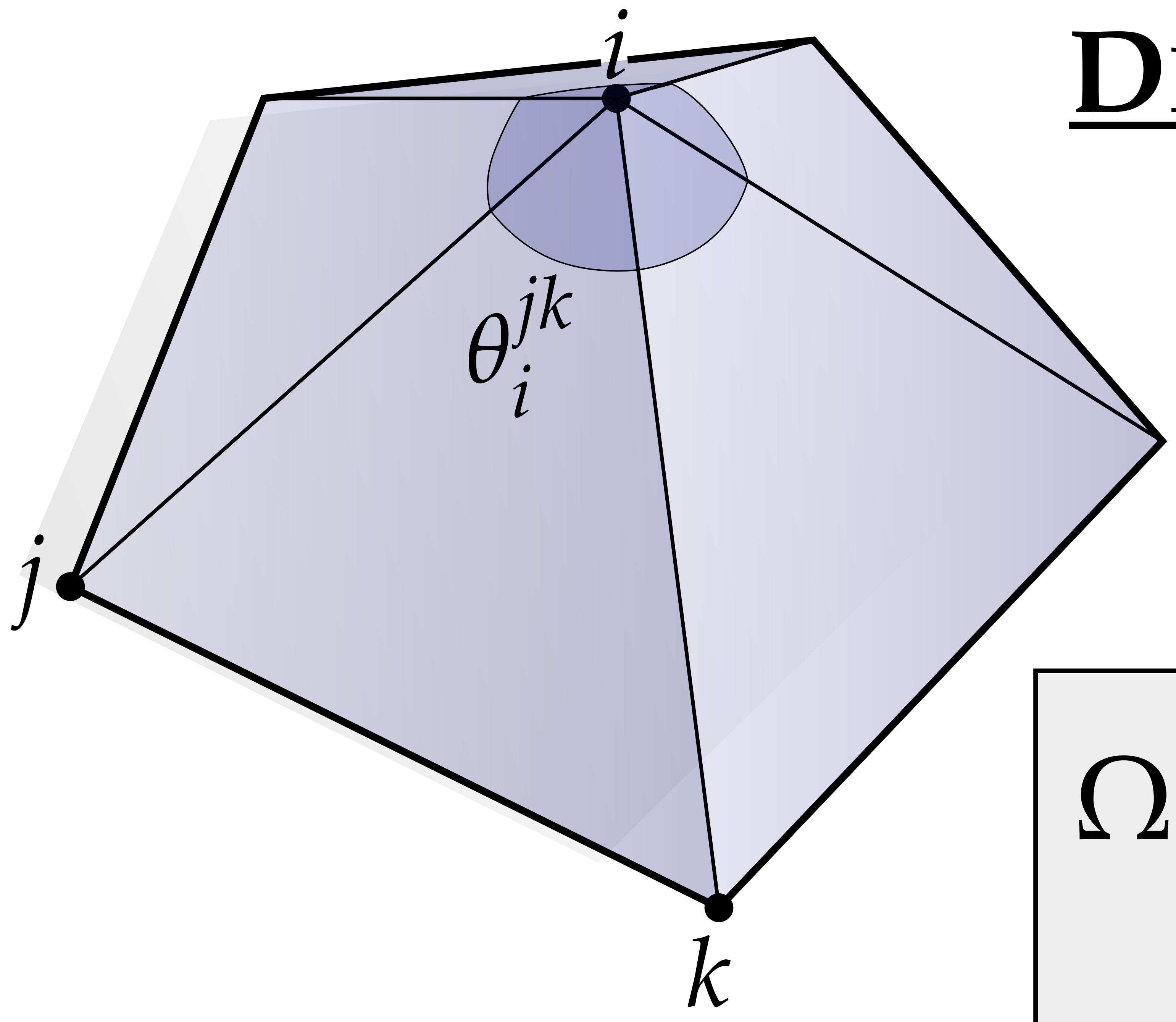
(Discrete) Gaussian Curvature

- Useful to take a moment to say what we mean by "flat"!
- *Gaussian curvature K* measures how hard it is to flatten a piece of material
- *Discrete* Gaussian curvature is just deviation from planar angle sum 2π:

SMOOTH (K)



DISCRETE



$$\Omega_i := 2\pi - \sum_{ijk \in F} \theta_i^{jk}$$

Yamabe Problem

- In the smooth setting, the *Yamabe equation* gives an explicit relationship between a conformal scaling of the metric, and the change in Gaussian curvature:

log scale factor

$$\Delta u = K - e^{2u}\tilde{K}$$

Laplacian *original curvature* *new curvature*

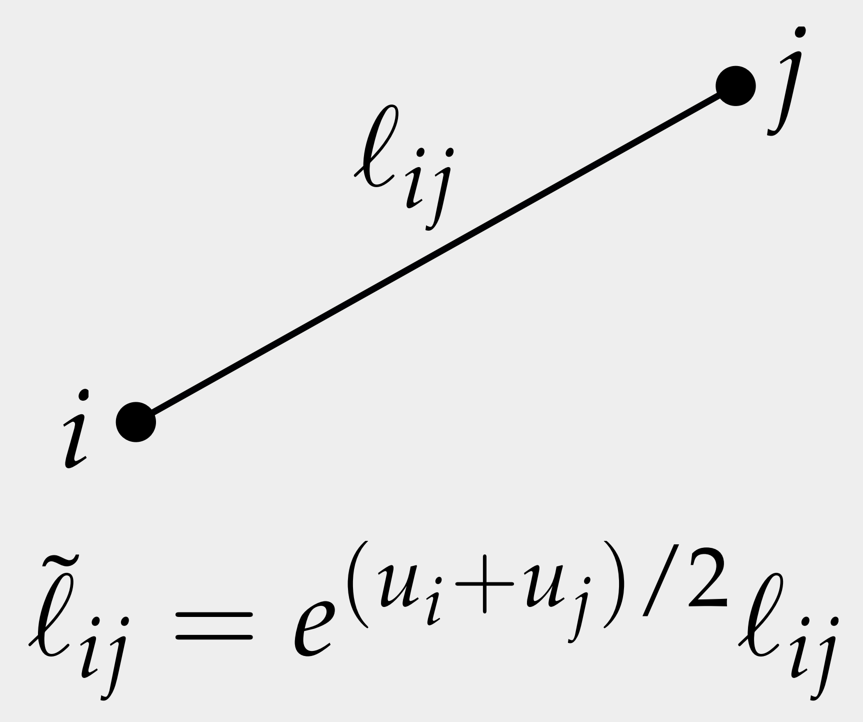
- Nonlinear due to e^{2u} term on right-hand side; hard to solve directly.

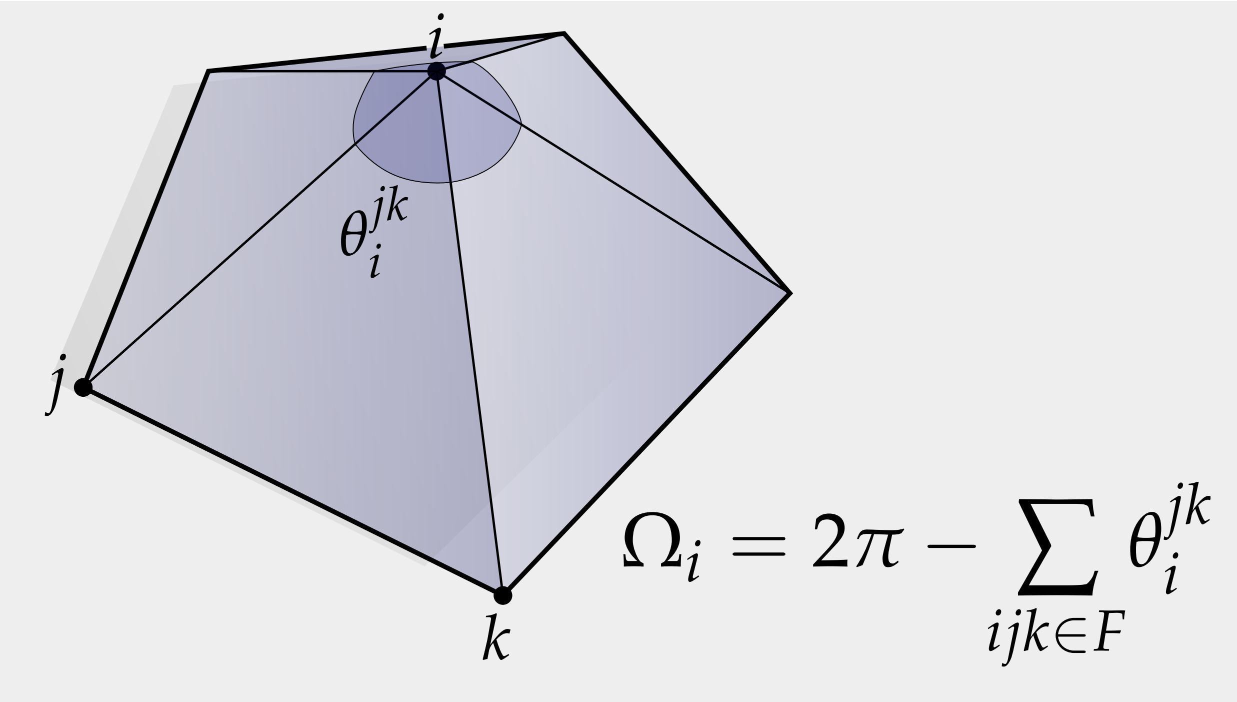
Discrete Yamabe Flow

- Instead, flow toward scale factors that give desired curvature
- *Discrete case*: scale factors determine new lengths, which determine new angles, which determine angle defect
- *Basic idea*: differentiate curvature with respect to u
- End up with so-called (discrete) *Yamabe flow:*

$$\boxed{\frac{d}{dt}u(t) = \Omega^* - \Omega(t)}$$

- (Here for *any* target curvature Ω^*, not just flat)

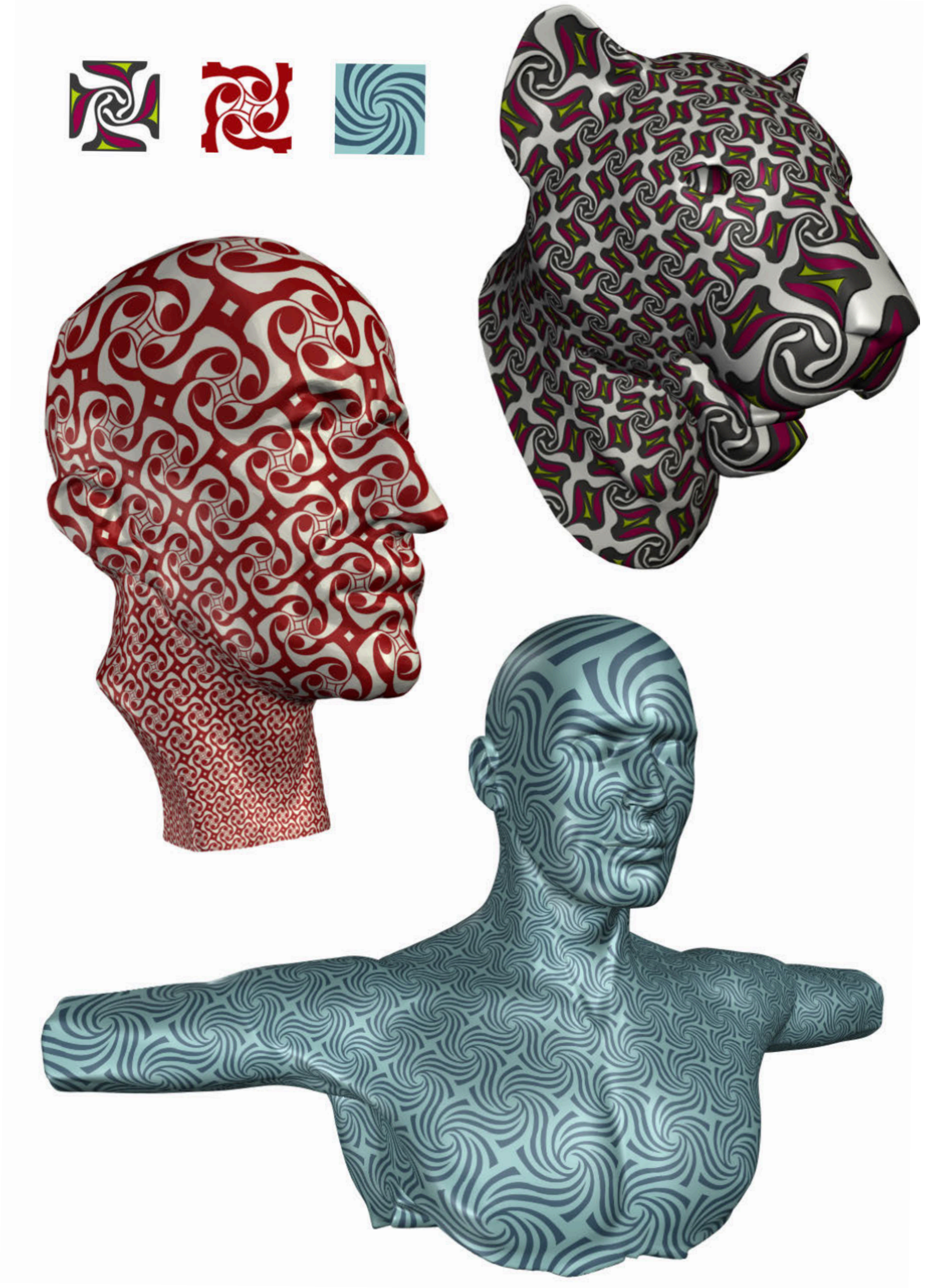




LUO, *"Combinatorial Yamabe Flow on Surfaces"* (2004)

CETM Algorithm

- Flow can also be interpreted as a gradient of convex energy
  - Hessian of this energy is infamous *"cotan Laplacian"*
- Makes the flow more practical for geometry processing algorithms
- Sophisticated control over boundary shape, cone singularities, *etc.*



SPRINGBORN, SCHRÖDER, PINKALL, *"Conformal Equivalence of Triangle Meshes"* (2008)

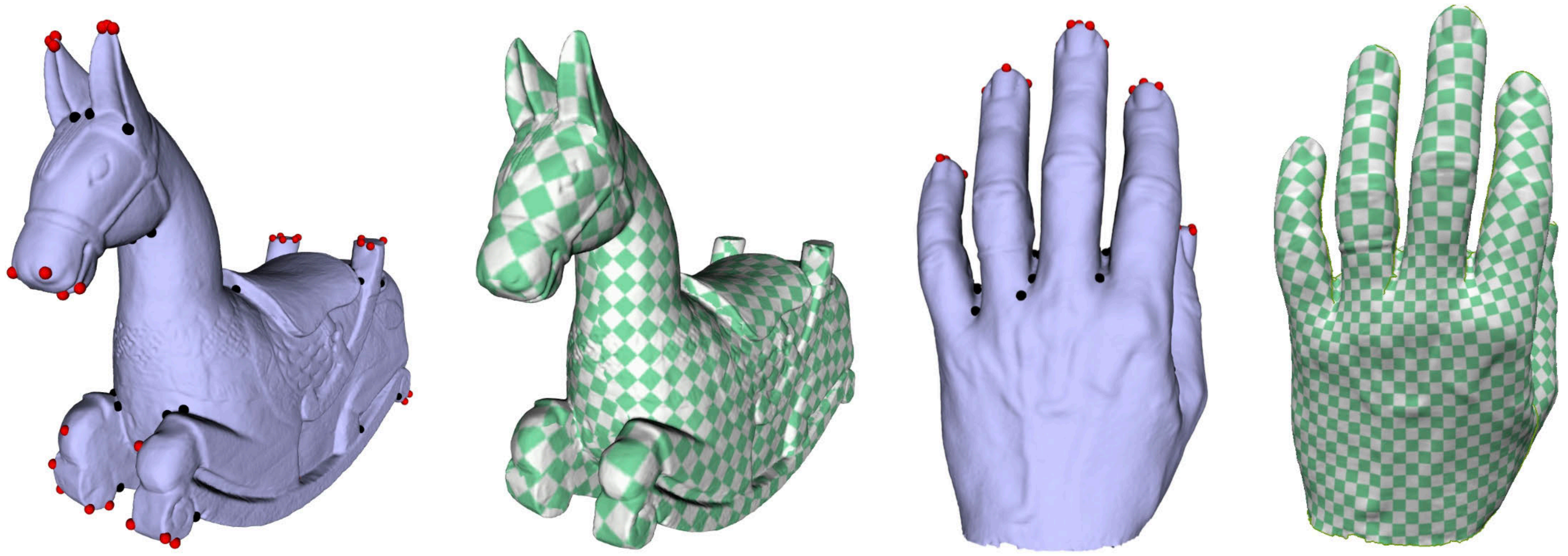
Curvature Prescription & Metric Scaling (CPMS)

- *Alternatively*: linearize Yamabe equation and solve in one step:

assume log factor is fixed, or zero

$$\Delta u = K - e^{2u}\tilde{K}$$

- Reasonable assumption when target curvature describes *cone metric*.

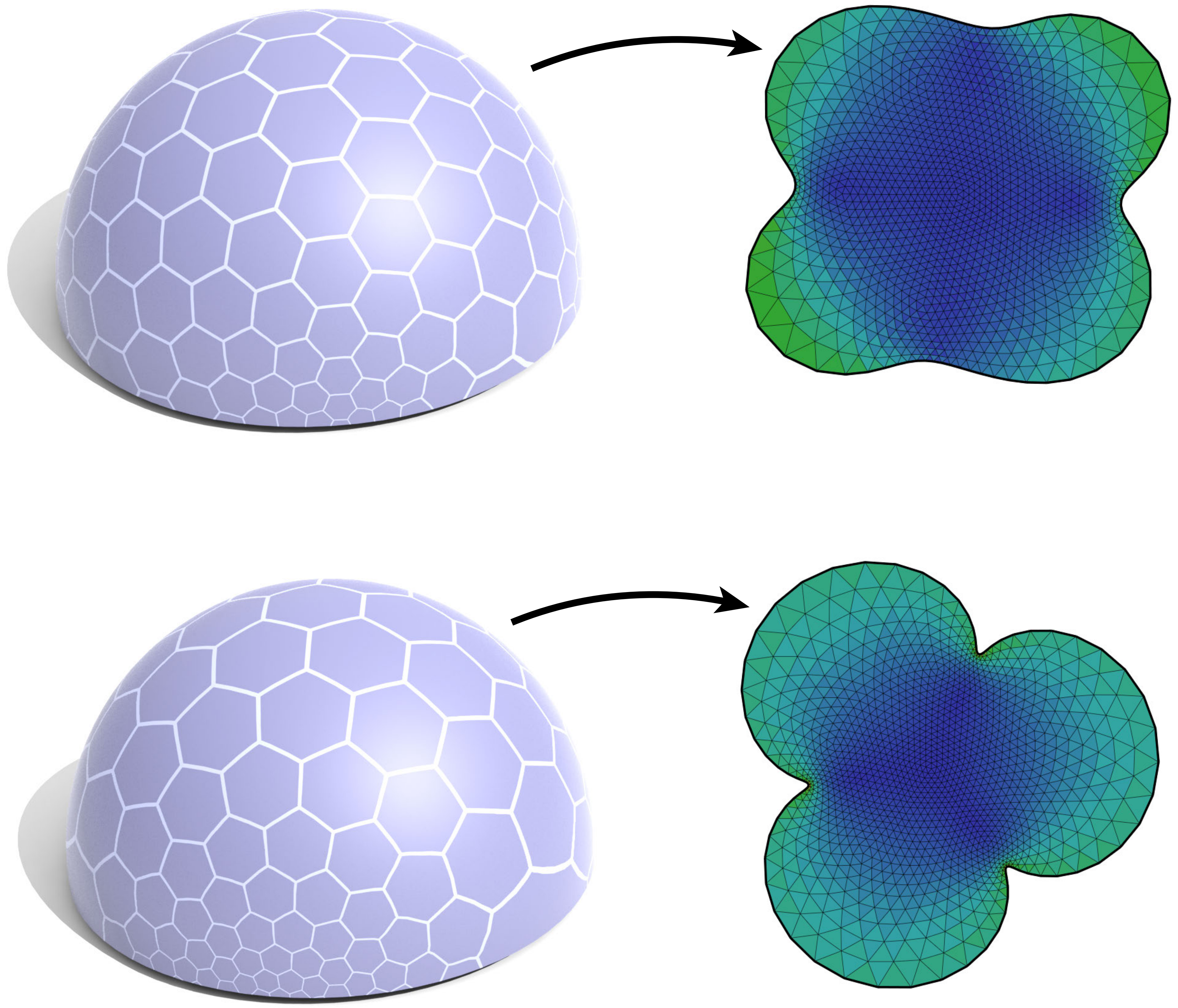


Cherrier Formula

- Yamabe equation was actually incomplete—what happens at the boundary?
- Answer given by *Cherrier equation*

$$\boxed{\begin{aligned} \Delta u &= K - e^{2u}\tilde{K} \quad \text{on } M \\ \frac{\partial u}{\partial n} &= \kappa - e^{u}\tilde{\kappa} \quad \text{on } \partial M \end{aligned}}$$

- Implies we can prescribe ***either*** the curvature κ ***or*** the scale factor u along the boundary—*but not both!*

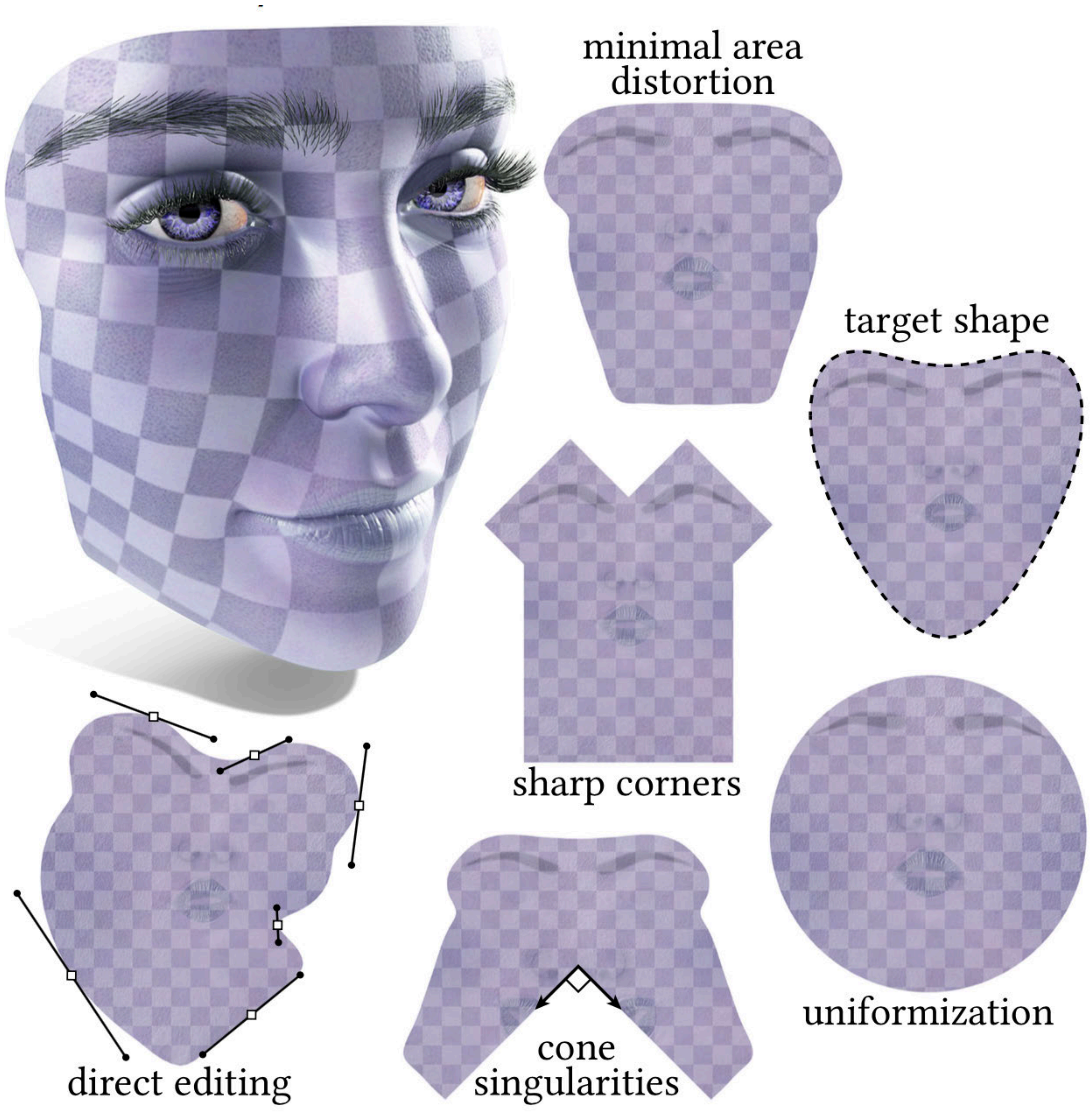


CHERRIER, *"Problèms de Neumann non linéaires sur les variétés Riemanniennes"* (1984)

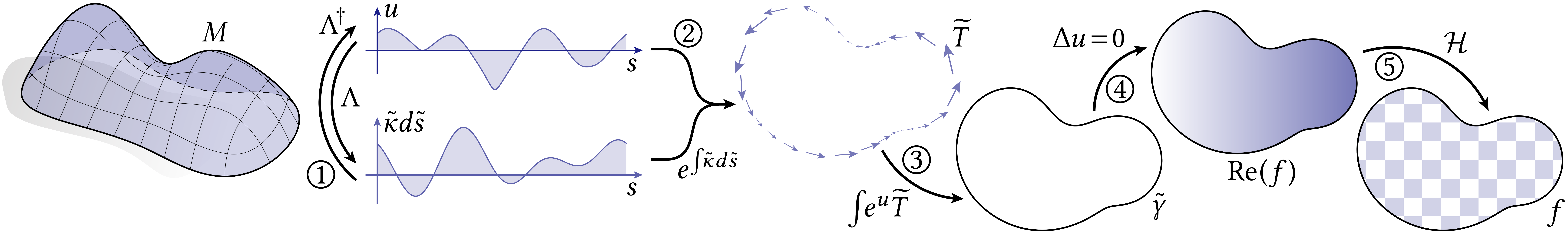
Boundary First Flattening (BFF)

- Brand new algorithm (2017) based on Cherrier plus some other tricks…
- Complete control over boundary shape
- Faster than LSCM; *much* faster than CETM (but with comparable quality)
- Lots of bonus features (optimal area distortion, cone singularities, …)

https://arxiv.org/abs/1704.06873



Boundary First Flattening—Rough Outline



- Given a surface, specify either length ***or*** curvature of target curve
- Solve *Cherrier problem* to get complementary data (curvature or length)
- Integrate boundary data to get boundary curve
- Extend boundary curve to a pair of *conjugate harmonic functions*

From Cauchy-Riemann to Conjugate Harmonic

- Starting with Cauchy-Riemann:

$$df(\mathcal{J}X) = \imath df(X)$$

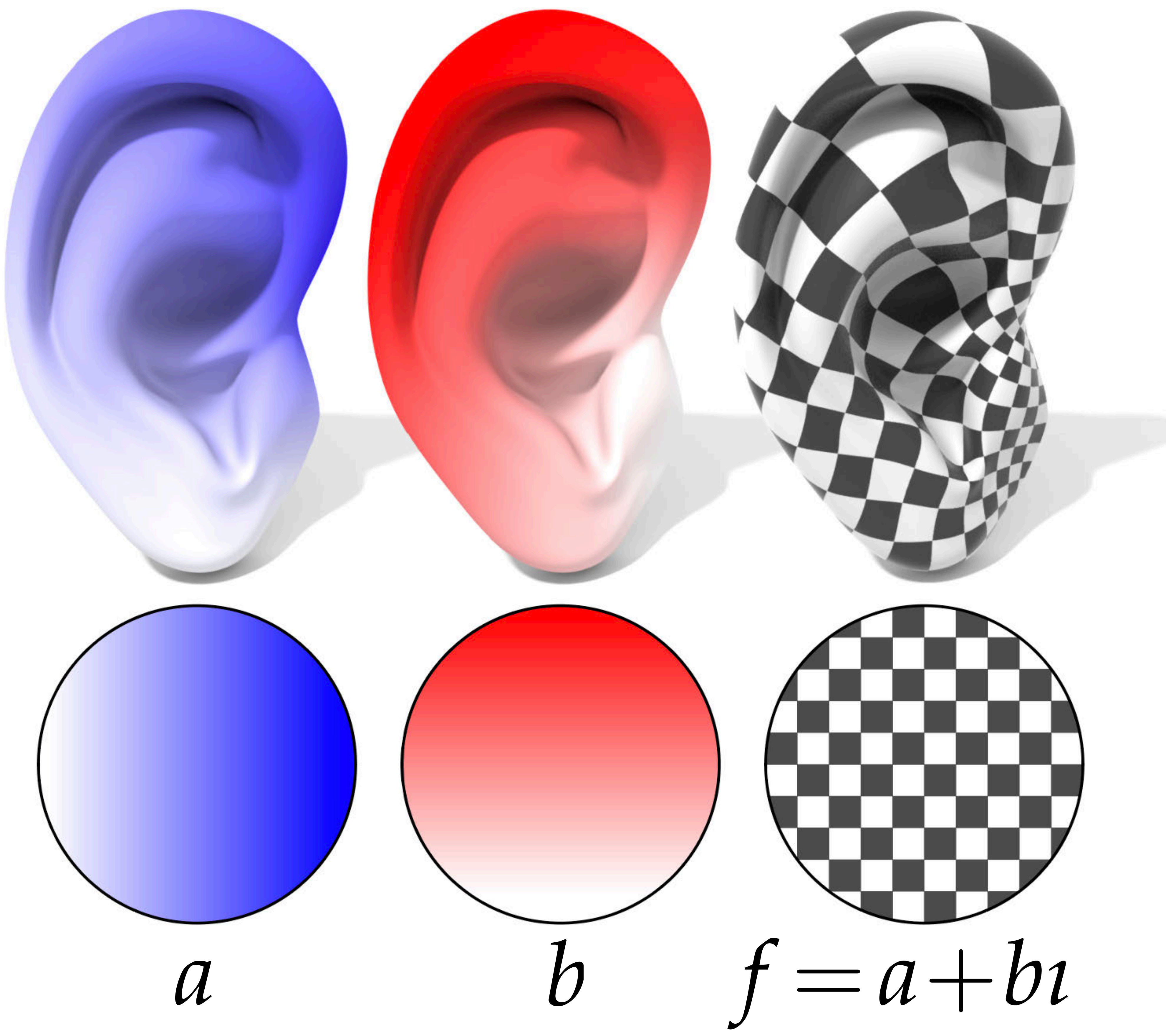
$$da(\mathcal{J}X) + \imath db(\mathcal{J}X) = \imath da(X) - \imath db(X)$$

$$\nabla a = -J\nabla b$$

$$\underbrace{\nabla \cdot \nabla}_{\Delta} a = -\underbrace{\nabla \cdot (\nabla b)}_{=0}$$

$$\boxed{\begin{array}{rcl} \Delta a & = & 0 \\ \Delta b & = & 0 \\ \mathcal{J}\nabla a & = & \nabla b \end{array}}$$

CONJUGATE HARMONIC PAIR



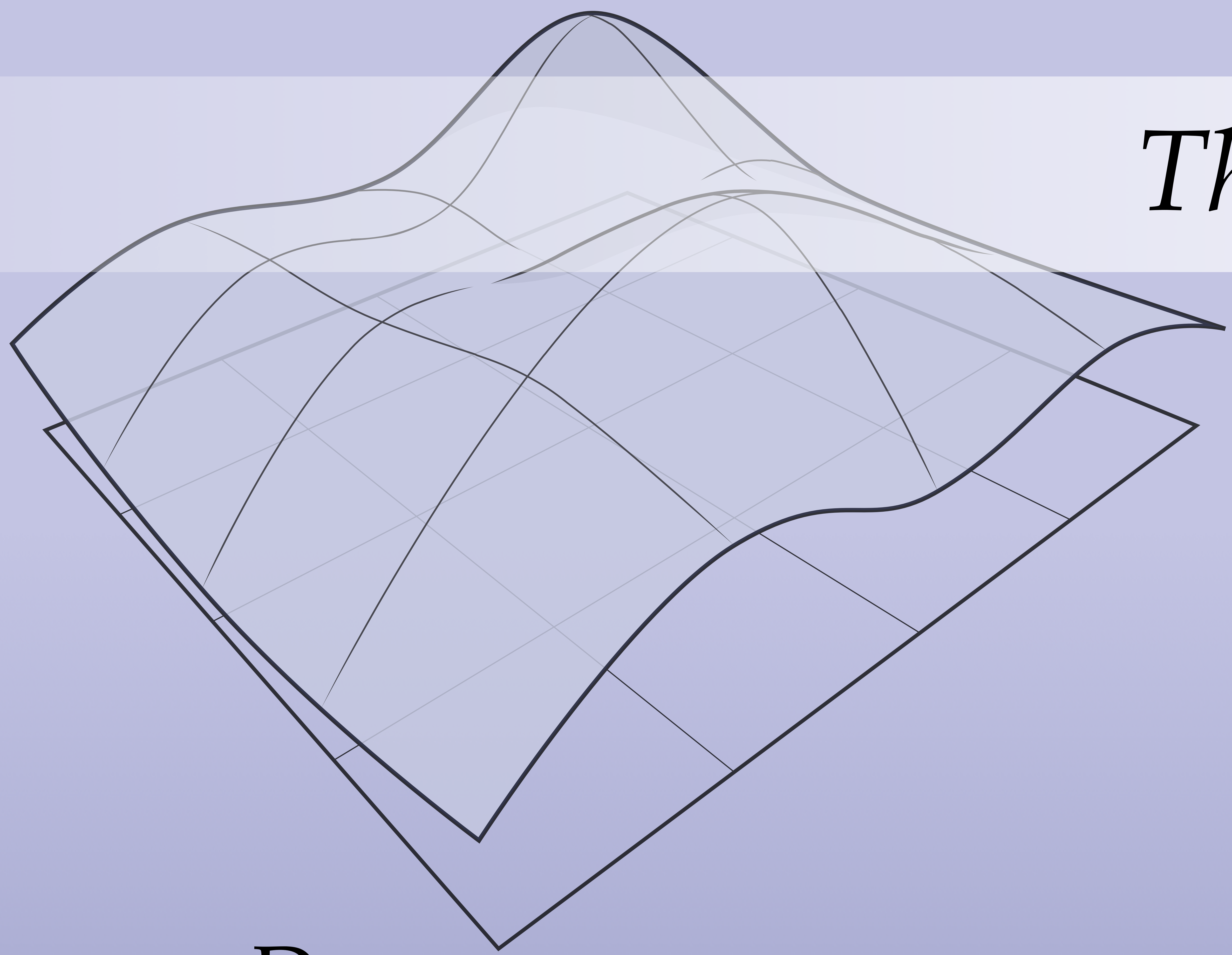
(How do you conjugate a piecewise linear function? See BFF paper!)

Other Methods

So much more!

- **Many** ideas/algorithms we didn't cover…
  - in the plane: *Schwarz-Christoffel, Cauchy-Green coordinates,* …
  - inversive distance [Guo et al 2009]
  - primal-dual length ratio / discrete Riemann surfaces [Mercat 2001]
  - facewise Möbius transformations [Vaxman et al 2015]
  - in the plane: *Schwarz-Christoffel, Cauchy-Green coordinates,* …
- Also, didn't get to see many of the (*beautiful!*) things people are doing with conformal maps. Hopefully you'll see a few here at SGP…

Thanks!



Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-869(J) • Spring 2016