Chapter 18
Tail Bounds

Tails

Defn: The **tail** of random variable X is $\boldsymbol{P}\{X > x\}$.

Examples of why we care about tails:

- Fraction of jobs that queue more than 24 hours
- Fraction of packets that find the router buffer full
- Fraction of hash buckets that have more than 10 items

Unfortunately, determining the tail of even simple r.v.s is often hard – much harder than determining the mean or transform!

Tails Example

Q: Suppose you're distributing n jobs among n servers at random. What's the probability that a particular server gets $\geq k$ jobs?

$$X \sim Binomial(n, p)$$

$$\boldsymbol{P}\{X \geq k\} = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

No closed-form known for this

Tails Example

Q: Jobs arrive to a datacenter according to a Poisson process with rate λ jobs/hour. What's the probability that $\geq k$ jobs arrive during the first hour?

$$X \sim Poisson(\lambda)$$

$$\boldsymbol{P}\{X \geq k\} = \sum_{i=k}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

No closed-form known for this

Tails Bounds

Rather than directly compute tails, we will derive upper bounds on the tails, called **tail bounds**!

$$\mathbf{P}\{X \geq k\} = \sum_{i=k}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad \qquad \mathbf{P}\{X \geq k\} = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

We'll soon have tail bounds for both of these!

Definition**:** An upper bound on $\mathbf{P}\{X \geq k\}$ is called a **tail bound**. An upper bound on $\mathbf{P}\{|X - \mu| \geq k\}$ where $\mu = \mathbf{E}[X]$ is called a **concentration bound** or **concentration inequality**.

Running Example

We will develop progressively better (tighter) **tail bounds**.

We will test each bound on the following running example:

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

Markov’s inequality

Theorem: (Markov’s inequality) If r.v. X is non-negative, with finite mean $\mu = \boldsymbol{E}[X]$, then $\forall a > 0$,

$$\boldsymbol{P}\{X \geq a\} \leq \frac{\mu}{a}$$

Proof:

$$\boldsymbol{E}[X] = \sum_{x=0}^{\infty} x \cdot p_X(x) \geq \sum_{x=a}^{\infty} x \cdot p_X(x)$$

$$\geq \sum_{x=a}^{\infty} a \cdot p_X(x)$$

$$= a \sum_{x=a}^{\infty} p_X(x) = a \cdot \boldsymbol{P}\{X \geq a\}$$

Markov's Inequality on Running Example

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

$$X = \text{Number Heads} \sim Binomial\left(n, \frac{1}{2}\right) \qquad \mu = \mathbf{E}[X] = \frac{n}{2}$$

$$\mathbf{P}\left\{X \geq \frac{3}{4}n\right\} \leq \frac{\mu}{\frac{3}{4}n} = \frac{\frac{n}{2}}{\frac{3}{4}n} = \frac{2}{3}$$

Clearly a terrible bound because doesn't involve n

Chebyshev's inequality

Theorem: (Chebyshev's inequality) Let X be any r.v. with finite mean, μ, and finite variance. Then $\forall a > 0$,

$$\mathbf{P}\{|X - \mu| \geq a\} \leq \frac{\mathbf{Var}(X)}{a^2}$$

Proof:

Q: Can you see how to apply Markov's inequality here?

$$\mathbf{P}\{|X - \mu| \geq a\} = \mathbf{P}\{(X - \mu)^2 \geq a^2\}$$

$$\leq \frac{\mathbf{E}[(X - \mu)^2]}{a^2}$$

$$= \frac{\mathbf{Var}(X)}{a^2}$$

Chebyshev's Bound on Running Example

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

$$X = \text{Number Heads} \sim Binomial\left(n, \frac{1}{2}\right) \qquad \mu = E[X] = \frac{n}{2} \qquad \boldsymbol{Var}(X) = \frac{n}{4}$$

At least decreases with n

$$\boldsymbol{P}\left\{X \geq \frac{3}{4}n\right\} = \boldsymbol{P}\left\{X - \frac{n}{2} \geq \frac{n}{4}\right\} = \frac{1}{2}\boldsymbol{P}\left\{\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right\} \leq \frac{1}{2} \cdot \frac{\boldsymbol{Var}(X)}{\left(\frac{n}{4}\right)^2} = \frac{1}{2} \cdot \frac{\frac{n}{4}}{\left(\frac{n}{4}\right)^2} = \frac{2}{n}$$

Why?

Chernoff Bound

In deriving the Chebyshev bound, we **squared** the r.v. and then applied Markov.

In deriving the Chernoff bound, we **exponentiate** the r.v. and then apply Markov.

$\forall t > 0$:

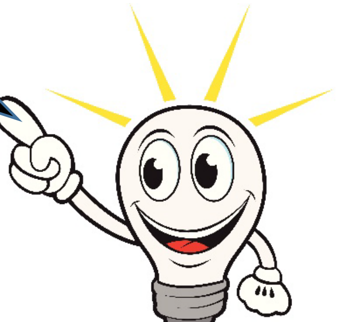
$$\boldsymbol{P}\{X \geq a\} = \boldsymbol{P}\{tX \geq ta\}$$

$$= \boldsymbol{P}\{e^{tX} \geq e^{ta}\}$$

$$\leq \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}}$$

Why are we allowed to apply Markov to this?

But because this bound holds $\forall t$, it also holds for the minimizing t.



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Theorem 18.3: (Chernoff bound) Let X be any r.v. and a be a constant. Then

$$\boldsymbol{P}\{X \geq a\} \leq \min_{t>0} \left\{\frac{\boldsymbol{E}[e^{tX}]}{e^{ta}}\right\}$$

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$$= \boldsymbol{P}\{e^{tX} \geq e^{ta}\}$$

$$\leq \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}}$$

Q: Why do we expect the Chernoff bound to be stronger than the others?

Theorem: (Chernoff bound) Let X be any r.v. and a be a constant. Then

$$\boldsymbol{P}\{X \geq a\} \leq \min_{t>0} \left\{ \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}} \right\}$$

A: Looks a lot like an onion!

Chernoff Bound on c.d.f.

Q: What do we do if we want to upper bound $\boldsymbol{P}\{X \leq a\}$?

Consider
$t < 0$

$\forall t < 0$:

$$\boldsymbol{P}\{X \leq a\} = \boldsymbol{P}\{tX \geq ta\}$$

$$= \boldsymbol{P}\{e^{tX} \geq e^{ta}\}$$

$$\leq \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}}$$

Theorem: (Chernoff bound on c.d.f.) Let X be any r.v. and a be a constant. Then

$$\boldsymbol{P}\{X \leq a\} \leq \min_{t<0} \left\{ \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}} \right\}$$

Chernoff Bound for Poisson Tail

Goal: Bound tail of $X \sim Poisson(\lambda)$

<u>Step 1</u>: Derive $\boldsymbol{E}[e^{tX}]$ where $t > 0$

$$\boldsymbol{E}[e^{tX}] = \sum_{i=0}^{\infty} e^{ti} \cdot \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda e^t)^i}{i!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

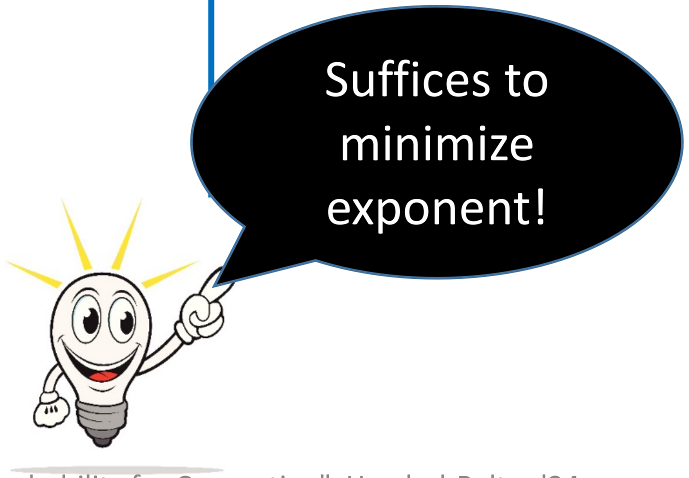
$$= e^{\lambda(e^t - 1)}$$

<u>Step 2</u>: Let $a > \lambda$. Bound $\boldsymbol{P}\{X \geq a\}$

$$\boldsymbol{P}\{X \geq a\} \leq \min_{t>0} \left\{ \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}} \right\}$$

$$= \min_{t>0} \left\{ \frac{e^{\lambda(e^t - 1)}}{e^{ta}} \right\}$$

$$= \min_{t>0} \left\{ e^{\lambda(e^t - 1) - ta} \right\}$$



Chernoff Bound for Poisson Tail

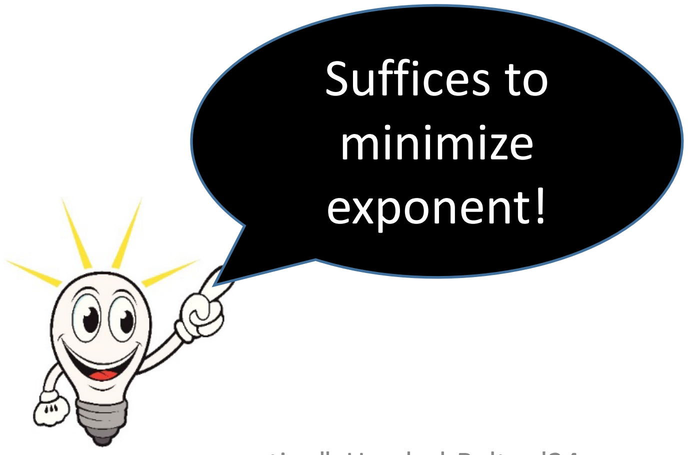
Goal: Bound tail of $X \sim Poisson(\lambda)$

<u>Step 2</u>: Let $a > \lambda$. Bound $\boldsymbol{P}\{X \geq a\}$

$$\boldsymbol{P}\{X \geq a\} \leq \min_{t>0} \left\{ \frac{\boldsymbol{E}[e^{tX}]}{e^{ta}} \right\}$$

$$= \min_{t>0} \left\{ \frac{e^{\lambda(e^t - 1)}}{e^{ta}} \right\}$$

$$= \min_{t>0} \left\{ e^{\lambda(e^t-1)-ta} \right\}$$



- Exponent is minimized at $t = ln\left(\frac{a}{\lambda}\right)$

  Thus:

- $\boldsymbol{P}\{X \geq a\} \leq e^{\lambda(e^t-1)-ta}$, at $t = ln\left(\frac{a}{\lambda}\right)$

$$= e^{\lambda\left(\frac{a}{\lambda}-1\right)-a ln\left(\frac{a}{\lambda}\right)}$$

$$= \boxed{e^{a-\lambda} \cdot \left(\frac{\lambda}{a}\right)^a}$$

Chernoff Bound for Binomial

Theorem 18.4: (Pretty Chernoff Bound for Binomial)

Let $X \sim Binomial(n,p)$ where $\mu = \boldsymbol{E}[X] = np$. Then, for any $\delta > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

$$\boldsymbol{P}\{X - np \leq -\delta\} \leq e^{-2\delta^2/n}$$

We will prove this soon, but let's try applying it first!

Bound is strongest when $\delta = \Theta(n)$
Try to use it in this regime.

Chernoff Bound on Running Example

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

$$X = \text{Number Heads} \sim Binomial\left(n, \frac{1}{2}\right) \qquad \mu = E[X] = \frac{n}{2}$$

$$\boldsymbol{P}\left\{X \geq \frac{3}{4}n\right\} = \boldsymbol{P}\left\{X - \frac{n}{2} \geq \frac{n}{4}\right\} \leq e^{-2\left(\frac{n}{4}\right)^2 \cdot \frac{1}{n}} = e^{-\frac{n}{8}}$$

Decreases exponentially fast in n

Note $\delta = \frac{n}{4} = \Theta(n)$

Comparing the bounds

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

Q: What is the exact answer?

$$\mathbf{P}\left\{X \geq \frac{3}{4}n\right\} = \sum_{i=\frac{3}{4}n}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{n-i} = \boxed{2^{-n} \sum_{i=\frac{3}{4}n}^{n} \binom{n}{i}}$$

Central Limit Theorem

Flip a fair coin n times: 

Q: What's a tail bound on the probability of getting at least $\frac{3}{4}n$ heads?

CLT applies because adding i.i.d. r.v.s

$$X \sim Binomial\left(n, \frac{1}{2}\right)$$

$$\mu = \boldsymbol{E}[X] = \frac{n}{2}$$

$$\boldsymbol{Var}(X) = \frac{n}{4}$$

$$\sigma_X = \sqrt{\frac{n}{4}}$$

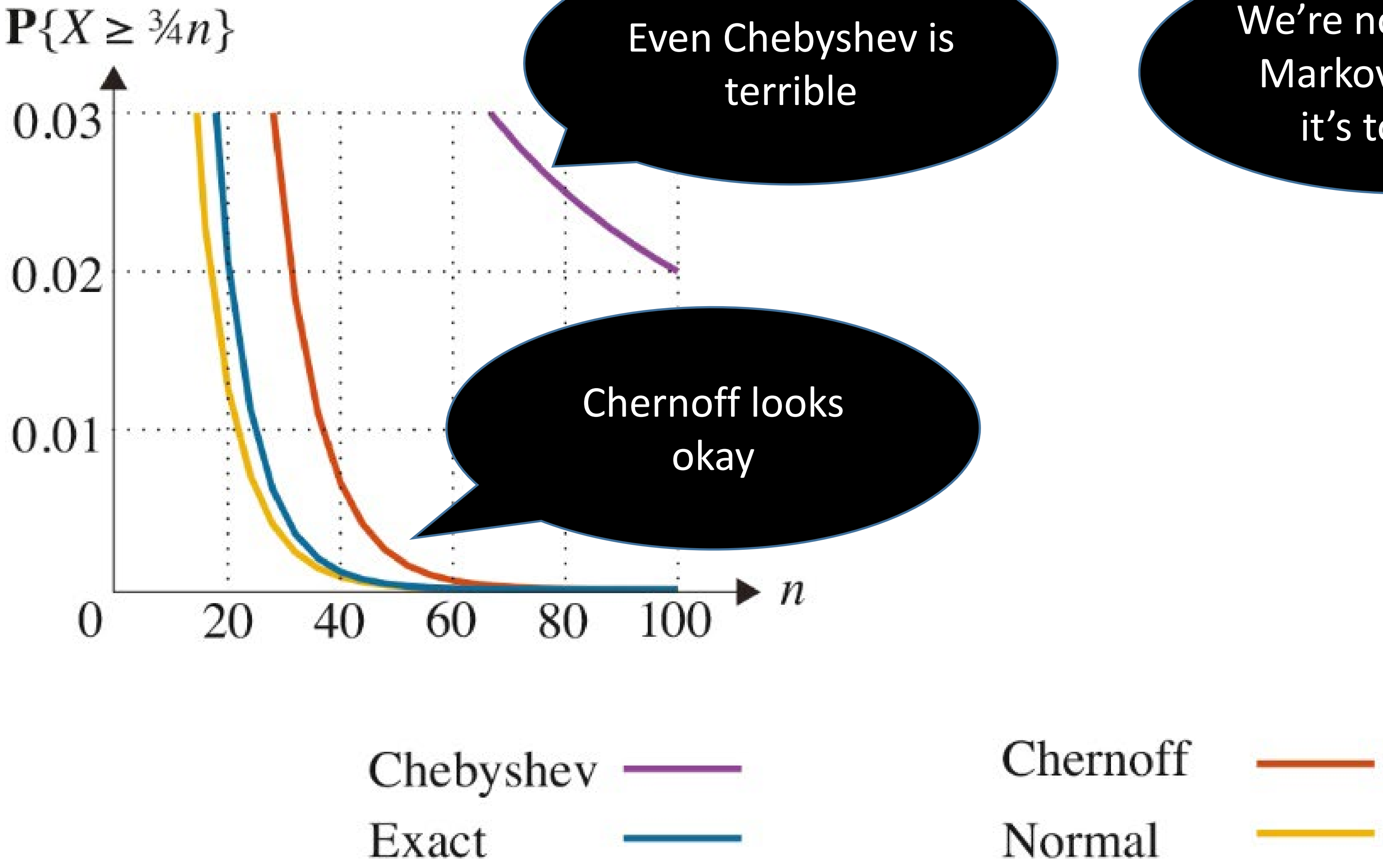
$$\boldsymbol{P}\left\{X \geq \frac{3}{4}n\right\} = \boldsymbol{P}\left\{X - \frac{n}{2} \geq \frac{n}{4}\right\}$$

$$= \boldsymbol{P}\left\{\frac{X - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \geq \frac{\frac{n}{4}}{\sqrt{\frac{n}{4}}}\right\}$$

$$\approx \boldsymbol{P}\left\{Normal(0,1) \geq \sqrt{\frac{n}{4}}\right\} = \boxed{1 - \boldsymbol{\Phi}\left(\sqrt{\frac{n}{4}}\right)}$$

Result is approximation not bound

Comparing the approximation and bounds



Even Chebyshev is terrible

We're not showing Markov because it's too high

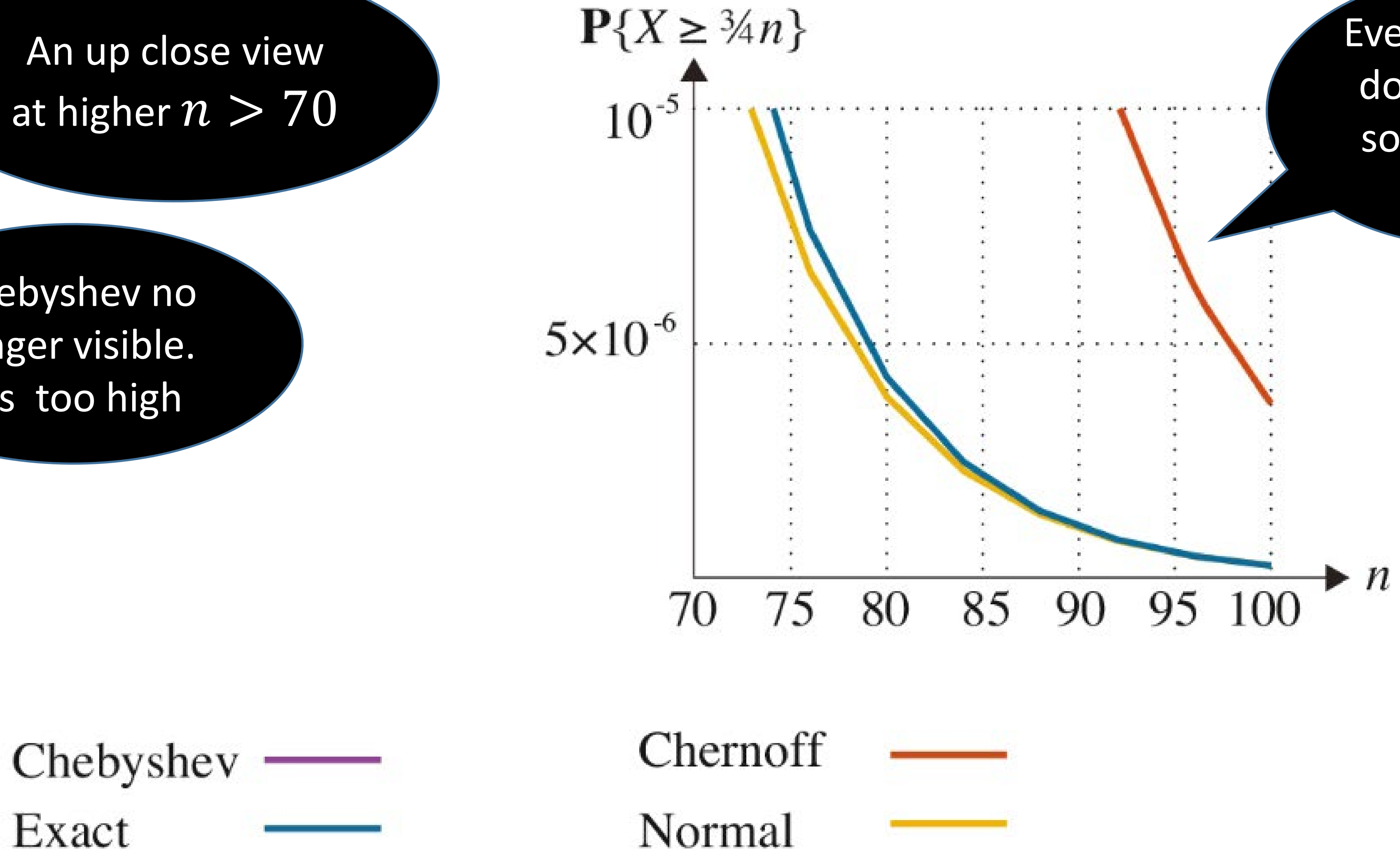
Chernoff looks okay

Comparing the approximation and bounds

An up close view at higher $n > 70$

Chebyshev no longer visible. It's too high

Even Chernoff doesn't look so great any more



"Introduction to Probability for Computing", Harchol-Balter '24

Proof of Thm 18.4 – Pretty Chernoff Bound

Theorem 18.4: (Pretty Chernoff Bound for Binomial)

Let $X \sim Binomial(n, p)$ where $\mu = \mathbf{E}[X] = np$. Then, for any $\delta > 0$,

$$\mathbf{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

$$\mathbf{P}\{X - np \leq -\delta\} \leq e^{-2\delta^2/n}$$

We will now prove Thm 18.4 (top half). The bottom half is an Exercise in your book.

Our proof requires using Lemma 18.5, which is proven in your book.

Lemma 18.5: For any $t > 0$ and $0 < p < 1$ and $q = 1 - p$, we have that:

$$pe^{tq} + qe^{-tp} \leq e^{t^2/8}$$

Proof of Thm 18.4 – Pretty Chernoff Bound

Theorem 18.4: Let $X \sim Binomial(n, p)$ where $\mu = \boldsymbol{E}[X] = np$. Then, for any $\delta > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

Proof: For any $t > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} = \boldsymbol{P}\{t(X - np) \geq t\delta\}$$

$$= \boldsymbol{P}\left\{e^{t(X-np)} \geq e^{t\delta}\right\}$$

$$\leq e^{-t\delta}\boldsymbol{E}\left[e^{t(X-np)}\right] = e^{-t\delta}\boldsymbol{E}\left[e^{t((X_1-p)+(X_2-p)+\cdots+(X_n-p))}\right]$$

$$= e^{-t\delta} \cdot \prod_{i=1}^{n} \boldsymbol{E}\left[e^{t(X_i-p)}\right]$$

We can break this up!
$X = \sum_{i=1}^{n} X_i$ where
$X_i \sim Bernoulli(p)$

Proof of Thm 18.4 – Pretty Chernoff Bound

Theorem 18.4: Let $X \sim Binomial(n, p)$ where $\mu = \boldsymbol{E}[X] = np$. Then, for any $\delta > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

Proof, cont: So, for any $t > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-t\delta} \prod_{i=1}^{n} \boldsymbol{E}\left[e^{t(X_i - p)}\right]$$

$$= e^{-t\delta} \prod_{i=1}^{n} \left(p \cdot e^{t(1-p)} + (1-p) \cdot e^{-tp}\right)$$

by Lemma 18.5

$$= e^{-t\delta} \prod_{i=1}^{n} \left(e^{t^2/8}\right) = e^{-t\delta + nt^2/8}$$

Q: What do we do next?

A: Find the minimizing t

Proof of Thm 18.4 – Pretty Chernoff Bound

Theorem 18.4: Let $X \sim Binomial(n, p)$ where $\mu = \boldsymbol{E}[X] = np$. Then, for any $\delta > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

Proof, cont: So, for any $t > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \quad \leq \boxed{e^{-t\delta + nt^2/8}}$$

The exponent is minimized at

$$t = \frac{4\delta}{n}$$

$$\Rightarrow \boldsymbol{P}\{X - np \geq \delta\} \;\leq\; e^{-\left(\frac{4\delta}{n}\right)\delta + n\left(\frac{4\delta}{n}\right)^2/8} = \boxed{e^{-2\delta^2/n}}$$

Stronger (?) Chernoff Bound for Binomial

Theorem 18.6 presents an alternative, sometime stronger, bound.
The bound holds for a more general definition of a Binomial.

Theorem 18.6: (Sometimes stronger Chernoff Bound)

Let $X = \sum_{i=1}^{n} X_i$ where $X_i \sim Bernoulli(p_i)$ and $\mu = \boldsymbol{E}[X] = \sum_{i=1}^{n} p_i$. Then, $\forall \epsilon > 0$,

$$\boldsymbol{P}\{X \geq (1-\epsilon)\mu\} < \left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{\mu}$$

Stronger (?) Chernoff Bound for Binomial

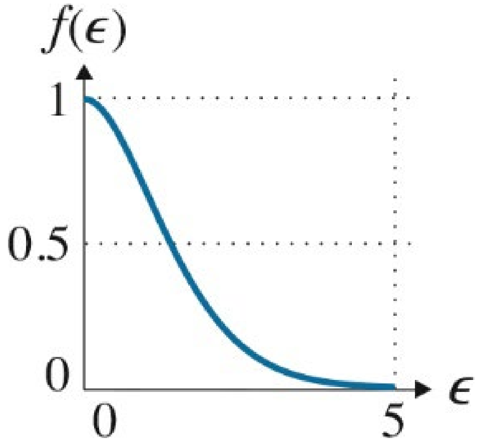
Theorem 18.6: (Sometimes stronger Chernoff Bound)

Let $X = \sum_{i=1}^{n} X_i$ where $X_i \sim Bernoulli(p_i)$ and $\mu = \boldsymbol{E}[X] = \sum_{i=1}^{n} p_i$. Then, $\forall \epsilon > 0$,

$$\boldsymbol{P}\{X \geq (1-\epsilon)\mu\} < \left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{\mu}$$

Plot of inner term:

$$f(\epsilon) = \frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}$$



Two observations:

1. $f(\epsilon) < 1$, so bound is exponentially decreasing.

2. Bound in Thm 18.6 is particularly strong when ϵ is high.

Comparison of Chernoff Bounds

Theorem 18.4: (Pretty bound)
Let $X \sim Binomial(n, p)$ where $\mu = \boldsymbol{E}[X] = np$. Then, for any $\delta > 0$,

$$\boldsymbol{P}\{X - np \geq \delta\} \leq e^{-2\delta^2/n}$$

Theorem 18.6: (Sometimes stronger bound)
Let $X = \sum_{i=1}^{n} X_i$ where $X_i \sim Bernoulli(p_i)$ and $\mu = \boldsymbol{E}[X] = \sum_{i=1}^{n} p_i$. Then, $\forall \epsilon > 0$,

$$\boldsymbol{P}\{X \geq (1+\epsilon)\mu\} < \left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{\mu}$$

Q: Which gives best bound on probability of getting $\geq \frac{3}{4}n$ heads, when flipping fair coin n times?

$$\boldsymbol{P}\left\{X \geq \frac{3n}{4}\right\} = \boldsymbol{P}\left\{X - \frac{n}{2} \geq \frac{n}{4}\right\}$$

$$\leq \boxed{e^{-\frac{n}{8}}}$$

This is the better bound!

$$\boldsymbol{P}\left\{X \geq \frac{3n}{4}\right\} = \boldsymbol{P}\left\{X \geq \left(1 + \frac{1}{2}\right) \cdot \frac{n}{2}\right\}$$

$$\leq \left(\frac{e^{0.5}}{1.5^{1.5}}\right)^{n/2} \approx \boxed{(1.54)^{-\frac{n}{8}}}$$

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$$\boldsymbol{P}\{X \geq (1+\epsilon)\mu\} < \left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{\mu}$$

Q: Which is the better bound on $\boldsymbol{P}\{X \geq 21\}$ if $p_i = p = \frac{1}{n}$?

$$\boldsymbol{P}\{X \geq 21\} = \boldsymbol{P}\{X - 1 \geq 20\} \leq e^{-\frac{2\cdot(20)^2}{n}} \leq e^{-\frac{800}{n}} \to \boxed{1}$$

$$\boldsymbol{P}\{X \geq 21\} = \boldsymbol{P}\{(X \geq (1+20)\cdot 1\} \leq \frac{e^{20}}{21^{21}} \approx \boxed{8.3 \cdot 10^{-20}}$$

Much better bound!

More general bound: Hoeffding's Inequality

Theorem 18.7: (Hoeffding's Inequality)

Let $X_1, X_2, \ldots, X_n$ be independent r.v.s, where $a_i \leq X_i \leq b_i, \ \forall i$.

Let:

$$X = \sum_{i=1}^{n} X_i$$

More general because X_i's don't have to be identically distributed

Then,

$$\boldsymbol{P}\{X - \boldsymbol{E}[X] \geq \delta\} \leq exp\left(-\frac{2\,\delta^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

$$\boldsymbol{P}\{X - \boldsymbol{E}[X] \leq -\delta\} \leq exp\left(-\frac{2\,\delta^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$