Chapter 1
Some Mathematical Basics

Review of Simple Series

$$S = 1 + x + x^2 + x^3 + \cdots + x^n$$

Q: What is S ?

Review of Simple Series

$$S = 1 + x + x^2 + x^3 + \cdots + x^n$$

$$\begin{aligned}
(1-x)S &= 1 + \cancel{x} + \cancel{x}^2 + \cancel{x}^3 + \cdots + \cancel{x}^n \\
&\quad - \cancel{x} - \cancel{x}^2 - \cancel{x}^3 + \cdots - \cancel{x}^n - x^{n+1} \\
&= 1 - x^{n+1}
\end{aligned}$$

$$S = \frac{1 - x^{n+1}}{1 - x} \qquad (\text{assuming } x \neq 1)$$

Review of Simple Series

$$S = 1 + x + x^2 + x^3 + \cdots, \qquad where\ |x| < 1$$

Q: What is S ?

Review of Simple Series

$$S = 1 + x + x^2 + x^3 + \cdots, \qquad where\ |x| < 1$$

$$S = \lim_{n\to\infty} (1 + x + x^2 + x^3 + \cdots + x^n)$$

$$= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x}$$

$$= \frac{1}{1 - x}$$ (because $|x| < 1$)

Review of Simple Series

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$$

Q: What is S ?

(Assume $x \neq 1$)

Review of Simple Series

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} \qquad (x \neq 1)$$

$$\begin{aligned} S &= \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots + x^n) \\ &= \frac{d}{dx}\left(\frac{1 - x^{n+1}}{1 - x}\right) \\ &= \frac{(1-x)\cdot(-(n+1)x^n) - (1 - x^{n+1})\cdot(-1)}{(1-x)^2} \\ &= \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} \end{aligned}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{g(x)^2}$$

Review of Simple Series

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad where\ |x| < 1$$

Q: What is S ?

Review of Simple Series

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad where\ |x| < 1$$

$$S = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots)$$

$$= \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{(1-x)^2}$$

Review of Double Integrals

Q: Derive: $\displaystyle\int_{y=0}^{y=\infty}\int_{x=0}^{x=y} e^{-y}\,dx\,dy$

Review of Double Integrals

Q: Derive: $\int_{y=0}^{y=\infty} \int_{x=0}^{x=y} e^{-y}\, dx dy$

$$\int_{y=0}^{y=\infty} \underbrace{\int_{x=0}^{x=y} e^{-y}\, dx}_{\text{Do inner integral first}} dy = \int_{y=0}^{y=\infty} xe^{-y}\Big|_{x=0}^{x=y} dy$$

$$= \int_{y=0}^{y=\infty} ye^{-y}\, dy$$

$$= 1 \qquad \text{(via integration by parts)}$$

Integration by parts

$$\int_{y=0}^{y=\infty} y\, e^{-y} dy = ?$$

$$\int_a^b u dv = uv \Big|_{y=a}^{y=b} - \int_a^b v du$$

$$\int_{y=0}^{y=\infty} \underbrace{y}_{u}\, \underbrace{e^{-y} dy}_{dv} = -ye^{-y} \Big|_{y=0}^{y=\infty} - \int_{y=0}^{y=\infty} (-e^{-y}) dy$$

$$= \ 0 - (-0) - e^{-y} \Big|_{y=0}^{y=\infty}$$

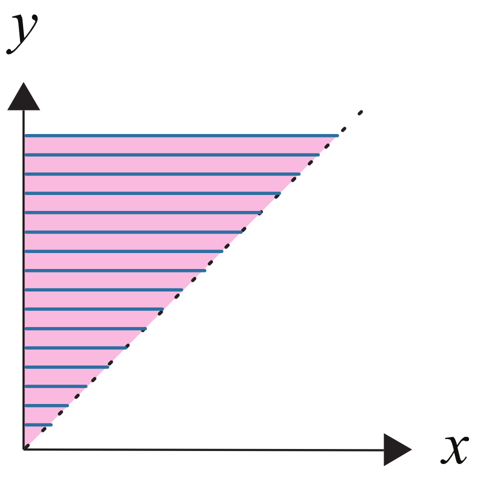
$$= \ 0 - (0 - 1)$$

$$= \ 1$$

Review of Double Integrals

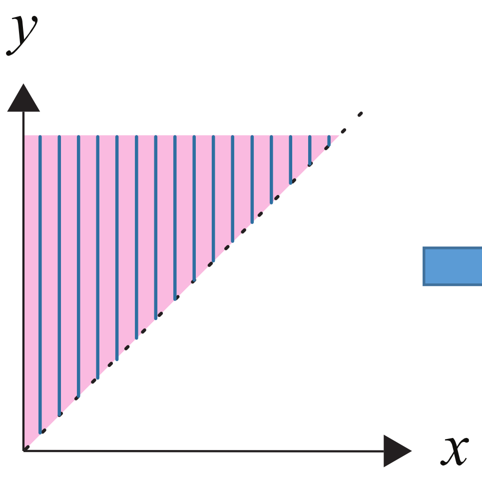
Q: Derive: $\int_{y=0}^{y=\infty} \int_{x=0}^{x=y} e^{-y}\, dx dy$ by first reversing the order of integration

Original integration space



y ranges from 0 to ∞.
For each particular value of y,
we let x range from 0 to y.

Equivalent integration space



$$\boxed{\int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} e^{-y}\, dy dx}$$

x ranges from 0 to ∞.
For each particular value of x,
we let y range from x to ∞.

Review of Double Integrals

Q: Derive: $\int_{y=0}^{y=\infty} \int_{x=0}^{x=y} e^{-y} dx dy$ by first reversing the order of integration

$$\int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} e^{-y} dy dx = \int_{x=0}^{x=\infty} -e^{-y}\Big|_{y=x}^{y=\infty} dx$$

$$= \int_{x=0}^{x=\infty} (0 + e^{-x}) dx$$

$$= -e^{-x}\Big|_{x=0}^{x=\infty} = 1$$

Fundamental Theorem of Calculus (FTC)

Theorem 1.8: (FTC) Let $f(t)$ be a continuous function defined on the interval $[a, b]$. Then, for any x, where $a < x < b$,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Furthermore, for any differentiable function $g(x)$,

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f\big(g(x)\big) \cdot g'(x)$$

Fundamental Theorem of Calculus (FTC)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$Box(x)$ is the area under $f(t)$ between $t = a$ and $t = x$.
We seek the rate
at which this area changes
for a small change in x.

Intuition:

Let $Box(x) = \int_a^x f(t)dt$

$$\frac{d}{dx}Box(x) = \lim_{\Delta\to 0}\frac{Box(x+\Delta) - Box(x)}{\Delta}$$

$$= \lim_{\Delta\to 0}\frac{\int_a^{x+\Delta} f(t)dt - \int_a^x f(t)dt}{\Delta}$$

$$= \lim_{\Delta\to 0}\frac{\int_x^{x+\Delta} f(t)dt}{\Delta} \approx \lim_{\Delta\to 0}\frac{f(x)\cdot\Delta}{\Delta} = f(x)$$

Since
$f(x) \approx f(x+\Delta)$

Fundamental Theorem of Calculus (FTC)

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f\big(g(x)\big)\cdot g'(x)$$

$$Box(x) = \int_a^{g(x)} f(t)dt$$

$$\frac{d}{dx}Box(x) = \lim_{\Delta\to 0}\frac{Box(x+\Delta) - Box(x)}{\Delta} = \lim_{\Delta\to 0}\frac{\int_a^{g(x+\Delta)} f(t)dt - \int_a^{g(x)} f(t)dt}{\Delta}$$

$$= \lim_{\Delta\to 0}\frac{\int_{g(x)}^{g(x+\Delta)} f(t)dt}{\Delta}$$

$$\approx \lim_{\Delta\to 0}\frac{f(g(x))\cdot(\mathrm{g}(x+\Delta) - g(x))}{\Delta}$$

$$= f\big(g(x)\big)\cdot\lim_{\Delta\to 0}\frac{\mathrm{g}(x+\Delta) - g(x)}{\Delta} = f\big(g(x)\big)\cdot g'(x)$$

Understanding e

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Q: How should we interpret e ?

A: Suppose you have m dollars. You are promised 100% interest yearly.

- If interest is compounded annually, we have $2m$ dollars after 1 year.
- If interest is compounded every 6 mo, we have $\left(1+\frac{1}{2}\right)^2 m = \frac{9}{4}m$ dollars after 1 year.
- If interest is compounded every 4 mo, we have $\left(1+\frac{1}{3}\right)^3 m = \frac{64}{27}m$ dollars after 1 year.
- If interest is compounded continuously, we have $\left(1+\frac{1}{n}\right)^n m \to e\cdot m$ dollars after 1 year.

Understanding e^x

Claim: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

Proof:

Let $a = \frac{n}{x}$. As $n \to \infty$, we have $a \to \infty$.

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{a\to\infty}\left(1+\frac{1}{a}\right)^{ax} = \lim_{a\to\infty}\left(\left(1+\frac{1}{a}\right)^a\right)^x = e^x$$

Review of Taylor/Maclaurin series

Let $0 < x < 1$.

Q: Which is bigger, $1 + x$ or e^x?

Q: Which is bigger, $1 - x$ or e^{-x}?

A: Recall, we can express $f(x)$ via its Taylor series expansion around $x = 0$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\Rightarrow \begin{cases} \mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \\ \mathrm{e}^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \end{cases} \Rightarrow \begin{cases} \mathrm{e}^x > 1 + x \\ \mathrm{e}^{-x} > 1 - x \end{cases}$$

Harmonic Number

Defn: The $\boldsymbol{n}$**th harmonic number** is denoted by H_n, where

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Harmonic Number Theorem:

$$\ln(n+1) < H_n < 1 + \ln(n)$$

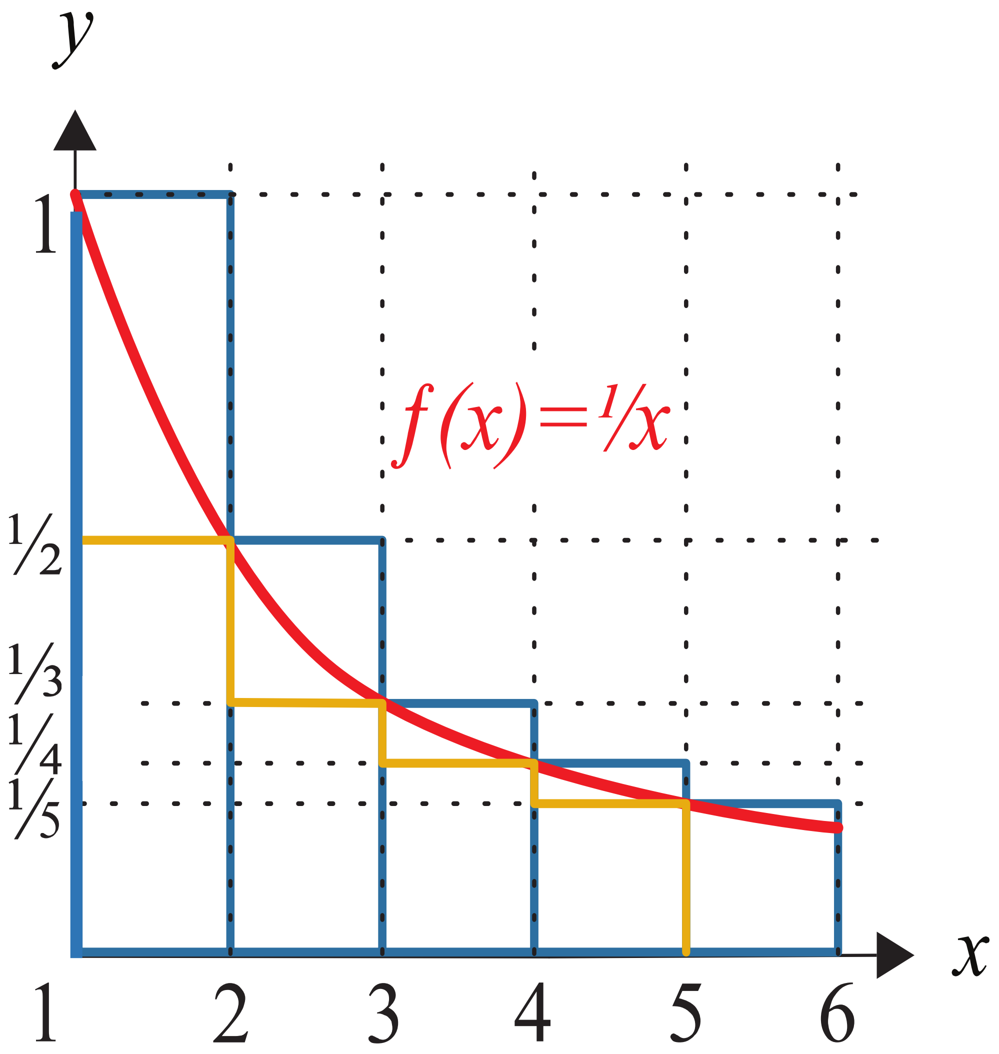
We will prove this next…

Cor:

$$H_n \approx \ln(n) \text{ for high } n$$

$$\lim_{n\to\infty} H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

Proof of Harmonic Number Theorem



Area under red curve $<$ Area in blue rectangles

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$$

Area in yellow rectangles $<$ Area under red curve up to yellow end

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n - 1 < \int_1^{n} \frac{1}{x}\,dx = \ln(n)$$

Counting: Combinations versus Permutations

Suppose Baskin-Robins has n flavors of ice cream. Your cone has $k < n$ scoops. How many different cones can you make if each flavor can only be used once?

Q: Answer the question if the order of the flavors matters.

Permutations

Q: Answer the question if the order of the flavors doesn’t matter.

Combinations

Counting: Combinations versus Permutations

Suppose Baskin-Robins has n flavors of ice cream. Your cone has $k < n$ scoops. How many different cones can you make if each flavor can only be used once?

Q: Answer the question if the order of the flavors matters.

Permutations

$$n\,(n-1)(n-2)\cdots\big(n-(k-1)\big) = \frac{n!}{(n-k)!}$$

Combinations

Q: Answer the question if the order of the flavors doesn't matter.

$$ABC = ACB = BCA = BAC = CBA = CAB$$

so divide #permutations by $k!$

$$\frac{n!}{(n-k)!\,k!} = \binom{n}{k} = \text{"}n\ choose\ k\text{"}$$

Sums of combinations

Q: Evaluate: $S_1 = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

Sums of combinations

Q: Evaluate: $S_1 = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

All subsets of size 0

All subsets of size 1

All subsets of size 2

All subsets of size n

$S_1 =$ total number of subsets of n elements $= 2^n$

Each element is either "in" or "out"

Sums of combinations

Q: Evaluate: $S_2 = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n$

A: This is the binomial expansion of $(x + y)^n$

Sums of combinations

Q: Evaluate: $S_3 = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$

A: This is the binomial expansion of $(x+1)^n$

Some useful bounds

Theorem 1.12:

$$\left(\frac{n}{k}\right)^k < \binom{n}{k} < \left(\frac{ne}{k}\right)^k$$

See book for proof!

Theorem (Stirling):

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < e\sqrt{n}\left(\frac{n}{e}\right)^n$$

Asymptotic notation: big-O

Asymptotic notation is a way to summarize rate at which function $f(n)$ grows with n.

- $O\big(g(n)\big)$ is the set of functions that grow no faster than $g(n)$.
  - $3n,\ \sqrt{n},\ \lg\lg(n) \in O(n)$
  - $n^2,\ n\lg(n)\ \notin O(n)$

we write
$3n = O(n)$

Defn: We say that $f(n) = O(g(n))$, pronounced as $f(n)$ is **big-O** of $g(n)$,
if there exists a constant $c \geq 0$, s.t.,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$

Asymptotic notation: little-o

- $o(g(n))$ is the set of functions that grow strictly slower than $g(n)$.

  - $2\sqrt{n},\ 15 \lg\lg(n) \in o(n)$
  - $\frac{n}{2},\ n \lg\lg(n),\ n^3 \notin o(n)$

we write
$\sqrt{n} = o(n)$

Defn: We say that $f(n) = o(g(n))$, pronounced as $f(n)$ is **little-o** of $g(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Cor: We say that $f(n) = o(1)$ if $\lim_{n\to\infty} f(n) = 0$.

Asymptotic notation: big-Omega

- $\Omega(g(n))$ is the set of functions that grow no slower than $g(n)$.

  - $\frac{n}{2}, n\lg n,\ n^3 \in \Omega(n)$
  - $\sqrt{n},\ 15\ \lg\lg(n),\ 25 \notin \Omega(n)$

we write
$n^2 = \Omega(n)$

<u>Defn</u>: We say that $f(n) = \Omega(g(n))$, pronounced as $f(n)$ is **big-Omega** of $g(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

Asymptotic notation: little-omega

- $\omega\big(g(n)\big)$ is the set of functions that grow strictly faster than $g(n)$.

  - $\frac{n^2}{2}, n\lg n,\ n^3 \in \omega(n)$
  - $n,\ 15\sqrt{n},\ \ 25n \notin \omega(n)$

we write
$n^2 = \omega(n)$

<u>Defn</u>: We say that $f(n) = \omega(g(n))$, pronounced as $f(n)$ is **little-omega** of $g(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Asymptotic notation: big-Theta

- $\Theta(g(n))$ is the set of functions that grow at the same rate as $g(n)$.

  - $15n,\ \frac{n}{2} \in \Theta(n)$
  - $n\lg n,\ \ 15\sqrt{n},\ \ n^2 \notin \Theta(n)$

we write
$2n = \Theta(n)$

Defn: We say that $f(n) = \Theta(g(n))$, pronounced as $f(n)$ is **Theta** of $g(n)$, if

$$f(n) = O(g(n)) \quad \textbf{and} \quad f(n) = \Omega(g(n))$$