$v^* = \text{opt}$

Convex program duality

f, g_i convex

- min $f(x)$ s.t.

  $Ax = b$

  $g_i(x) \le 0 \qquad i \in I$

$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \end{pmatrix}$

for feasible x

$y^T(Ax-b) = 0$

$z^T g(x) \le 0$ if $z \ge 0$

$f(x) \ge f(x) + y^T(Ax-b) + z^T g(x)$

$\equiv L(x,y,z)$

$v^* = \inf_{\text{feas } x} f(x) \ge \inf_{\text{feas } x} L(x,y,z) \ge \inf_x L(x,y,z) \equiv L(y,z)$

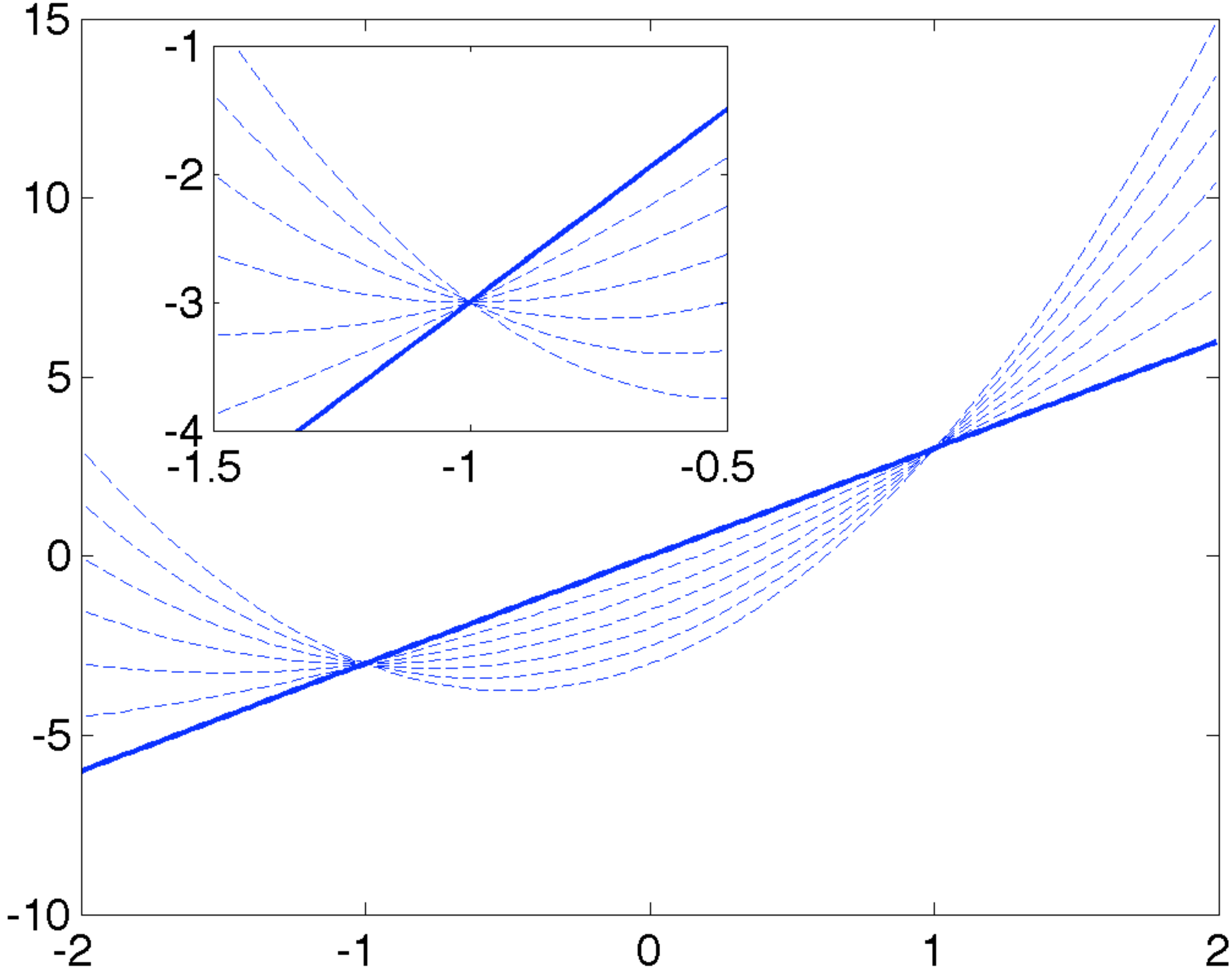
Dual prob: $\max L(y,z)$ s.t. $z \ge 0$

Properties

- Weak duality
- L(y, z) is closed, concave

Duality example

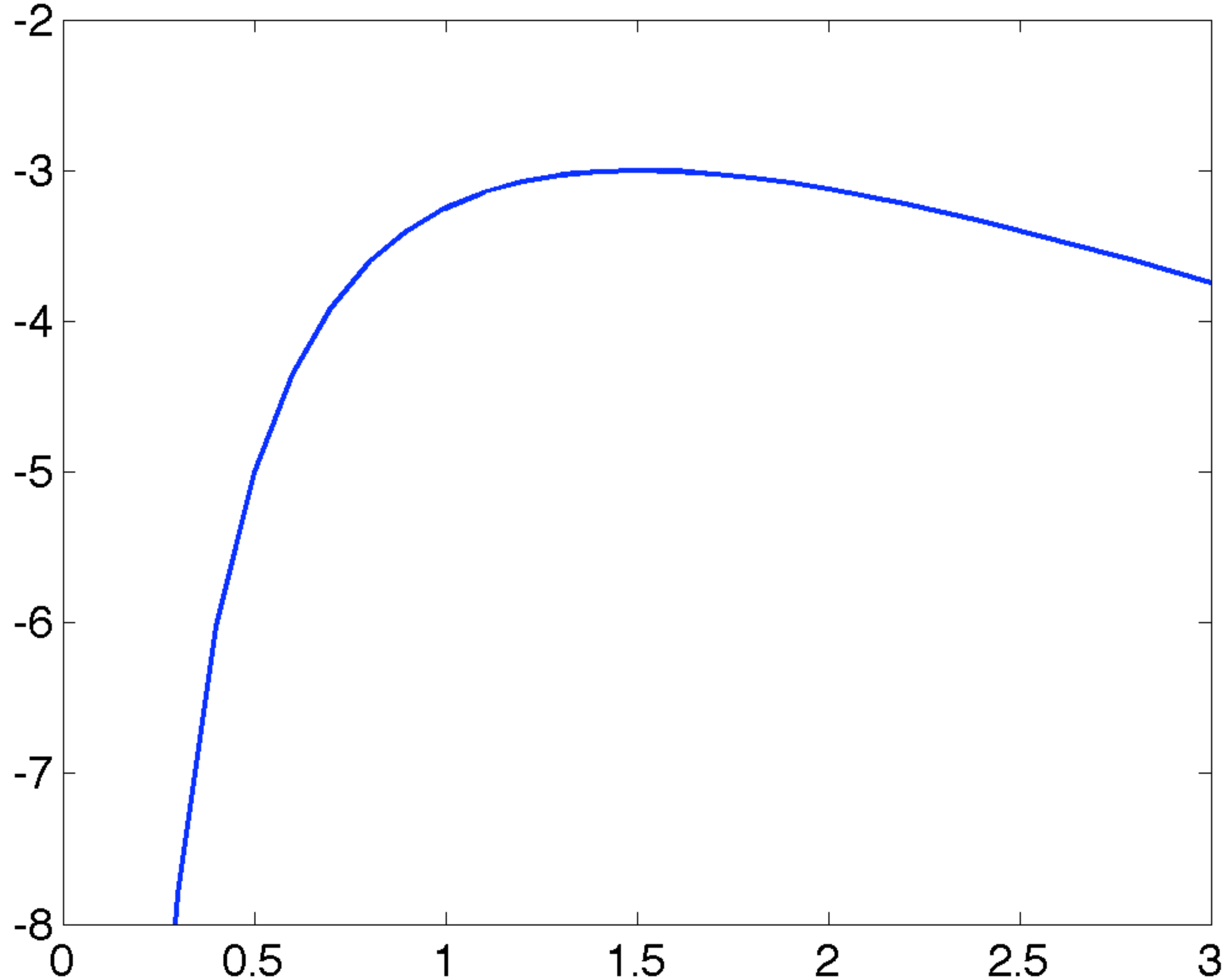
- min $3x$ s.t. $x^2 \leq 1$
- $L(x, y) = 3x + y(x^2 - 1)$


15
10
5
0
-5
-10
-2
-1
0
1
2
-1
-2
-3
-4
-1.5
-1
-0.5

Dual function

- $L(y) = \inf_x L(x,y) = \inf_x 3x + y(x^2 - 1)$

Dual function



Duality w/ linear constraints

- min $f(x)$ s.t. $Ax = b$, $Cx \le d$
- $L(x, y, z) =$
- $L(y, z) = \inf_x$

- Dual problem

CP duality w/ cone constraints

- min f(x) s.t.

  $A_0 x + b_0 = 0$

  $A_i x + b_i \in K \quad i \in I$

- Dual:

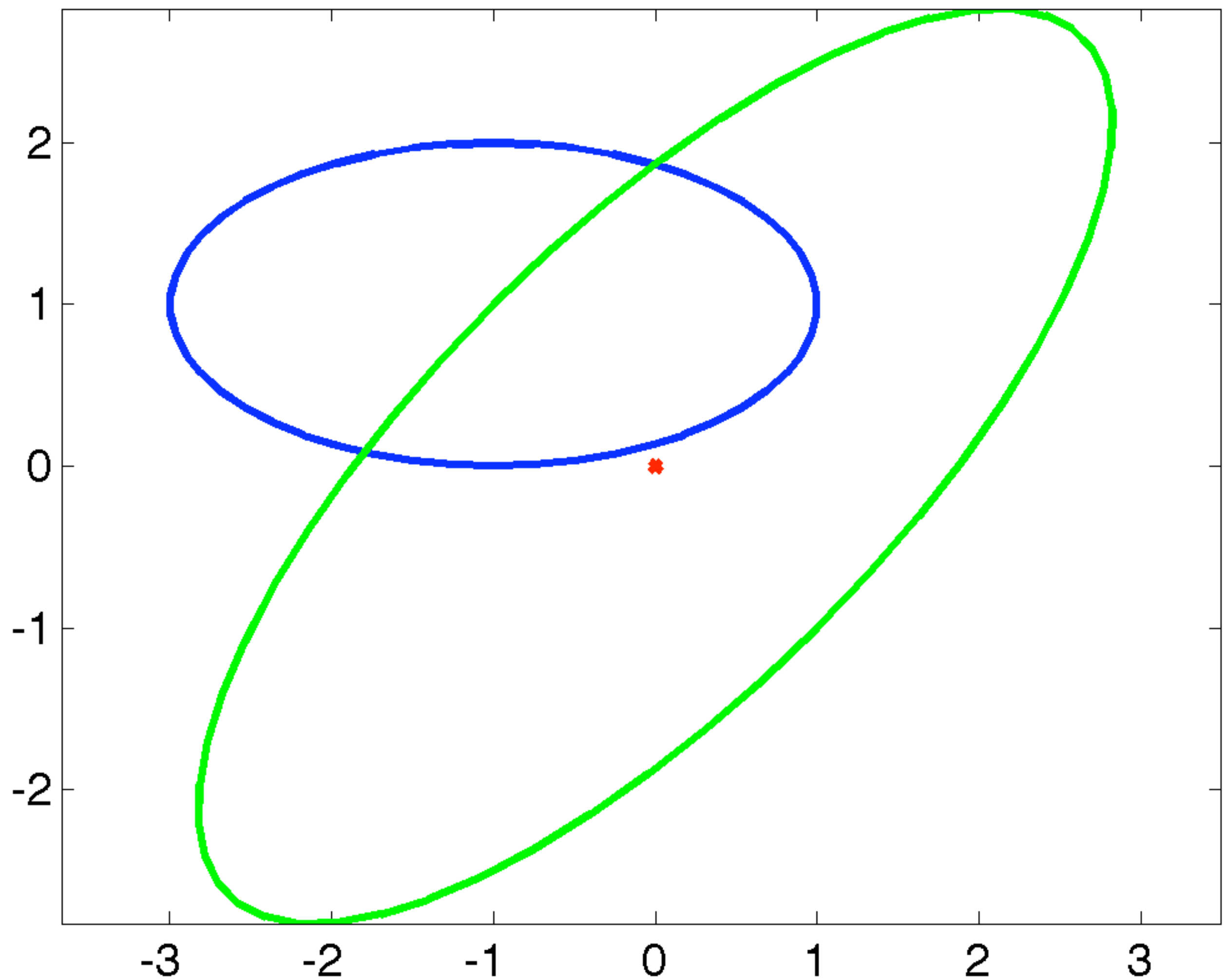
Ex: 2nd order cone program

$A_1 =$

$b_1 =$

$A_2 =$

$b_2 =$



Strong duality

- $v^* = \inf_x f(x)$ s.t. $Ax=b$, $g(x) \le 0$

  $L(x, y, z) =$

  $L(y, z) =$

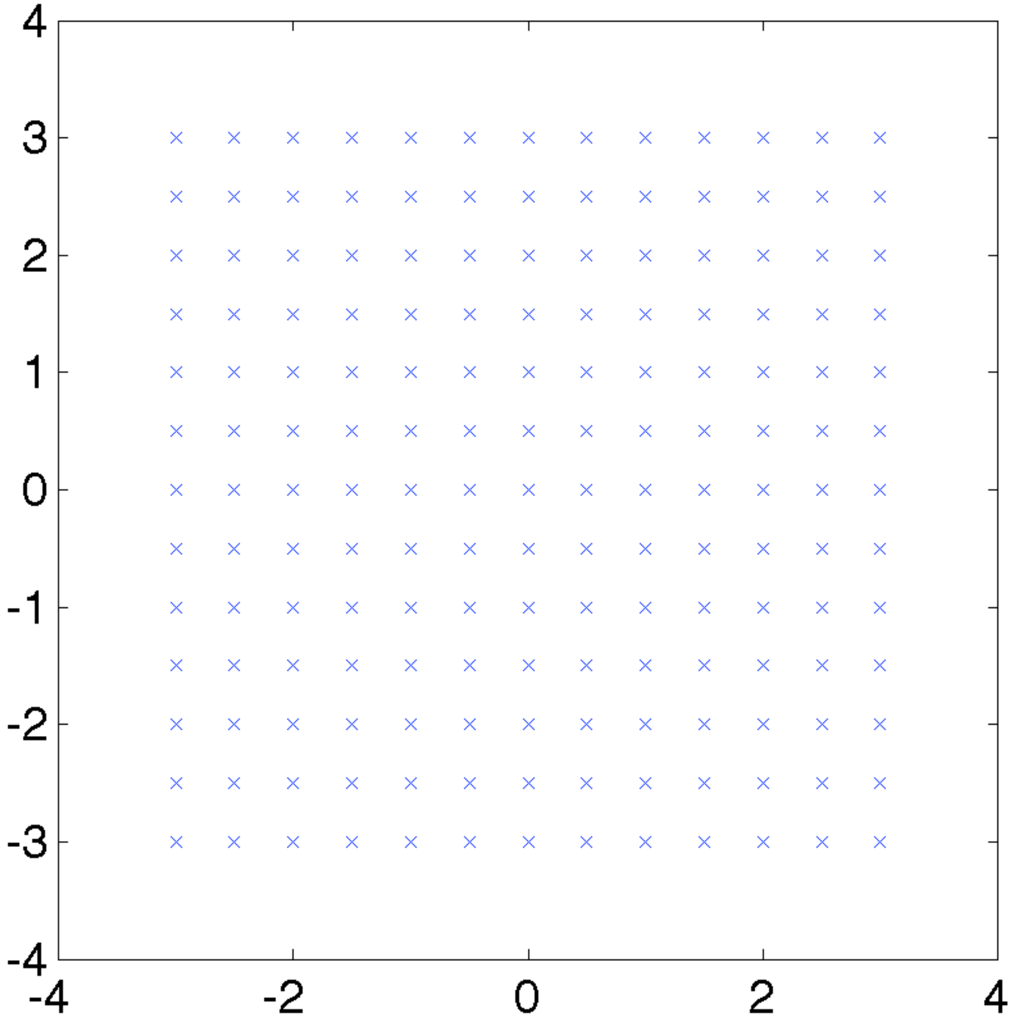
- $d^* = \sup_{yz} L(y, z)$ s.t. $z \ge 0$
- Strong duality:
- Slater's condition:
  –
  –

Slater's condition, part II

- If $g_i(x)$ affine, only need

- E.g., for $\inf_x f(x)$ s.t. $Ax = b$, $Cx \leq d$

Example: maxent



Example: maxent

- Maxent problem:

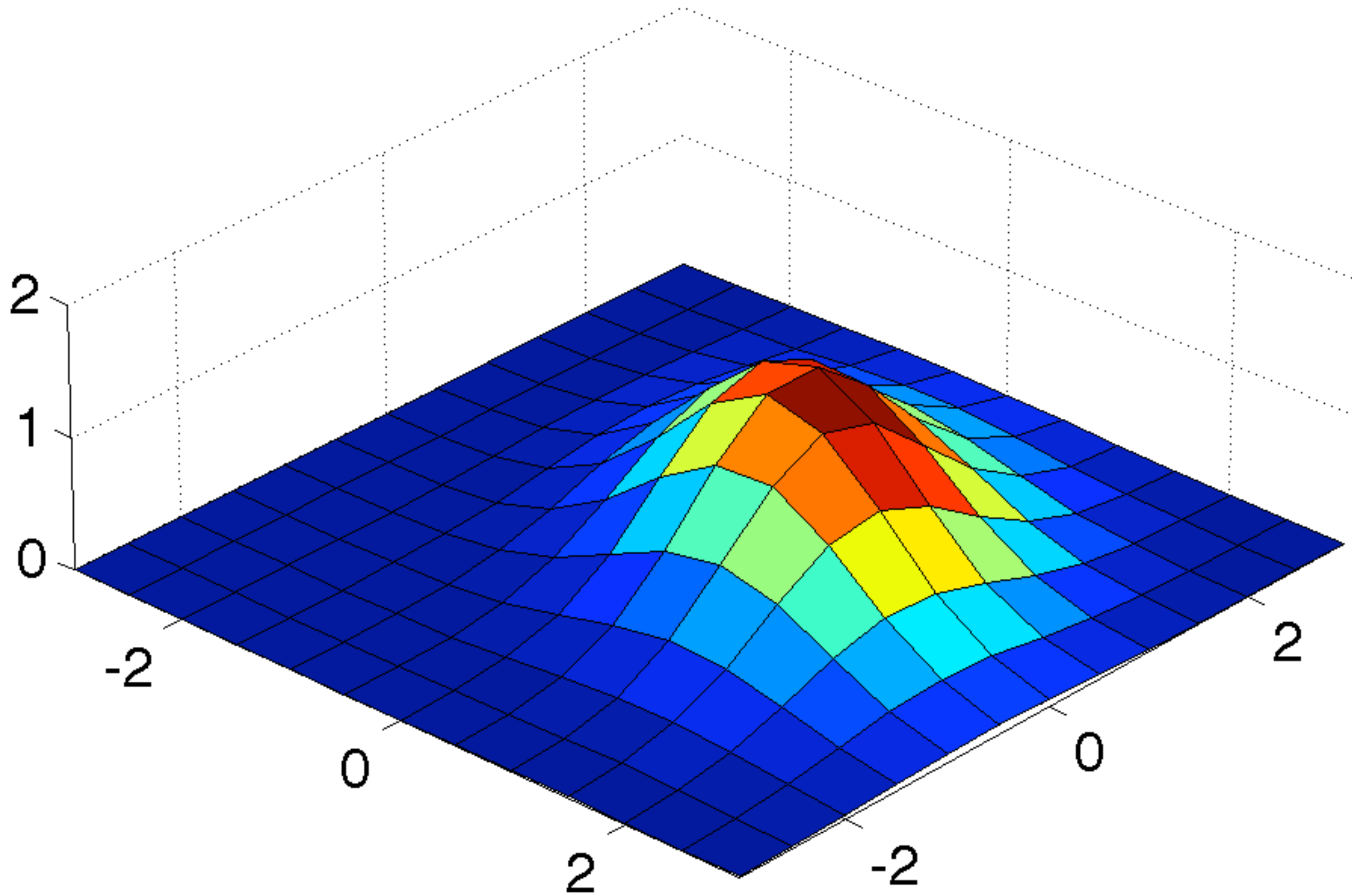
  max H(p) s.t. T’p = b

  H(p) =

  H’(p) =

- Slater’s condition:

Example of maxent solution



Dual of maxent

- max $H(p)$ s.t. $T^T p = b$
- $L(p, y) =$
- $L(y) = \inf_p L(p, y) =$

Is Slater necessary?

- $\min_x \; x^T A x + 2b^T x \;$ s.t. $\; ||x||^2 \leq 1$

Slater's condition: proof

- $v^* = \inf_x f(x)$ s.t. $Ax = b$, $g(x) \le 0$

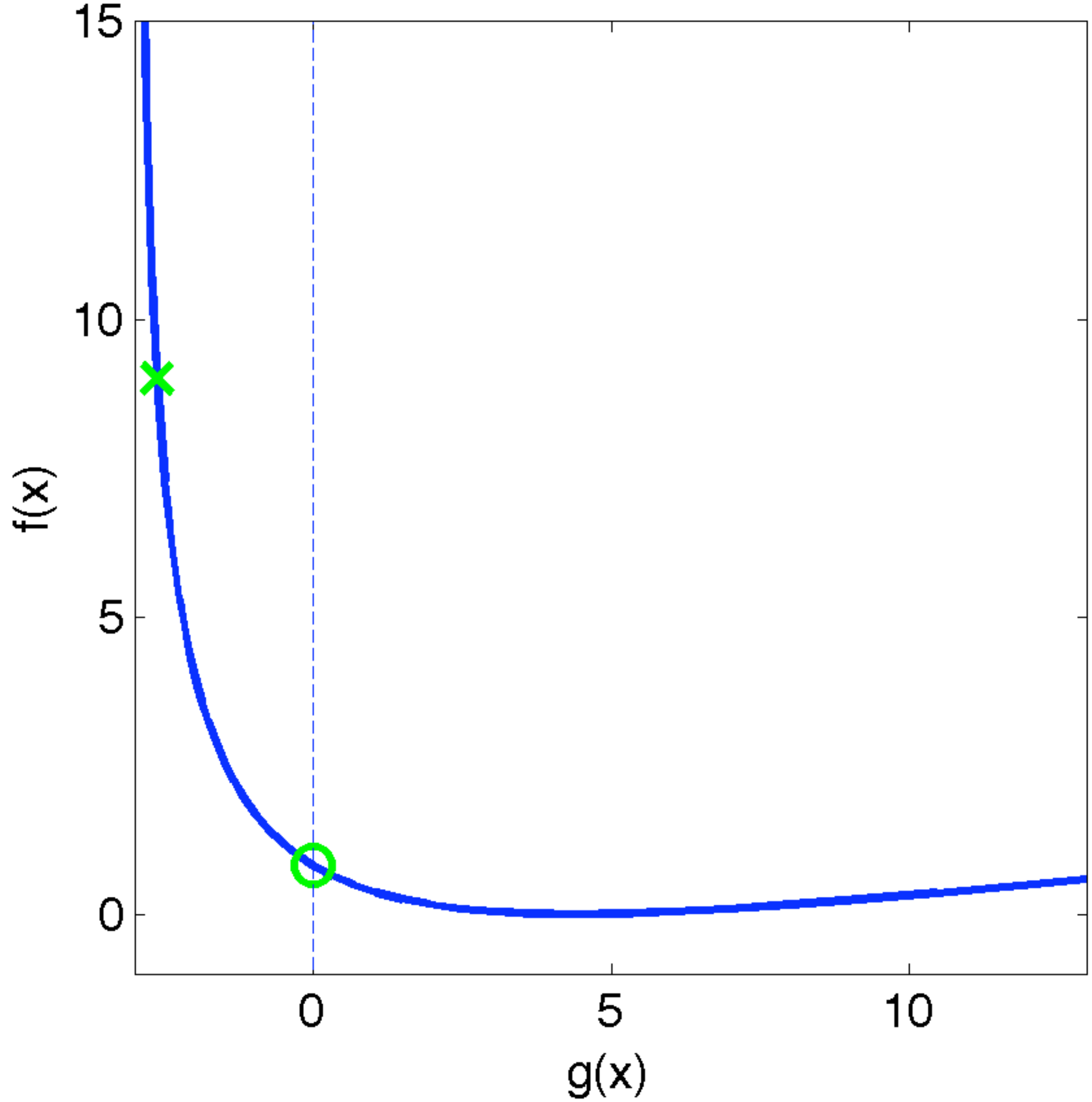
  e.g., $\inf x^2$ s.t. $e^{x+2} - 3 \le 0$

- A =

  e.g., A =

Picture of set A

L(y,z) =



Nonconvex example

Interpretations

$$L(x, y, z) = f(x) + y^T(Ax - b) + z^T g(x)$$

- Prices or sensitivity analysis

- Certificate of optimality

Optimality conditions

- $L(x, y, z) = f(x) + y^T(Ax-b) + z^Tg(x)$
- Suppose strong duality, (x, y, z) optimal

Optimality conditions

- $L(x, y, z) = f(x) + y^T(Ax-b) + z^T g(x)$
- Suppose (x, y, z) satisfy KKT:

$$Ax = b \qquad g(x) \le 0$$

$$z \ge 0 \qquad z^T g(x) = 0$$

$$0 \in \partial f(x) + A^T y + \textstyle\sum_i z_i \, \partial g_i(x)$$

Using KKT

- Can often use KKT to go from primal to dual optimum (or vice versa)
- E.g., in SVM:

  $\alpha_i > 0 \Longleftrightarrow y_i(x_i^T w + b) = 1$
- Means $b = y_i - x_i^T w$ for any such i
  - typically, average a few in case of roundoff