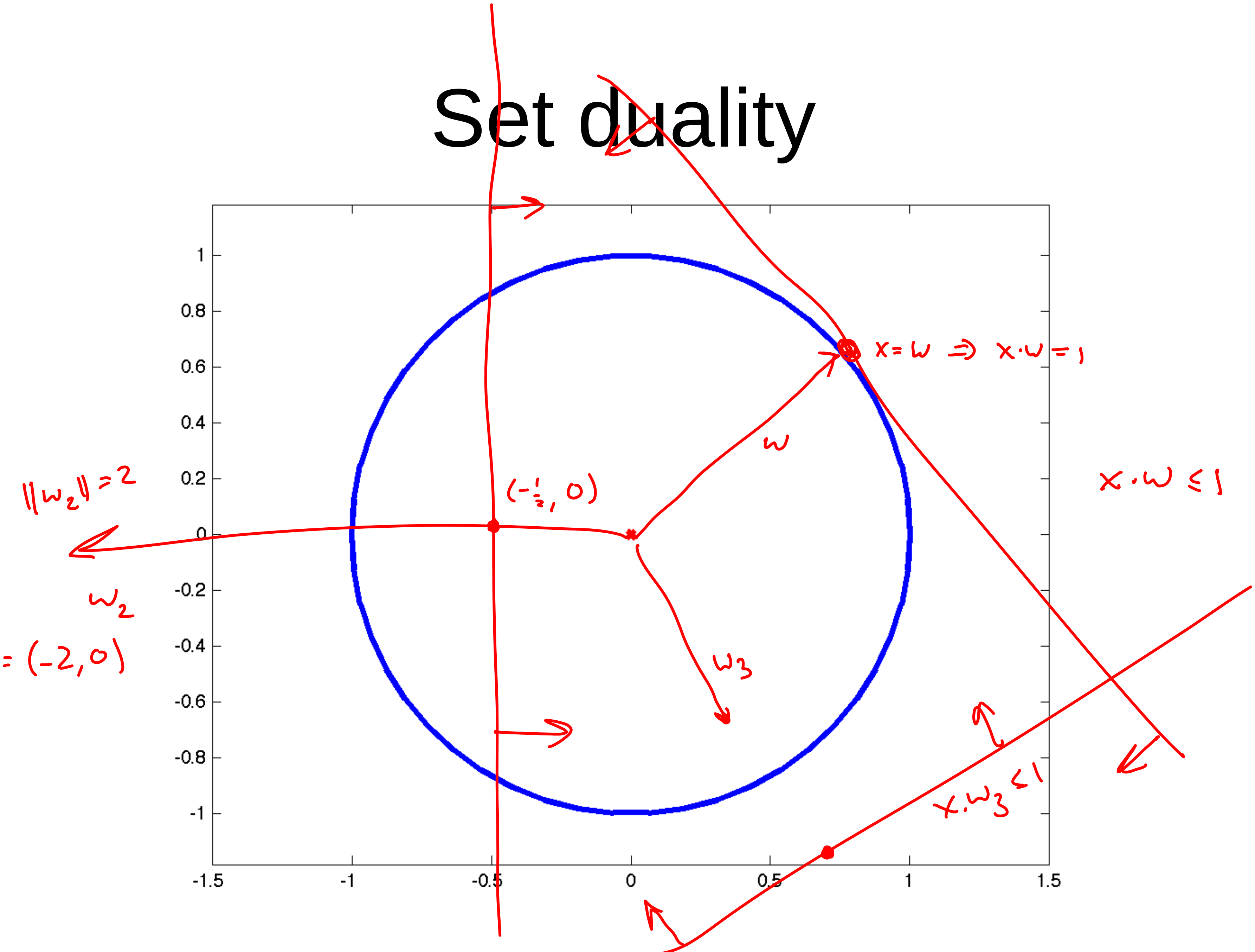
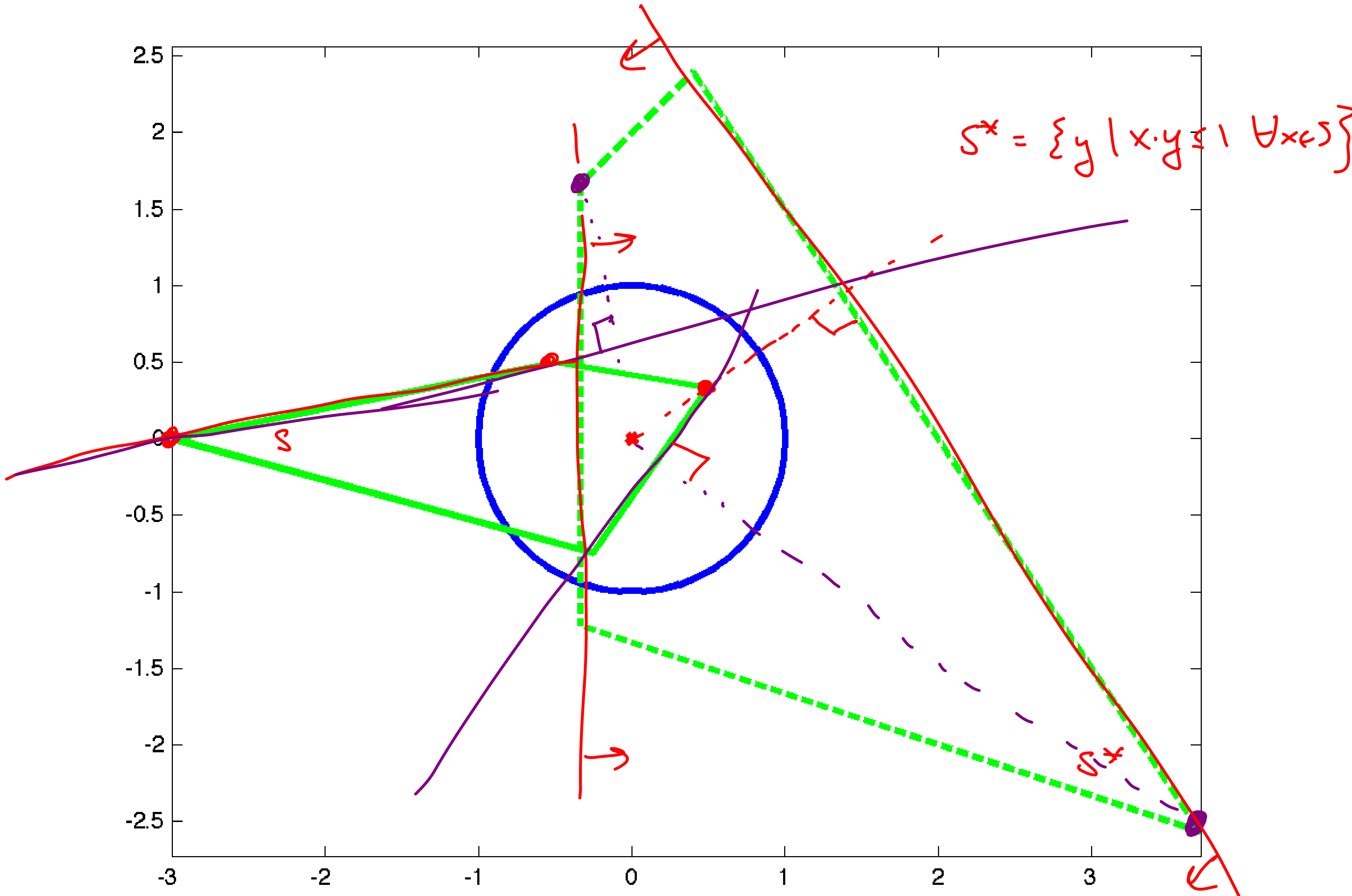
Review of duality so far

- LP/QP duality, cone duality, set duality
- All are **halfspace bounds**
  - on a cone
  - on a set
  - on objective of LP/QP

Set duality



Set duality



$S^* = \{y \mid x \cdot y \le 1 \ \forall x \in S\}$
S
S^*
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3

LP/QP objective

min z s.t.

a $z \geq x - 1$

b $z \geq 3 - 2x$

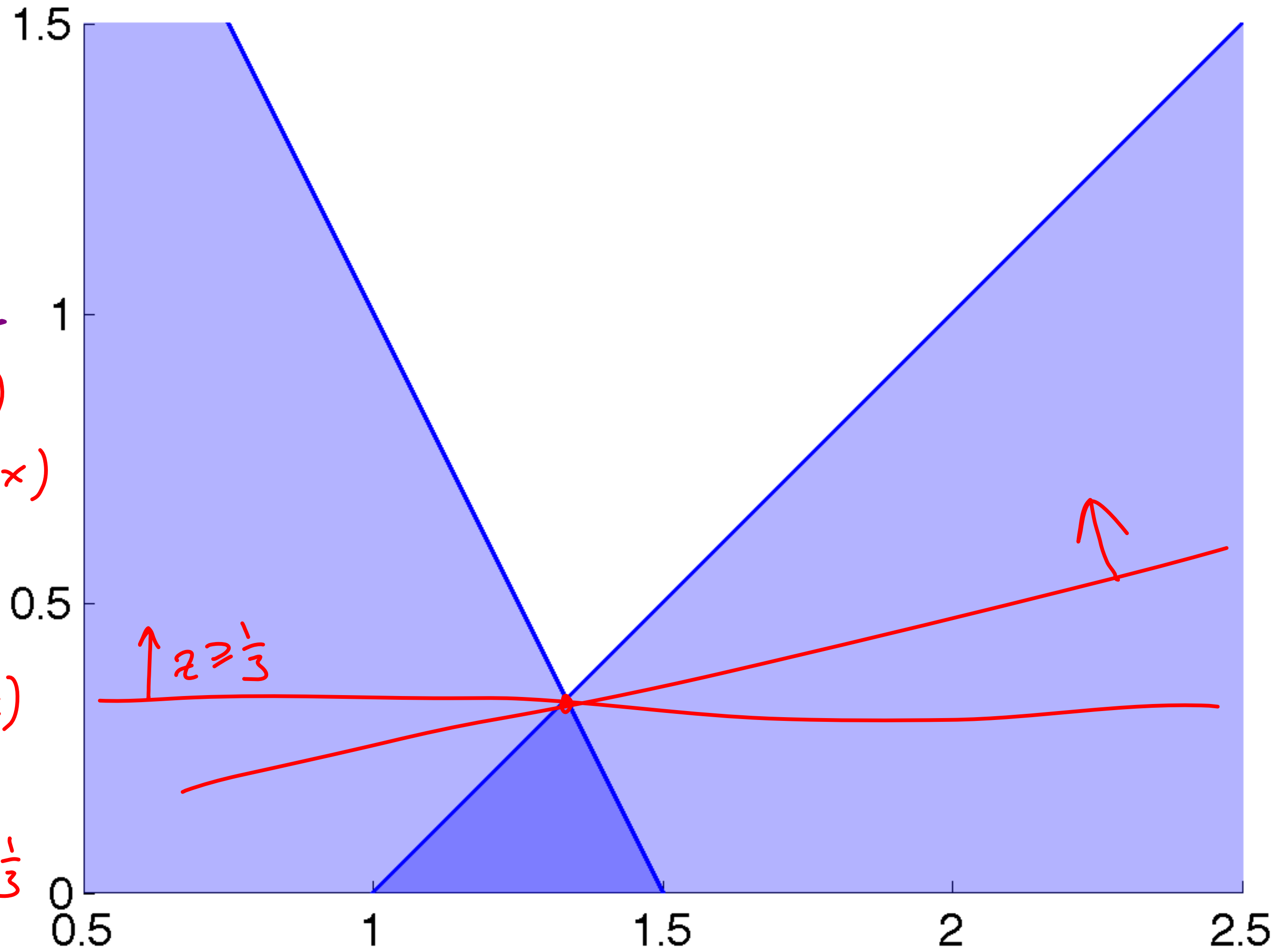
$az + bz \geq a(x-1) + b(3-2x)$

if $a + b = 1$

$z \geq a(x-1) + b(3-2x)$

if $a = 2/3$ $b = 1/3$

$z \geq -\frac{2}{3} + 1 = \frac{1}{3}$



Dual functions

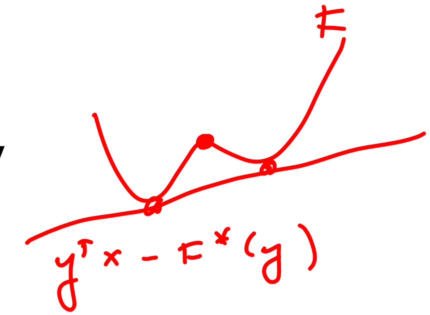
- Arbitrary function F(x)
- Dual is F*(y) = $\sup_x \left(x \cdot y - F(x) \right)$

$\partial F = x$

- For example: F(x) = $\underline{x^T x/2}$
- F*(y) = $\sup_x \left(x \cdot y - x \cdot x/2 \right) = y \cdot y - y \cdot y/2 = \frac{y \cdot y}{2}$

$\hookrightarrow \partial F^* = y \qquad 0 = \frac{d}{dx} = y - x \Rightarrow x = y$

Fenchel's inequality

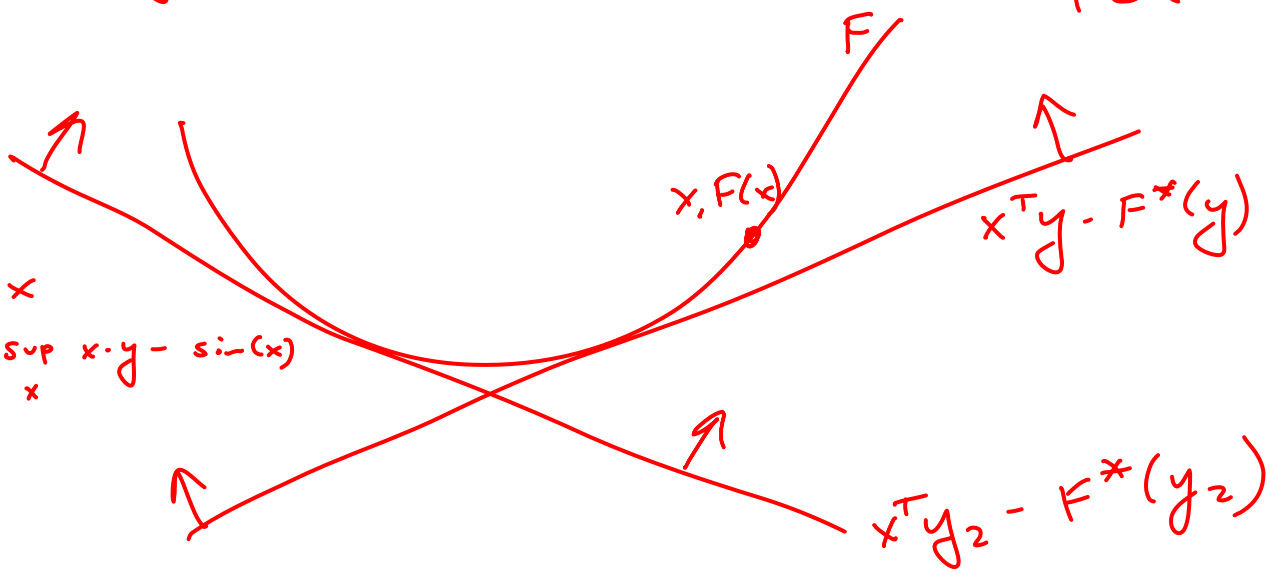


$F(x) \geq x^T y - F^*(y)$

- $F^*(y) = \sup_x [x^T y - F(x)]$ $\geq x^T y - F(x)$

$\forall x, y.\ F^*(y) + F(x) - x^T y \geq 0$

Fenchel's ineq.



$F(x) = \sin x$

$F^*(y) = \sup_x x \cdot y - \sin(x)$

$y, F^*(y)$ → halfspace bound on $F(x)$

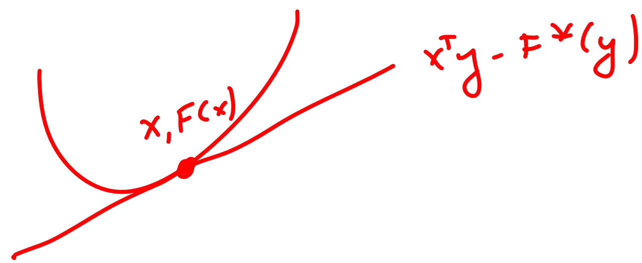
Duality and subgradients

Choose x, y

- Suppose $F(x) + F^*(y) - x^T y = 0$

$$F(x) = x^T y - F^*(y)$$

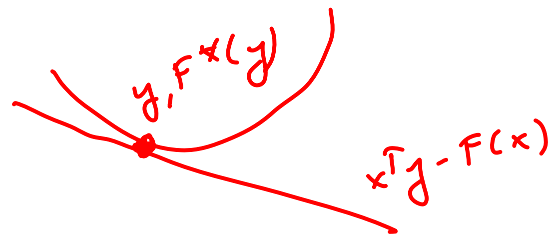
$$\iff y \in \partial F(x)$$



defn of generalized inverse

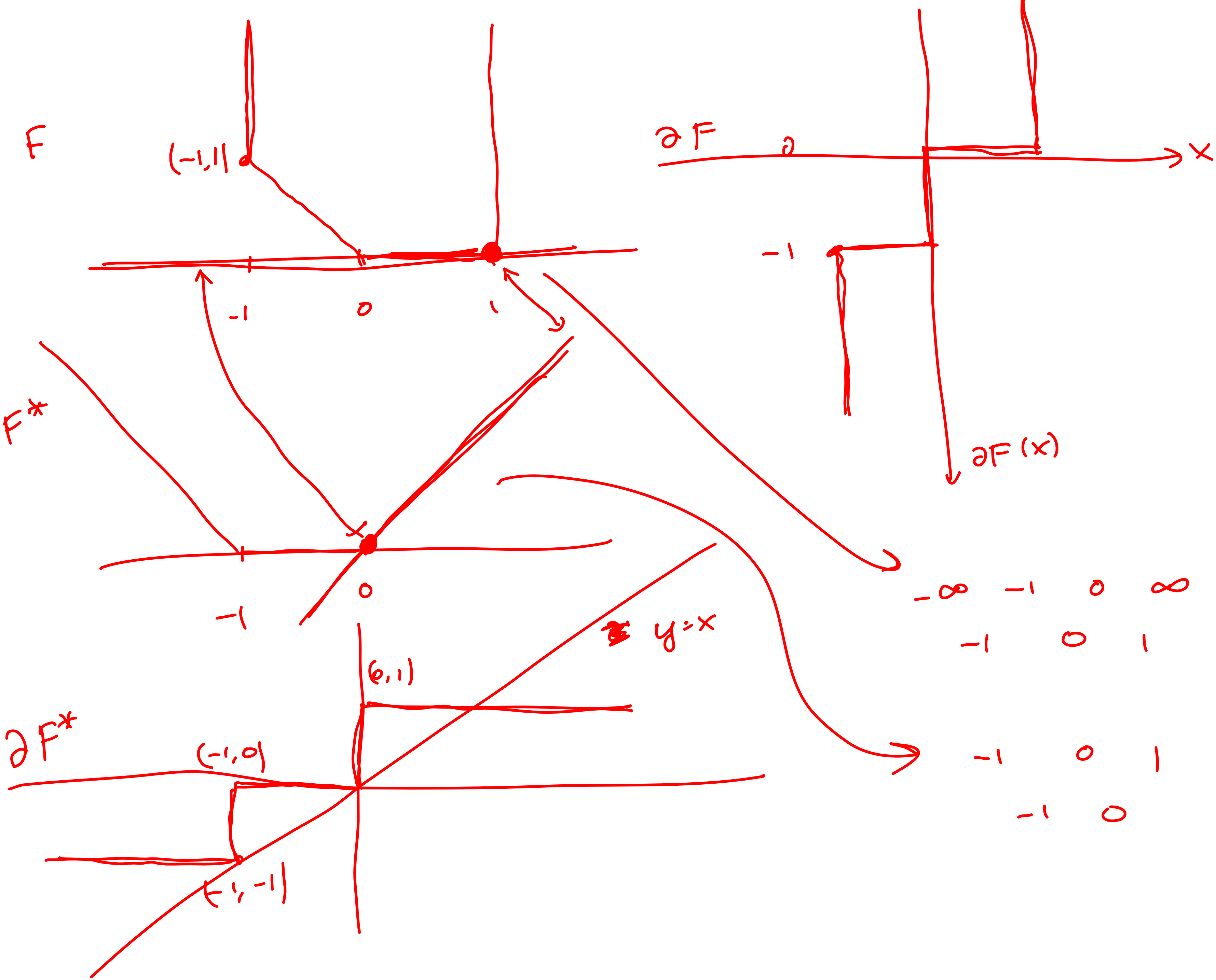
$$F^*(y) = x^T y - F(x)$$

$$\iff x \in \partial F^*(y)$$

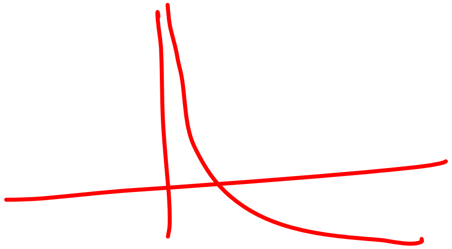


$\partial F = (\partial F^*)^{-1}$ if inverse exists

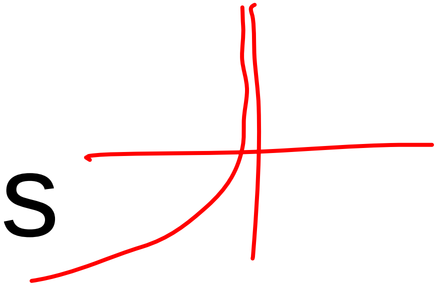
$\partial F = (\partial F^*)^{\text{gen.} -1}$ always — if F convex



F
(-1,1)
-1
0
1
∂F
∂
x
-1
∂F(x)
F*
-1
0
y=x
(0,1)
∂F*
(-1,0)
(-1,-1)
-∞ -1 0 ∞
-1 0 1
-1 0 1
-1 0



Duality examples



- $F(x) = -1/2 - \ln(x)$

$F^*(y) = \sup_x \left(yx - F(x)\right) = \sup_x \left(yx + \frac{1}{2} + \ln x\right)$

$0 = \frac{d}{dx}(\sim) = y + 1/x \Rightarrow x = -1/y$

$F^*(y) = y \cdot \left(-\frac{1}{y}\right) + \frac{1}{2} + \ln -\frac{1}{y}$

$= -1 + \frac{1}{2} - \ln(-y)$

$= -\frac{1}{2} - \ln(-y)$

- $F(x) = e^x$

$F^*(y) = \sup_x \left(xy - e^x\right) = y \cdot \ln y - y$

$0 = \frac{d}{dx} \sim = y - e^x \Rightarrow \ln y = x$

- $F(x) = x \ln(x) - x$

$F^*(y) = e^y$

More examples

$Q = Q^T$

- $F(x) = x^TQx/2 + c^Tx$, Q psd:

$F^*(y) = \sup_x \; y^Tx - x^TQx/2 - c^Tx$

$0 = \frac{d}{dx} \sim = y - Qx - c \Rightarrow x = Q^{-1}(y-c) \quad R = Q^{-1}$

$F^*(y) = y^TR(y-c) - \frac{1}{2}(y-c)^T \cancel{R^T}\,\cancel{Q}\,R(y-c) - c^TR(y-c)$

$= \frac{1}{2}(y-c)^TR(y-c)$

- $F(X) = -\ln|X|$, X psd:

$F^*(Y) = \sup_X \left(\mathrm{tr}(X^TY) + \ln|X|\right) = \left(\mathrm{tr}(-Y^{-1}Y) + \ln|-Y^{-1}|\right)$

$= -n - \ln|-Y|$

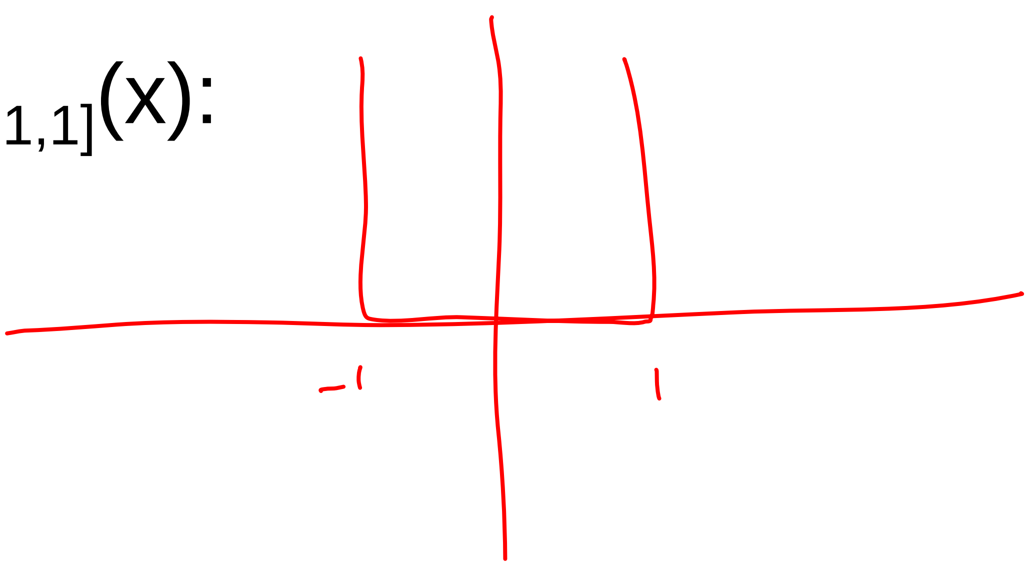
$0 = \frac{d}{dX} = Y + X^{-1} \Rightarrow X = -Y^{-1}$

Indicator functions

- Recall: for a set S,

  $$I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases}$$

- E.g., $I_{[-1,1]}(x)$:



Duals of indicators

- $I_{\{a\}}(x)$, point a: $I_a^*(y) = \sup_x \left(x^T y - I_a(x)\right) = a^T y$

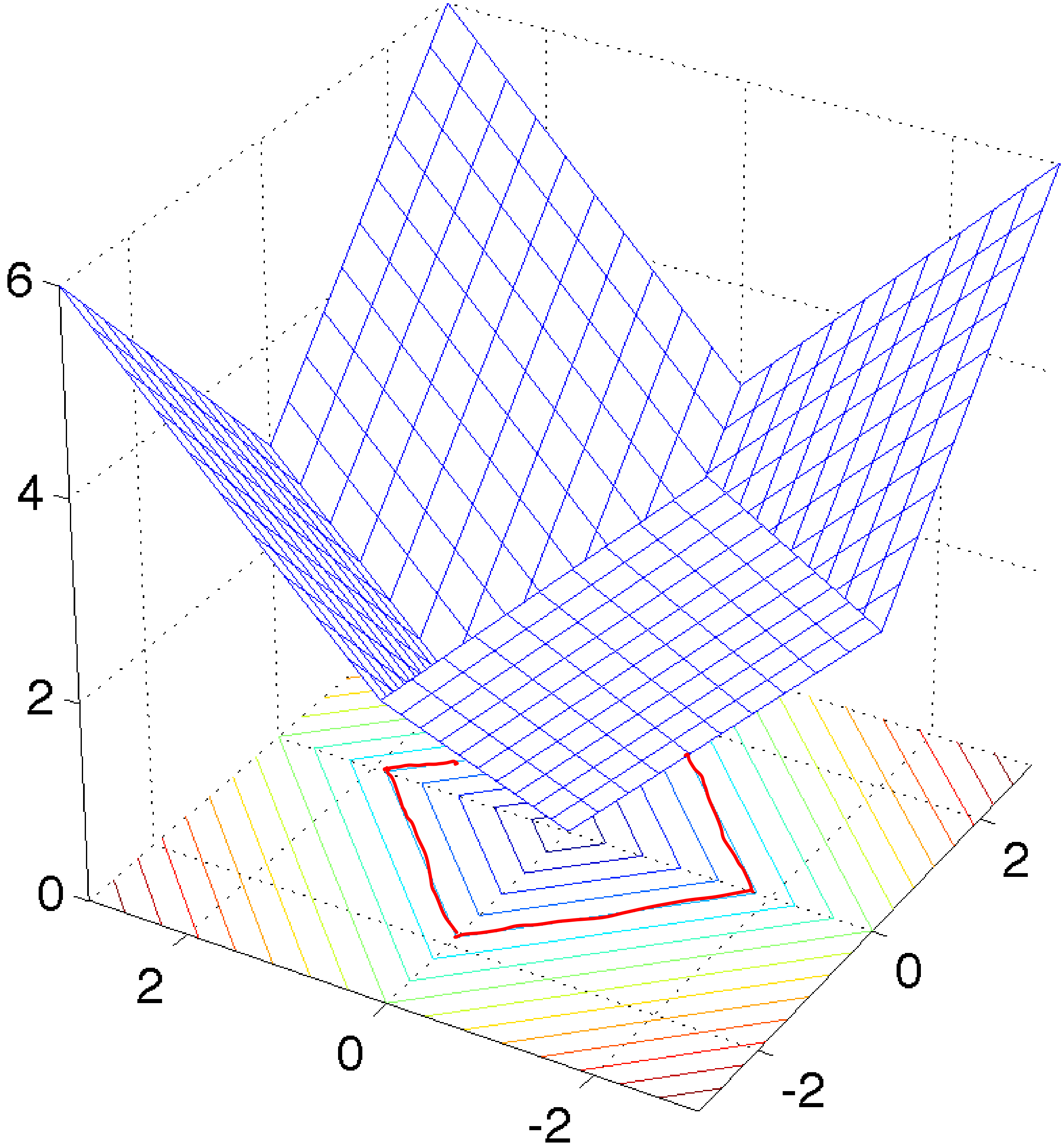
- $I_K(x)$, cone K: $I_K^*(y) = \sup_x \left(x^T y - I_K(x)\right)$ $\quad (I_K)^* = I_{-K^*}$

  if $y \in -K^* \Rightarrow x = 0 \Rightarrow (I_K)^*(y) = 0$

  if $y \notin -K^* \Rightarrow x$ big $\Rightarrow (I_K)^*(y) = \infty$

- $I_C(x)$, set C: $C = [-1, 1]^2 \Rightarrow (I_C)^* = |y_1| + |y_2|$

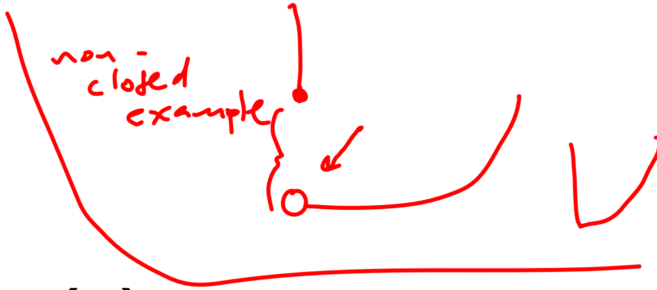
  $I_C^*(y) = \sup_x \left(x^T y - I_C(x)\right) = \sup_{x \in C} x^T y \rightarrow$ support fn. of C



6
4
2
0
2
0
-2
2
0
-2

$-\ln(x) - \ln(1-x)$

Properties



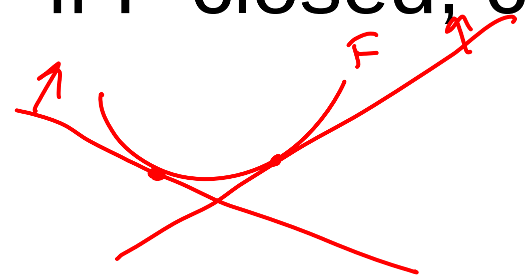
- $\underline{F(x) \geq G(x)}$ $\qquad\qquad F^*(y) \leq G^*(y)$

$F^*(y) = \sup_x (x^T y - F(x)) \leq \sup_x (x^T y - G(x)) = G^*(y)$

- F^* is closed, convex

- $\underline{F^{**}} = \underline{\text{cl conv } F}$ (= F if F closed, convex)

$F^{**}(x) = \sup_y \underbrace{(y^T x - F^*(y))}$



- If F is differentiable:

$F^*(y) = \sup_x (x^T y - F(x))$

$y = F'(x) \qquad x = (F')^{-1}(y)$

$F^*(y) = y(F')^{-1}(y) - F((F')^{-1}(y))$

Working with dual functions

- $G(x) = F(x) + k$

$$G^*(y) = \sup_x \left(x^T y - F(x) - k\right) = \sup_x \left(x^T y - F(x)\right) - k$$
$$= F^*(y) - k$$

- $G(x) = k\,F(x) \qquad k > 0$

$$G^*(y) = \sup_x \left(x^T y - kF(x)\right) = k \sup_x \left(\frac{x^T y}{k} - F(x)\right) = kF^*(y/k)$$

- $G(x) = F(x) + a^T x$

$$G^*(y) = \sup_x \left(x^T y - F(x) - a^T x\right) = \sup_x \left((y-a)^T x - F(x)\right) = F^*(y-a)$$

Working with dual functions

- $G(x_1, x_2) = F_1(x_1) + F_2(x_2)$

$$G^*(y_1, y_2) = \sup_{x_1, x_2} \left(x_1^T y_1 + x_2^T y_2 - F_1(x_1) - F_2(x_2) \right)$$

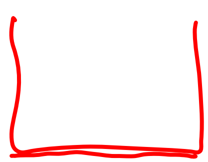
$$= F_1^*(y_1) + F_2^*(y_2)$$

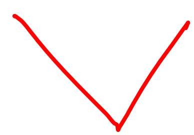
An odd-looking operation

- Definition: **infimal convolution**

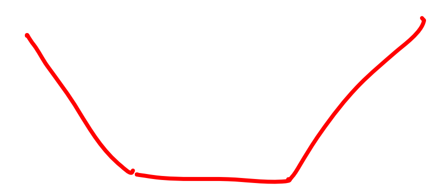
$$(F_1 \square F_2)(x) = \inf_{a+b=x} \left(\underline{F_1(a)} + \underline{F_2(b)} \right)$$

- E.g., $F_1(x) = \underline{I_{[-1,1]}(x)}$, $\underline{F_2(x) = |x|}$

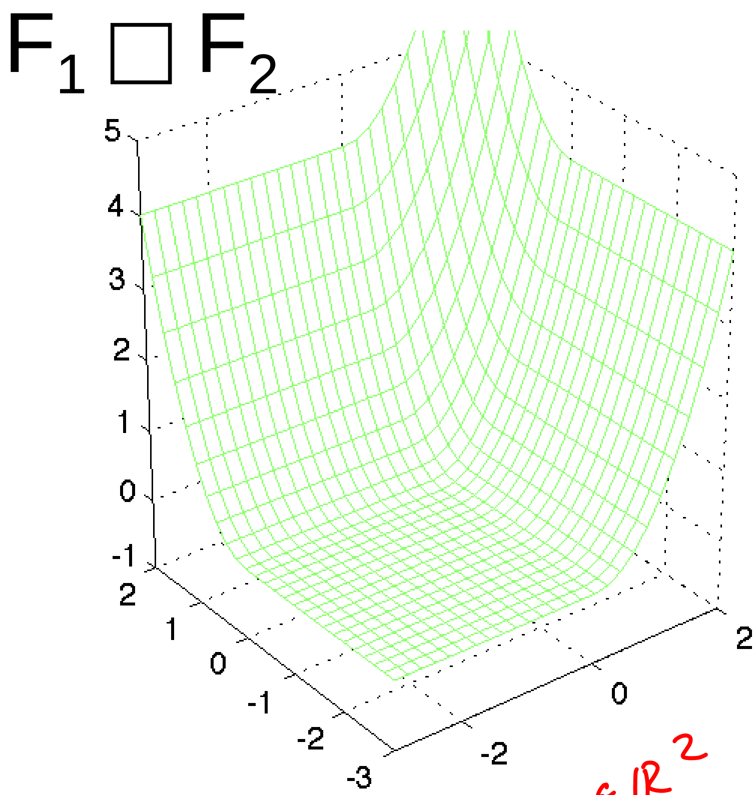


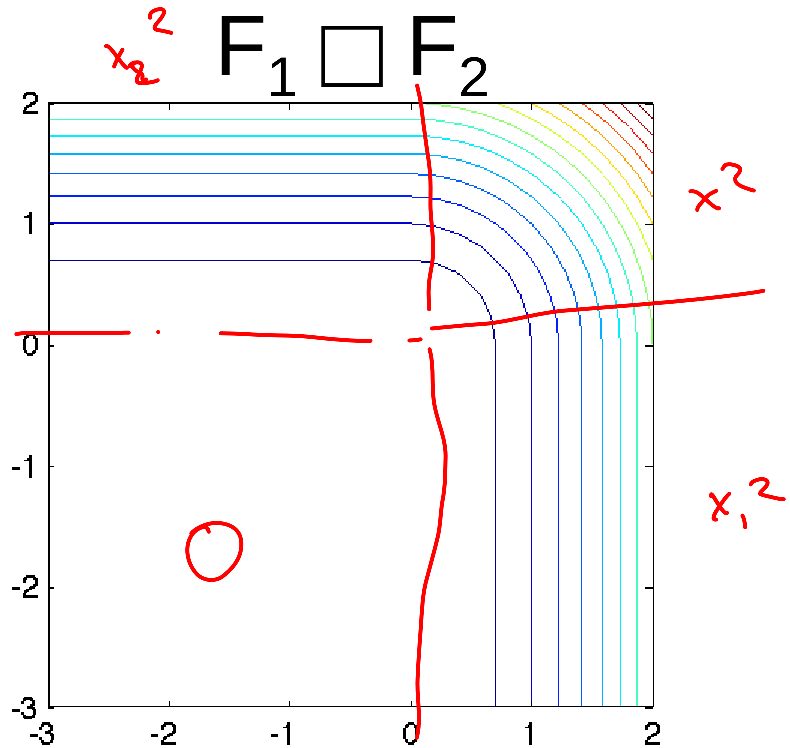


$F_1 \square F_2 =$



Infimal convolution example





$\in \mathbb{R}^2$

- $F_1(x) = I_{\le 0}(x)$, $F2(x) = x^2$ $= \|x\|^2$

Dual of infimal convolution

- $G(x) = F_1(x) \,\square\, F_2(x)$
- $G^*(y) = \sup_{x} \left(x^T y - \inf_{a+b=x} \left(F_1(a) + F_2(b) \right) \right)$

$$= \cancel{\sup_{x}} \ \sup_{\substack{\cancel{a+b=x} \\ a,b}} \underbrace{\left((a+b)^T y - F_1(a) - F_2(b) \right)}$$

$$= \sup_{a,b} \left(a^T y + b^T y - F_1(a) - F_2(b) \right)$$

$$= F_1^*(y) + F_2^*(y)$$

- $G(x) = F_1(x) + F_2(x)$ $\qquad$ $G^*(y) = F_1^*(y) \,\square\, F_2^*(y)$

$v^* = \text{opt}$

Convex program duality

f, g_i: convex

- min $\underline{f(x)}$ s.t.

  $\underline{Ax = b}$

  $\underline{g_i(x) \le 0}$ $i \in I$

$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \end{pmatrix}$

for feasible x

$y^T(Ax-b) = 0$

$z^T g(x) \le 0$ if $z \ge 0$

$f(x) \ge f(x) + \underline{y^T(Ax-b)} + \underline{z^T g(x)}$

$\equiv L(x,y,z)$

$v^* = \inf_{\text{feas } x} f(x) \ge \inf_{\text{feas } x} L(x,y,z) \ge \inf_{x} L(x,y,z) \equiv L(y,z)$

Dual prob: $\max L(y,z)$ s.t. $z \ge 0$