Chapter 5

The Inverse Method

After the definition of logic via natural deduction, we have developed a succession of techniques for theorem proving based on sequent calculi. We considered a sequent $\Gamma \Longrightarrow C$ as a goal, to be solved by backwards-directed search which was modeled by the bottom-up construction of a derivation. The critical choices were disjunctive non-determinism (resolved by guessing and backtracking) and existential non-determinism (resolved by introducing existential variables and unification). The limiting factor in more refined theorem provers based on this method is generally the number of disjunctive choices which have to be made. It is complicated by the fact that existential variables are global in a partial derivation, which means that choices in one conjunctive branch have effects in other branches. This effects redundancy elimination, since subgoals are not independent of each other.

The diametrically opposite approach would be to work forward from the initial sequents until the goal sequent is reached. If we guarantee a fair strategy in the selection of axioms and inference rules, every goal sequent can be derived this way. Without further improvements, this is clearly infeasible, since there are too many derivations for us to hope that we can find one for the goal sequent in this manner.

The *inverse method* is based on the property that in a cut-free derivation of a goal sequent, we only need to consider subformulas of the goal and their substitution instances. For example, when we have derived both A and B in the forward direction, we only derive their conjunction $A \wedge B$ if $A \wedge B$ is a subformula of the goal sequent.

The nature of forward search under these restrictions is quite different from the backward search. Since we always add new consequences to the sequents already derived, knowledge grows monotonically and no disjunctive non-determinism arises. Similarly for existential non-determinism, if we keep sequents in their maximally general form. On the other hand, there is a potentially very large amount of conjunctive non-determinism, since we have to apply all applicable rules to all sequents in a fair manner in order to guarantee completeness. The critical factor in forward search is to limit conjunctive non-determinism. We can view this as redundancy elimination: among the many ways that a given sequent may be derived, we try to actually consider a few as possible. The techniques developed in the preceding chapters, with some modifications, can be applied in this new setting.

Historically, the inverse method is due to Maslov [Mas64]. It has been adapted to intuitionistic and other non-classical logics by Voronkov [Vor92], Mints [Min94], and Tammet [Tam96, Tam97].

5.1 Forward Sequent Calculus

As a first step towards the inverse method, we write out a sequent calculus appropriate for forward search. This stems from a basic reinterpretation of a sequent during search. Previously, $\Gamma \Longrightarrow C$ expressed that we may use all hypotheses in Γ to prove that C is true. Now we will think of $\Gamma \longrightarrow C$ to mean that we needed all the hypotheses in Γ in order to prove that C is true.

This means that weakening is no longer valid for sequents $\Gamma \longrightarrow C$ and we have to take special care when we formulate correctness theorems. Secondly, we do not need to keep duplicate assumptions, so we view Γ in the sequent $\Gamma \longrightarrow C$ as a *set* of assumptions. We write $\Gamma_1 \cup \Gamma_2$ for the union of two sets of assumptions, and Γ, A stands for $\Gamma \cup \{A\}$.[1]

Initial Sequents. Previously, we allowed $\Gamma, A \Longrightarrow A$, since the assumptions in Γ can be used, but are just not needed in this case. In the forward calculus, initial sequents

$$\frac{}{A \longrightarrow A}\,\text{init}$$

express that only the hypothesis A is needed to derive the truth of A and nothing else.

Conjunction. In the right rule for conjunction, we previously concluded $\Gamma \Longrightarrow A \wedge B$ from $\Gamma \Longrightarrow A$ and $\Gamma \Longrightarrow B$. This expressed that all assumptions Γ are available in both branches. Now we need to take the union of the two sets of assumptions, expressing that both are needed to prove the conclusion.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2 \longrightarrow B}{\Gamma_1 \cup \Gamma_2 \longrightarrow A \wedge B}\,\wedge\text{R}$$

On the left rules, so such considerations arise.

$$\frac{\Gamma, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\,\wedge\text{L}_1 \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\,\wedge\text{L}_2$$

Note that if $A \wedge B$ is already present in Γ in the two left rules, it will not be added again.

[1] In the language of judgments, $\Gamma \longrightarrow A$ is a *strict hypothetical judgment*.

Truth. As in the backward sequent calculus, there is only a right rule. Unlike the backward sequent calculus, it does not permit any hypotheses.

$$\frac{}{\cdot \longrightarrow \top}\top R$$

Implication. In the backward sequent calculus, the right rule for implication has the form

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\supset R.$$

In the forward direction this would not be sufficient, because it would allow us to conclude $A \supset B$ only if A is actually needed in the proof of B. To account for this case, we introduce two separate rules.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\supset R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}\supset R_2$$

Another, more efficient possibility combines these rules into one which removes A from the context of the premise if it is there and otherwise leaves it unchanged (see Section **??**). In the left rule we have to take a union as in the right rule for conjunction.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2, B \longrightarrow C}{\Gamma_1 \cup \Gamma_2, A \supset B \longrightarrow C}\supset L$$

Note that the principal proposition $A \supset B$ does not occur in the premises. However, it might occur in Γ_1 or Γ_2, in which case it is not added again in the conclusion.

Disjunction. This introduces no new considerations.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\vee R_2$$

$$\frac{\Gamma_1, A \longrightarrow C \qquad \Gamma_2, B \longrightarrow C}{\Gamma_1, \Gamma_2, A \vee B \longrightarrow C}\vee L$$

Falsehood. There is only a left rule.

$$\frac{}{\perp \longrightarrow C}\perp L$$

We postpone the consideration of negation and quantifiers.

The soundness of the forward sequent calculus is easy to establish.

Theorem 5.1 (Soundness of Forward Sequent Calculus)
If $\Gamma \longrightarrow C$ *then* $\Gamma \Longrightarrow C$

Proof: By induction on the structure of the derivation $\mathcal{F}$ of $\Gamma \longrightarrow C$. We show only some of the cases, since the patterns are very similar in the remaining ones. In order to avoid confusion, we write Γ, A and $\Gamma \cup \{A\}$ for forward sequents to be more explicit about possible contractions.

Case:

$$\mathcal{F} = \frac{}{C \longrightarrow C}\,\text{init}$$

$C \Longrightarrow C$ By rule init

Case:

$$\mathcal{F} = \frac{\begin{array}{c}\mathcal{F}_1\\ \Gamma_1 \longrightarrow C_1\end{array} \qquad \begin{array}{c}\mathcal{F}_2\\ \Gamma_2 \longrightarrow C_2\end{array}}{\Gamma_1 \cup \Gamma_2 \longrightarrow C_1 \wedge C_2}\,\wedge\text{R}$$

$\Gamma_1 \Longrightarrow C_1$	By i.h. on $\mathcal{F}_1$
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_1$	By weakening
$\Gamma_2 \Longrightarrow C_2$	By i.h. on $\mathcal{F}_2$
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_2$	By weakening
$\Gamma_1 \cup \Gamma_2 \Longrightarrow C_1 \wedge C_2$	By rule $\wedge$R

Case:

$$\mathcal{F} = \frac{\begin{array}{c}\mathcal{F}_1\\ \Gamma_1 \longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{F}_2\\ \Gamma_2, B \longrightarrow C\end{array}}{\Gamma_1 \cup \Gamma_2 \cup \{A \supset B\} \longrightarrow C}\,\supset\text{L}$$

$\Gamma_1 \Longrightarrow A$	By i.h. on $\mathcal{F}_1$
$\Gamma_1 \cup \Gamma_2, A \supset B \Longrightarrow A$	By weakening
$\Gamma_2, B \Longrightarrow C$	By i.h. on $\mathcal{F}_2$
$\Gamma_1 \cup \Gamma_2, A \supset B, B \Longrightarrow C$	By weakening
$\Gamma_1 \cup \Gamma_2, A \supset B \Longrightarrow C$	By rule $\supset$L
$\Gamma_1 \cup \Gamma_2 \cup \{A \supset B\} \Longrightarrow C$	By contraction (if needed)

□

Completeness is more difficult. In fact, it is false! For example, for atomic propositions P and Q we can not derive $P, Q \Longrightarrow P$. Fortunately, the absence of weakening is the only source of difficulty and can easily be taken into account.

Theorem 5.2 (Completeness of Forward Sequent Calculus)
If $\Gamma \Longrightarrow C$ *then* $\Gamma' \longrightarrow C$ *for some* $\Gamma' \subseteq \Gamma$.

Proof: By induction on the structure of $\mathcal{S}$ for $\Gamma \Longrightarrow C$.

Case:

$$\mathcal{S} = \frac{}{\Gamma_1, C \Longrightarrow C}\,\text{init}$$

$C \longrightarrow C$	By rule init
$\{C\} \subseteq \Gamma_1, C$	By definition of $\subseteq$

Case:

$$\mathcal{S} = \frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma, A \Longrightarrow B\end{array}}{\Gamma \Longrightarrow A \supset B}\,{\supset}\text{R}$$

$\Gamma'' \longrightarrow B$ for some $\Gamma'' \subseteq \Gamma, A$	By i.h. on $\mathcal{S}_1$
$\Gamma'' = \Gamma', A$ and $\Gamma' \subseteq \Gamma$	First subcase
$\Gamma' \longrightarrow A \supset B$	By rule ${\supset}\text{R}_1$
$\Gamma'' \subseteq \Gamma$	Second subcase
$\Gamma'' \longrightarrow A \supset B$	By rule ${\supset}\text{R}_2$

Case:

$$\mathcal{S} = \frac{\begin{array}{c}\mathcal{S}_1\\ \Gamma_1, A \supset B \Longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{S}_2\\ \Gamma_1, A \supset B, B \Longrightarrow C\end{array}}{\Gamma_1, A \supset B \Longrightarrow C}\,{\supset}\text{L}$$

$\Gamma_1' \longrightarrow A$ for some $\Gamma_1' \subseteq \Gamma_1, A \supset B$	By i.h. on $\mathcal{S}_1$
$\Gamma_2' \longrightarrow C$ for some $\Gamma_2' \subseteq \Gamma_1, A \supset B, B$	By i.h. on $\mathcal{S}_2$
$\Gamma_2' = \Gamma_2'', B$ and $\Gamma_2'' \subseteq \Gamma_1, A \supset B$	First subcase
$\Gamma_1' \cup \Gamma_2'' \cup \{A \supset B\} \longrightarrow C$	By rule ${\supset}\text{L}$
$\Gamma_1' \cup \Gamma_2'' \cup \{A \supset B\} \subseteq \Gamma_1 \cup \{A \supset B\}$	By properties of $\subseteq$
$\Gamma_2' \subseteq \Gamma_1, A \supset B$	Second subcase
$\Gamma' = \Gamma_2'$ satisfies claim	

□

5.2 Negation and Empty Succedents

In the backward sequent calculus, the rules for negation

$$\frac{\Gamma, A \Longrightarrow p}{\Gamma \Longrightarrow \neg A}\,\neg\text{R}^p \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow C}\,\neg\text{L}$$

require propositional parameters p. In Gentzen's original formulation of the sequent calculus he avoided this complication by allowing an empty right-hand side. A sequent of the form

$$\Gamma \Longrightarrow \cdot$$

can then be interpreted as

$$\Gamma \Longrightarrow p \qquad \text{for a parameter } p \text{ not in } \Gamma$$

As a result we can substitute an arbitrary proposition for the right-hand side (the defining property for parametric judgments) and obtain an evident judgment. In the sequent calculus with empty right-hand sides, this can be accomplished by weakening on the right:

If $\Gamma \Longrightarrow \cdot$ then $\Gamma \Longrightarrow C$ for any proposition C.

When the right-hand side can be empty or a singleton we write $\Gamma \Longrightarrow \gamma$, where $\gamma = C$ or $\gamma = \cdot$.

In a forward sequent calculus we can take advantage of this in order to avoid overcommitment in the rules for negation and falsehood. We first show the forward rules for negation; then we reexamine all the previous rules.

Negation. We just take advantage of the new form of judgment, avoiding, for example, a commitment to a particular proposition C in the $\neg$L rule.

$$\frac{\Gamma, A \longrightarrow \cdot}{\Gamma \longrightarrow \neg A}\neg\text{R} \qquad \frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow \cdot}\neg\text{L}$$

Interestingly, we do not need a second right rule for negation as for implication (see Exercise **??**).

Falsehood. Falsehood can similarly benefit from avoiding commitment. Note that previously the rule stated $\bot \longrightarrow C$, although there are many possible choices for C. Now we just replace this by

$$\frac{}{\bot \longrightarrow \cdot}\bot\text{L}$$

There still is no right rule.

Initial Sequents. They do not change.

$$\frac{}{A \longrightarrow A}\text{init}$$

Conjunction. The right rule requires no change.

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2 \longrightarrow B}{\Gamma_1 \cup \Gamma_2 \longrightarrow A \wedge B}\wedge\text{R}$$

On the left rules simply need to allow for an empty right-hand side.

$$\frac{\Gamma, A \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\wedge\text{L}_1 \qquad \frac{\Gamma, B \longrightarrow \gamma}{\Gamma, A \wedge B \longrightarrow \gamma}\wedge\text{L}_2$$

Truth. Does not change.

$$\frac{}{\cdot \longrightarrow \top}\top R$$

Implication. The possibility of empty right-hand sides requires a third right rule for implication. Again, in an implementation the three rules might be combined into a more efficient one.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\supset R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}\supset R_2 \qquad \frac{\Gamma, A \longrightarrow \cdot}{\Gamma \longrightarrow A \supset B}\supset R_3$$

$$\frac{\Gamma_1 \longrightarrow A \qquad \Gamma_2, B \longrightarrow \gamma}{\Gamma_1 \cup \Gamma_2, A \supset B \longrightarrow \gamma}\supset L$$

Disjunction. The rule for disjunction on the right remains the same, but the left rule now has to account for several possibilities, depending on whether the right-hand sides of the premises are empty. Essentially, we take the union of the right-hand sides of the two premises, except that the result must be a singleton or empty for the sequent to be well-formed.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\vee R_2$$

$$\frac{\Gamma_1, A \longrightarrow \gamma_1 \qquad \Gamma_2, B \longrightarrow \gamma_2}{\Gamma_1 \cup \Gamma_2, A \vee B \longrightarrow \gamma_1 \cup \gamma_2}\vee L$$

In detail, either γ_1 or γ_2 is empty, or $\gamma_1 = \gamma_2 = C = \gamma_1 \cup \gamma_2$. The rule does not apply otherwise.

The statement of the soundness theorem does not change much with empty succedents.

Theorem 5.3 (Soundness of Forward Sequent Calculus)

1. *If* $\Gamma \longrightarrow C$ *then* $\Gamma \Longrightarrow C$*, and*
2. *if* $\Gamma \longrightarrow \cdot$ *then* $\Gamma \Longrightarrow C$ *for all* C*.*

Proof: By induction on the derivation $\mathcal{F}$ of $\Gamma \longrightarrow \gamma$. □

In the completeness theorem, we now need to allow possible weakening on the left or on the right.

Theorem 5.4 (Completeness of Forward Sequent Calculus)

1. *If* $\Gamma \Longrightarrow C$ *then* $\Gamma' \longrightarrow C$ *or* $\Gamma' \longrightarrow \cdot$ *for some* $\Gamma' \subseteq \Gamma$*.*

Proof: By induction on the derivation $\mathcal{S}$ of $\Gamma \Longrightarrow C$. □

5.3 The Subformula Property

It is a general property of cut-free sequent calculi that all propositions occurring in a derivation are subformulas of the endsequent. In the forward direction we can therefore restrict the application of a rule to the case where the principal formula in the conclusion is a subformula of the goal sequent. We refine this property further by tracking positive and negative subformula occurrences. We then restrict left rule to introduce only negative subformulas of the goal sequent and right rules to positive subformulas of the goal sequent. To this end we introduce signed formulas.

$$\begin{array}{rrcl} \text{Positive} & A^+ & ::= & P^+ \mid A_1^+ \wedge A_2^+ \mid A_1^- \supset A_2^+ \mid A_1^+ \vee A_2^+ \mid \top^+ \mid \bot^+ \mid \neg A^- \\ \text{Negative} & A^- & ::= & P^- \mid A_1^- \wedge A_2^- \mid A_1^+ \supset A_2^- \mid A_1^- \vee A_2^- \mid \top^- \mid \bot^- \mid \neg A^+ \end{array}$$

It is obvious that every proposition can be annotated both positively and negatively, and that such an annotation is unique. We write Γ^- for a context $A_1^-, \ldots, A_n^-$ and γ^+ for an empty succedent or C^+. All inference rules for the sequent calculus can be annotated so that for a goal sequent $\Gamma^- \longrightarrow \gamma^+$, each sequent arising in the derivation has the same form, with only negative propositions on the left and positive propositions on the right (see Exercise 5.1). We say that A is a subformula of Γ or γ if A is a subformula of an element of Γ or γ, respectively, and similarly for signed propositions.

Theorem 5.5 (Signed Subformula Property)
Given a derivation $\mathcal{S}$ of $\Gamma^- \longrightarrow \gamma^+$. Then each sequent in $\mathcal{S}$ has the form $A_1^-, \ldots, A_n^- \longrightarrow B^+$ or $A_1^-, \ldots, A_n^- \longrightarrow \cdot$ where all A_i^- and B^+ are signed subformulas of Γ^- or γ^+.

Proof: By straightforward induction on the structure of $\mathcal{S}$. □

Note that this is a very strong theorem, since it asserts not only that every provable goal sequent has a derivation consisting of subformulas, but that *all* derivations of a provable sequent consist only of subformulas. A sequent not consisting of subformulas cannot contribute to a derivation of a goal sequent in the (cut-free) forward sequent calculus.

The subformula property immediately gives rise to a procedure for forward theorem proving. We start with all initial sequents of the form $A^- \longrightarrow A^+$ where both A^- and A^+ are signed subformulas of the goal sequent. We also have to add $\cdot \longrightarrow \top^+$ and $\bot^- \longrightarrow \cdot$ if $\top^+$ or $\bot^-$ are subformulas of the goal sequent, respectively.

Then we apply all possible inference rules where the principal proposition in the conclusion is a subformula of the goal sequent. We stop with success when we have generated the goal sequent, or if the goal sequent can be obtained from a generated sequent by weakening. We fail if any possible way of applying inference rules yields only sequents already in the database. In that case the goal sequent cannot be derivable if we have not encountered it (or a strengthened form of it) already.

We now show an example derivation in a linearized format. The goal sequent is $A \supset (B \supset C) \longrightarrow ((A \wedge B) \supset C)$. After signing each subformula we obtain

$$(A^+ \supset (B^+ \supset C^-)^-)^- \longrightarrow (((A^- \wedge B^-)^-) \supset C^+)^+$$

If show only the top-level sign, this leads to the following list of signed subformulas.

$$\begin{array}{l} A^+, B^+, C^-, A^-, B^-, C^+, \\ (B \supset C)^-, (A \wedge B)^-, \\ (A \supset (B \supset C))^-, ((A \wedge B) \supset C)^+ \end{array}$$

This means we have both positive and negative occurrences of A, B, and C and we have to consider three initial sequents.

1	$A^- \longrightarrow A^+$	init
2	$B^- \longrightarrow B^+$	init
3	$C^- \longrightarrow C^+$	init
4	$(A \wedge B)^- \longrightarrow A^+$	$\wedge L_1$ 1
5	$(A \wedge B)^- \longrightarrow B^+$	$\wedge L_1$ 2
6	$(A \wedge B)^-, (B \supset C)^- \longrightarrow C^+$	$\supset L$ 5 3
7	$(A \wedge B)^-, (A \supset (B \supset C))^- \longrightarrow C^+$	$\supset L$ 4 6
8	$(A \supset (B \supset C))^- \longrightarrow ((A \wedge B) \supset C)^+$	$\supset R_1$ 7

We use the horizontal lines to indicate iterations of an algorithm which derives all possible new consequences from the sequents already established. We have elided those sequents that do not contribute to the final derivation. For example, in the first step we can use $\supset R_2$ to conclude $C^- \longrightarrow ((A \wedge B) \supset C)^+$, from $C^- \longrightarrow C^+$, since the succedent is a positive subformula of the goal sequent.

Note that the inference of line 7 contains an implicit contraction, since $(A \wedge B)^-$ is an assumption in both premises (4 and 6).

5.4 Naming Subformulas

Without any further optimizations, the check if a given inference rule should be used in the forward direction is complicated, since we have to repeatedly scan the goal sequent for subformula occurrences. An integral part of the inverse method is to avoid this scan by introducing names for non-atomic subformulas and then specialize the inference rules to work only the names. We will not be formal about this optimization, since we view it as an implementation technique, but not an improvement of a logical nature. By expanding all newly defined names we obtain a derivation as in the previous section.

We return to the previous example to illustrate the technique. The goal sequent is $A \supset (B \supset C) \longrightarrow (A \wedge B) \supset C$. After naming each subformula we obtain the signed atomic propositions

$$A^+, B^+, C^-, A^-, B^-, C^+,$$

and the new names

$$\begin{array}{rcl} L_1^- & = & B^+ \supset C^- \\ L_2^- & = & A^- \wedge B^- \\ L_3^- & = & A^+ \supset L_1^- \\ L_4^+ & = & L_2^- \supset C^+ \end{array}$$

We can now write out the general sequent calculus inference rules, specialized to the above labels. Since the goal sequent contains no negative occurrence of negation or falsehood, we may restrict the right-hand sides of all rules to be non-empty. This means only two implication right rules are necessary instead of three for L_4^+.

$$\frac{\Gamma_1 \longrightarrow B^+ \qquad \Gamma_2, C^- \longrightarrow \gamma}{\Gamma_1 \cup \Gamma_2, L_1^- \longrightarrow \gamma} \supset\!\mathrm{L}\ (L_1^-)$$

$$\frac{\Gamma, A^- \longrightarrow \gamma}{\Gamma, L_2^- \longrightarrow \gamma} \wedge\mathrm{L}_1\ (L_2^-) \qquad \frac{\Gamma, B^- \longrightarrow \gamma}{\Gamma, L_2^- \longrightarrow \gamma} \wedge\mathrm{L}_2\ (L_2^-)$$

$$\frac{\Gamma_1 \longrightarrow A^+ \qquad \Gamma_2, L_1^- \longrightarrow \gamma}{\Gamma_1 \cup \Gamma_2, L_3^- \longrightarrow \gamma} \supset\!\mathrm{L}\ (L_3^-)$$

$$\frac{\Gamma, L_2^- \longrightarrow C^+}{\Gamma \longrightarrow L_4^+} \supset\!\mathrm{R}_1\ (L_4^+) \qquad \frac{\Gamma \longrightarrow C^+}{\Gamma \longrightarrow L_4^+} \supset\!\mathrm{R}_2\ (L_4^+)$$

In its labeled form, the derivation above looks as follows.

1	$A^- \longrightarrow A^+$	init
2	$B^- \longrightarrow B^+$	init
3	$C^- \longrightarrow C^+$	init
4	$L_2^- \longrightarrow A^+$	$\wedge\mathrm{L}_1$ 1
5	$L_2^- \longrightarrow B^+$	$\wedge\mathrm{L}_1$ 2
6	$L_2^-, L_1^- \longrightarrow C^+$	$\supset\!\mathrm{L}$ 5 3
7	$L_2^-, L_3^- \longrightarrow C^+$	$\supset\!\mathrm{L}$ 4 6
8	$L_3^- \longrightarrow L_4^+$	$\supset\!\mathrm{R}_1$ 7

In the algorithm for labeling subterms we can avoid some redundancy if we give identical subterms the same label. However, this is not required for soundness and completeness, it only trims the search space.

Another choice arises for initial sequents. As in backwards search, we may restrict ourselves to atomic initial sequents or we may allow arbitrary labeled subformulas as long as they occur both negatively and positively. Tammet [Tam96] reports that allowing non-atomic initial sequents led to significant speed-up on a certain class of test problems. Of course, in their named form, even non-atomic sequents have the simple form $L^- \longrightarrow L^+$ for a label L.