3.7 Classical Sequent Calculus

We briefly mentioned in Section 2.2 that there are several ways to add a rule or axiom schema to natural deduction to obtain a classical interpretation of the connectives. As the example of $A \vee \neg A$ illustrates, this changes the interpretation of the propositions and our method of explaining the meaning of a proposition via its introduction and elimination rules fails. In this section we explore an alternative, judgmental approach to classical logic. Rather than starting from natural deduction we start from the sequent calculus, because Gentzen [Gen35] has already proposed a sequent calculus for classical logic that has a strong subformula property and thereby satisfies at least the requirement that the meaning of a proposition (if we can define what that means) depends only on the meaning of its constituents.

Recall the basic judgment form for the (intuitionistic) sequent calculus,

$$u_1{:}A_1 \ \mathit{left}, \ldots, u_n{:}A_n \ \mathit{left} \vdash A \ \mathit{right},$$

which arises by splitting the basic judgment A *true* into $A\,\mathit{left}$ (truth as an assumption, only in the antecedent) and $A\,\mathit{right}$ (truth as a conclusion, only in the succedent), which we abbreviated as

$$A_1, \ldots, A_n \Longrightarrow A$$

In order to formulate classical logic, we add a new basic judgment, A *false*, which we use only as an assumption. Furthermore, we have the judgment of contradiction, *contr*, expressing that a collection of assumptions is contradictory. The hypothetical judgment form we consider is

$$u_1{:}A_1 \ \mathit{true}, \ldots, u_n{:}A_n \ \mathit{true}, z_1{:}B_1 \ \mathit{false}, \ldots, z_m{:}B_m \ \mathit{false} \vdash \mathit{contr}$$

stating that the assumptions about truth and falsehood are contradictory. The basic **rule of contradiction** relating truth and falsehood is

$$\frac{}{\Psi, u{:}A \ \mathit{true}, z{:}A \ \mathit{false} \vdash \mathit{contr}}\ \text{contra}$$

which states that a proposition cannot be simultaneously true and false. There are further unused assumptions about truth and falsehood are allowed in Ψ. Interestingly, many theorem proving procedures for classical logic are presented in this style: instead of proving a proposition we derive a contradiction from the negated assumptions. Perhaps our analysis provides some hints why this is indeed the right view of classical logic.

Conversely, we have a principle that states any proposition A must be either true or false.

> **Principle of excluded middle.**
> If $\Psi, u{:}A \ \mathit{true} \vdash \mathit{contr}$ and $\Psi, z{:}A \ \mathit{false} \vdash \mathit{contr}$ then $\Psi \vdash \mathit{contr}$.

The argument for this principle, from the rule of contradiction, goes as follows: if the assumption A *true* is contradictory, then either A must be false (assuming Ψ) or Ψ itself is contradictory. In the latter case we are done. But if A *false* follows from Ψ then we can discharge the assumption that A is false from the second given derivation.

It is important that this principle must hold for the logic, rather than being assumed as an inference rule. This means that the law of excluded middle is not an arbitrary assumption, but arises from the nature of falsehood as the opposite of truth in a systematic way, at the level of judgments.

We abbreviate the the judgment

$$u_1{:}A_1\ \mathit{true}, \ldots, u_n{:}A_n\ \mathit{true}, z_1{:}B_1\ \mathit{false}, \ldots, z_m{:}B_m\ \mathit{false} \vdash \mathit{contr}$$

as

$$A_1, \ldots, A_n \# B_1, \ldots, B_m.$$

We have to keep in mind that A_i are assumptions about truth, and B_j are assumptions about falsehood, with the overall goal to derive a contradiction.

In the literature one finds two other common notations for this judgment, first and foremost Gentzen's multiple-conclusion sequent calculus.

$$A_1, \ldots, A_n \Longrightarrow B_1, \ldots, B_m$$

Gentzen observed that we can capture the difference between classical and intuitionistic reasoning by either allowing or disallowing multiple conclusions. However, the only way we have been able to explain this from the judgmental point of view is in the manner indicated above. The other notation sometimes used, in the presentation of tableaux, resolution and other theorem proving techniques is

$$A_1^\top, \ldots A_n^\top, B_1^\perp, \ldots, B_m^\perp$$

where we mark assumption A *true* as $A^\top$ and assumptions B *false* as $B^\perp$ instead of segregating them in the manner of a sequent calculus. There is no essential difference between these notations as long as we keep in mind their correct interpretation.

We first restate our judgmental rules and principles in the new notation, using Γ for truth assumptions and Δ for falsehood assumptions.

Rule of Contradiction.

$$\frac{}{\Gamma, A \# A, \Delta}\,\text{contra}$$

Principle of Excluded Middle.

If $\Gamma \# A, \Delta$ and $\Gamma, A \# \Delta$ then $\Gamma \# \Delta$.

We also have the expected weakening and contraction properties, both for truth and falsehood, which follow from the general nature of hypothetical reasoning.

For a multiple-conclusion view of sequents, these are much more difficult to explain.

Since we do not change the meaning of truth (or the meaning of the connectives), all the left rules from the intuitionistic sequent calculus carry over to analogous rules here. We have to derive the rules for assumptions A *false* from the principle of excluded middle (which was in turn justified by the rule of contradiction that defined falsehood).

Conjunction. The (left) rules for truth are as usual. We write the names of the rules as T in this context.

$$\frac{\Gamma, A \wedge B, A \# \Delta}{\Gamma, A \wedge B \# \Delta} \wedge T_1 \qquad \frac{\Gamma, A \wedge B, B \# \Delta}{\Gamma, A \wedge B \# \Delta} \wedge T_2$$

To determine the rules for falsehood we have to think about what we can conclude from the assumption that $A \wedge B$ *false*. If $A \wedge B$ is false, then either A or B must be false, so we must be able to obtain a contradiction in both cases.

$$\frac{\Gamma \# A, A \wedge B, \Delta \qquad \Gamma \# B, A \wedge B, \Delta}{\Gamma \# A \wedge B, \Delta} \wedge F$$

We use F to mark rules operating on falsehood assumptions.

The fact that the truth and falsehood rules mesh in a way predicted by the principle of excluded middle is the subject of Theorem 3.17. Intuitively, we should verify that if we use excluded middle for a conjunction we can reduce it to uses on excluded middle for the conjuncts.

Truth. Truth is straightforward: there is no left rule and the rule for $\top$ *false* simply succeeds.

$$\frac{}{\Gamma \# \top, \Delta} \top F$$

Disjunction. The left rule becomes the rule for truth assumptions.

$$\frac{\Gamma, A \vee B, A \# \Delta \qquad \Gamma, A \vee B, B \# \Delta}{\Gamma, A \vee B \# \Delta} \vee T$$

How can we proceed if we know that $A \vee B$ *false*? Intuitively, it means that both A and B must be false.

$$\frac{\Gamma \# A, A \vee B, \Delta}{\Gamma \# A \vee B, \Delta} \vee F_1 \qquad \frac{\Gamma \# B, A \vee B, \Delta}{\Gamma \# A \vee B, \Delta} \vee F_2$$

Falsehood. This is dual to truth: there is a left rule but no rule for $\perp$ *false*, which provides no information.

$$\frac{}{\Gamma, \perp \# \Delta} \perp T$$

Implication. We can use an assumption $A \supset B$ *true* only by proving A *true* (which licenses us to assume B *true*). Unfortunately, our classical sequent calculus does not allow us to derive the truth of any proposition, only contradictions. This means we cannot give a judgmental explanation of the constructive implication in the classical sequent calculus without destroying its meaning. However, there is a classical form of implication $A \Rightarrow B$ meaning that either A is false or B is true. This leads to the following rules for this new classical connective

$$\frac{\Gamma, A \Rightarrow B \mathrel{\#} A, \Delta \qquad \Gamma, A \Rightarrow B, B \mathrel{\#} \Delta}{\Gamma, A \Rightarrow B \mathrel{\#} \Delta}{\Rightarrow}\mathrm{T} \qquad \frac{\Gamma, A \mathrel{\#} B, A \Rightarrow B, \Delta}{\Gamma \mathrel{\#} A \Rightarrow B, \Delta}{\Rightarrow}\mathrm{F}$$

Negation. As for implication, we cannot formulate a rule for constructive negation in the classical sequent calculus. Instead, we have a new form of negation that flips between truth and falsehood. That is, $\sim A$ is true if A is false and $\sim A$ is false if A is true. We obtain the following rules

$$\frac{\Gamma, {\sim}A \mathrel{\#} A, \Delta}{\Gamma, {\sim}A \mathrel{\#} \Delta}{\sim}\mathrm{T} \qquad \frac{\Gamma, A \mathrel{\#} {\sim}A, \Delta}{\Gamma \mathrel{\#} {\sim}A, \Delta}{\sim}\mathrm{F}$$

We conclude that the difference between intuitionistic and classical logic does not lie in the nature of conjunction or disjunction, but in the nature of implication and negation. Moreover, if we accept that a notion of falsehood of a proposition as being contradictory with its truth, then the principle of excluded middle seems fully justified for proofs of contradiction. This does not answer any question about a possible computational interpretation of classical logic or about a faithful system of natural deduction or about a possible integration of intuitionistic and classical logic in a single system.

Theorem 3.17 (Principle of Excluded Middle)
If $\Gamma \mathrel{\#} A, \Delta$ *and* $\Gamma, A \mathrel{\#} \Delta$ *then* $\Gamma \mathrel{\#} \Delta$

Proof: By induction on the structure of A and, for each A simultaneously on the structure of the two given derivations. This means, as for the proof of the admissibility of cut (Theorem 3.11), we can appeal to the induction hypothesis on a smaller formulas and arbitrary derivations, or on the same formula such that one of the derivations gets smaller and the other one remains the same.

The division into cases, and the idea of the proof in each case is quite similar to the admissibility of cut, so we elide any details here. □

At this point it might seem like intuitionistic logic and classical logic are simply different, with classical logic somewhat impoverished. It only appears to have conjunction, disjunction, and a form of negation, while intuitionistic logic also has a constructive implication that does not appear expressible in classical logic.

However, the situation is more complicated. It turns out that there is a uniform way to translate classical logic to intuitionistic logic that preserves truth. This means intuitionistic logic can simulate A *false*, contradiction, and negation. The idea is due to Kolmogorov [**?**] who, however, did not prove its correctness in the modern sense.

The translation A^o maps atomic formulas to themselves, classical negation to intuitionistic negation, and prefixes any other subformula by a double negation. Some optimization are possible, but not necessary.

$$\begin{array}{rcl}
P^o & = & P \\
(A \wedge B)^o & = & \neg\neg A^o \wedge \neg\neg B^o \\
(\top)^o & = & \top \\
(A \vee B)^o & = & \neg\neg A^o \vee \neg\neg B^o \\
(\bot)^o & = & \bot \\
(\sim A)^o & = & \neg A^o \\
(A \Rightarrow B)^o & = & \neg\neg A \supset \neg\neg B
\end{array}$$

Then we interpret A *false* and $\neg A^o$ *true*. We also write Γ^o for translating each formula A in Γ to A^o, and $\neg\Gamma$ for applying $\neg$ to each formula in Γ.

Lemma 3.18
If $\Gamma \# \Delta$ then $\Gamma^o, \neg\Delta^o \Longrightarrow p$ for a parameter p not in Γ or Δ.

Proof: Part (1) follows by induction on the derivation $\mathcal{D}$ of $\Gamma \# \Delta$. Each case is straightforward; we show some representative ones. After the first case, we silently apply weakening when necessary. The proofs in each case may be easier to think about if we read them from the last line upwards.

Case:

$$\mathcal{D} = \frac{}{\Gamma_1, A \# A, \Delta_1}\,\text{contra}$$

$\Gamma_1^o, A^o, \Delta_1^o \Longrightarrow A^o$	By rule init
$\Gamma_1^o, A^o, \neg A^o, \Delta_1^o \Longrightarrow A^o$	By weakening
$\Gamma_1^o, A^o, \neg A^o, \Delta_1^o \Longrightarrow p$	By rule $\neg$L

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma_1, A \wedge B, A \# \Delta\end{array}}{\Gamma_1, A \wedge B \# \Delta}\,\wedge\mathrm{T}_1$$

$\Gamma_1^o, \neg\neg A^o \wedge \neg\neg B^o, A^o, \neg\Delta^o \Longrightarrow p$	By i.h. on $\mathcal{D}_1$
$\Gamma_1^o, \neg\neg A^o \wedge \neg\neg B^o, \neg\Delta^o \Longrightarrow \neg A^o$	By rule $\neg$R
$\Gamma_1^o, \neg\neg A^o \wedge \neg\neg B^o, \neg\neg A^o, \neg\Delta^o \Longrightarrow p$	By rule $\neg$L
$\Gamma_1^o, \neg\neg A^o \wedge \neg\neg B^o, \neg\Delta^o \Longrightarrow p$	By rule $\wedge$L$_1$
$\Gamma_1^o, (A \wedge B)^o, \neg\Delta^o \Longrightarrow p$	By defn of $()^o$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \mathrel{\#} A, A \wedge B, \Delta_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \mathrel{\#} B, A \wedge B, \Delta_1\end{array}}{\Gamma \mathrel{\#} A \wedge B, \Delta_1} \wedge\mathrm{F}$$

$\Gamma^o, \neg A^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow p$	By i.h. on $\mathcal{D}_1$
$\Gamma^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow \neg\neg A^o$	By rule $\neg$R
$\Gamma^o, \neg B^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow p$	By i.h. on $\mathcal{D}_2$
$\Gamma^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow \neg\neg B^o$	By rule $\neg$R
$\Gamma^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow \neg\neg A^o \wedge \neg\neg B^o$	By rule $\wedge$R
$\Gamma^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow (A \wedge B)^o$	By defn. of $()^o$
$\Gamma^o, \neg(A \wedge B)^o, \neg\Delta_1^o \Longrightarrow p$	By rule $\neg$L

□

For part (2), an induction over the structure of the given deduction will not work, because subdeductions will not necessarily have a conclusion of the same form. Instead we employ a simpler backward translation, $()^e$ and show that if $\Gamma \Longrightarrow C$ then $\Gamma^e \mathrel{\#} C^e$

$$\begin{array}{rcl}
P^e & = & P \\
(A \wedge B)^e & = & A^e \wedge B^e \\
(\top)^e & = & \top \\
(A \vee B)^e & = & A^e \vee B^e \\
(\bot)^e & = & \bot \\
(\neg A)^e & = & {\sim} A^e \\
(A \supset B)^e & = & A^e \Rightarrow B^e
\end{array}$$

Because the double-negation translation $()^o$ inserts double negations and the backward translation $()^e$ keeps the structure of the formula intact, if we translate back and forth we obtain an equivalent proposition.

Lemma 3.19
$A \mathrel{\#} (A^o)^e$ and $(A^o)^e \mathrel{\#} A$ for any (classical) proposition A.

Proof: By induction on the structure of A. We show two cases, eliding appeals to weakening.

Case: $A = P$. Then $(A^o)^e = (P^o)^e = P$ and $P \mathrel{\#} P$ by rule contra.

Case: $A = A_1 \wedge A_2$.

$A_1 \mathrel{\#} (A_1^o)^e$	By i.h. on A_1
$A_1 \wedge A_2 \mathrel{\#} (A_1^o)^e$	By $\wedge\mathrm{T}_1$
$A_1 \wedge A_2, {\sim}(A_1^o)^e \mathrel{\#} \cdot$	By $\sim$T
$A_1 \wedge A_2 \mathrel{\#} {\sim}{\sim}(A_1^o)^e$	By $\sim$F
$A_2 \mathrel{\#} (A_2^o)^e$	By i.h. on A_2

$A_1 \wedge A_2 \mathrel{\#} (A_2^o)^e$	By $\wedge$T$_2$
$A_1 \wedge A_2, \sim(A_2^o)^e \mathrel{\#} \cdot$	By $\sim$T
$A_1 \wedge A_2 \mathrel{\#} \sim\sim(A_2^o)^e$	By $\sim$F
$A_1 \wedge A_2 \mathrel{\#} \sim\sim(A_1^o)^e \wedge \sim\sim(A_2^o)^e$	By $\wedge$F
$A_1 \wedge A_2 \mathrel{\#} (\neg\neg A_1^o \wedge \neg\neg A_2^o)^e$	By defn. of $()^e$
$A_1 \wedge A_2 \mathrel{\#} ((A_1 \wedge A_2)^o)^e$	By defn. of $()^o$

□

The truth and falsehood rules of the classical sequent calculus can simulate the left and right rules of the intuitionistic sequent calculus on corresponding propositions.

Lemma 3.20 *If* $\Gamma \Longrightarrow C$ *then* $\Gamma^e \mathrel{\#} C^e$.

Proof: By induction on the structure of the given derivation $\mathcal{D}$. We show some representative cases.

Case:

$$\mathcal{D} = \frac{}{\Gamma_1, C \Longrightarrow C}\,\text{init}$$

$\Gamma_1^e, C^e \mathrel{\#} C^e$	By rule contra

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \Longrightarrow B\end{array}}{\Gamma \Longrightarrow A \supset B}\,{\supset}\text{R}$$

$\Gamma^e, A^e \mathrel{\#} B^e$	By i.h. on $\mathcal{D}_1$
$\Gamma^e, A^e \mathrel{\#} B^e, A^e \Rightarrow B^e$	By weakening
$\Gamma^e \mathrel{\#} A^e \Rightarrow B^e$	By rule $\Rightarrow$F
$\Gamma^e \mathrel{\#} (A \supset B)^e$	By defn. of $()^e$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1, A \supset B \Longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1, A \supset B, B \Longrightarrow C\end{array}}{\Gamma_1, A \supset B \Longrightarrow C}\,{\supset}\text{L}$$

$\Gamma_1^e, A^e \Rightarrow B^e \mathrel{\#} A^e$	By i.h. on $\mathcal{D}_1$
$\Gamma_1^e, A^e \Rightarrow B^e \mathrel{\#} A^e, C^e$	By weakening
$\Gamma_1^e, A^e \Rightarrow B^e, B^e \mathrel{\#} C^e$	By i.h. on $\mathcal{D}_2$
$\Gamma_1^e, A^e \Rightarrow B^e \mathrel{\#} C^e$	By rule $\Rightarrow$F
$\Gamma_1^e, (A \supset B)^e \mathrel{\#} C^e$	By defn. of $()^e$

□

Theorem 3.21 (Interpretation of Classical in Intuitionistic Logic)
$\Gamma \mathrel{\#} \Delta$ iff $\Gamma^o, \neg\Delta^o \Longrightarrow p$ for a parameter p not in Γ or Δ.

Proof: The forward direction is precisely the subject of Lemma 3.18.

In the backward direction, we know from Lemma 3.20, that $(\Gamma^o)^e, {\sim}(\Delta^o)^e \mathrel{\#} p$. The we repeatedly use $\sim$T to conclude $(\Gamma^o)^e \mathrel{\#} (\Delta^o)^e, p$. This derivation is parametric in p, so we instantiate p with $\perp$ and then use the law of excluded middle with the one-step derivation of $(\Gamma^o)^e, \perp \mathrel{\#} (\Delta^o)^e$ to conclude $(\Gamma^o)^e \mathrel{\#} (\Delta^o)^e$. Now we can repeatedly apply excluded middle, using Lemma 3.19, to arrive at $\Gamma \mathrel{\#} \Delta$. □