- the last rule of Π is (C);
- the last rule of Π is $(\mathrm{F}\wedge)$;
- the last rule of Π is $(\mathrm{F}\vee_l)$;
- the last rule of Π is $(\mathrm{F}\forall)$;
- the last rule of Π is $(\mathrm{F}\exists)$.

At the beginning we note the following useful property of sequents in $\mathbf{C}^{\xi}_{inv}$-derivations.

> For every rule of $\mathbf{C}^{\xi}_{inv}$ if a pair $\alpha\cdot\tau$ is in the premise then $dom(\tau) = free(\alpha)$. Furthermore we can suppose that the same property holds for all pairs $\alpha\cdot\tau$ in sequents derived by induction hypothesis because of the renaming rule. (3.2)

Indeed, $dom(\tau) \subseteq free(\alpha)$ because we assume the substitution τ to be relevant for α. Likewise, $free(\alpha) \subseteq dom(\tau)$ because we require the premises to be variable-disjoint from $var(\xi)$, and therefore, from $free(\alpha)$.

In order to simplify the presentation we suppose that every sequent Γ in the rules of $\mathbf{C}_{inv}$, except for (Ax) and (C) consists only of one formula, and the sequents in (Ax) and (C) consist of at most two formulas.

CASE (Ax). In this case the last inference has the form

$$\frac{}{\mathrm{T}\,A_1\theta_1, \mathrm{F}\,A_2\theta_2}\ (Ax),$$

where $\mathrm{T}\,A_1, \mathrm{F}\,A_2$ are free signed subformulas of ξ and $A_1\theta_1 = A_2\theta_2$. Without loss of generality we can assume that $dom(\theta_1) \subseteq free(A_1)$ $(dom(\theta_2) \subseteq free(A_2))$, otherwise we can take the substitutions $\theta_j\restriction_{free(A_j)}$ instead of θ_j, for $j = 1, 2$. Let a renaming ρ rename A_1 away from A_2. Since $A_1\rho(\rho^{-1}\theta_1) = A_2\theta_2$ we infer that $((\rho^{-1}\theta_1)\restriction_{free(A_1\rho)} \cup \theta_2)$ is a unifier of $A_1\rho$ and A_2. Therefore there exists a most general unifier σ of $A_1\rho$ and A_2 such that $((\rho^{-1}\theta_1)\restriction_{free(A_1\rho)} \cup \theta_2) = \sigma\lambda$ for a substitution λ. We can conclude that the figure

$$\frac{}{\mathrm{T}\,A_1\cdot\rho\sigma, \mathrm{F}\,A_2\cdot\sigma}\ (Ax)$$

is the intended refutation in $\mathbf{C}^{\xi}_{inv}$. Indeed,

$$A_1\theta_1 = A_1\rho((\rho^{-1}\theta_1)\restriction_{free(A_1\rho)} \cup \theta_2) = A_1\rho\sigma\lambda,$$
$$A_2\theta_2 = A_2((\rho^{-1}\theta_1)\restriction_{free(A_1\rho)} \cup \theta_2) = A_2\sigma\lambda.$$

CASE (C). The last inference in the $\mathbf{C}_{inv}$-refutation is

$$\frac{\alpha_1\theta_1, \alpha_2\theta_2}{\alpha_1\theta_1}\ (C),$$

such that $\alpha_1\theta_1 = \alpha_2\theta_2$. By the induction hypothesis, there exist substitutions τ_1, τ_2, λ' such that $\alpha_1\theta_1 = \alpha_1\tau_1\lambda'$, $\alpha_2\theta_2 = \alpha_1\tau_2\lambda'$ and the sequent $\alpha_1\cdot\tau_1, \alpha_1\cdot\tau_2$

is derivable in $\mathbf{C}^{\xi}_{inv}$. But since $\alpha_1\theta_1 = \alpha_2\theta_2$, we have $\alpha_1\tau_1\lambda' = \alpha_1\tau_2\lambda'$, so λ' is a unifier of τ_1, τ_2. Therefore, there exists a most general unifier σ of τ_1 and τ_2 and a substitution λ such that $\sigma\lambda = \lambda'$. The following inference in $\mathbf{C}^{\xi}_{inv}$ completes the proof of this case

$$\frac{\alpha_1 \cdot \tau_1, \alpha_1 \cdot \tau_2}{\alpha_1 \cdot \tau_1\sigma}\ (C).$$

CASE (F $\wedge$). The last inference in the $\mathbf{C}_{inv}$-refutation is

$$\frac{\mathrm{F}\, A_1\theta_1 \quad \mathrm{F}\, A_2\theta_2}{\mathrm{F}\, (A_1 \wedge A_2)\theta}\ (\mathrm{F}\wedge),$$

where, $\theta_j = \theta\lceil_{free(A_j)}$, for $j = 1, 2$. By the induction hypothesis, there are sequents $\mathrm{F}\, A_j \cdot \tau_j$ derivable in $\mathbf{C}^{\xi}_{inv}$ and the substitutions λ_j such that $\mathrm{F}\, A_j\theta_j = \alpha_j\tau_j\lambda_j$, for $j = 1, 2$. By using the renaming rule, we can assume $dom(\lambda_1) \cap dom(\lambda_1) = \emptyset$.
Let $\{x_1, \ldots, x_n\} = dom(\tau_1)\backslash dom(\tau_2)$ and $\{y_1, \ldots, y_m\} = dom(\tau_2)\backslash dom(\tau_1)$. Let $\lambda_1' = \lambda_1 \cup \{y_1 \mapsto y_1\tau_2\lambda_2, \ldots, y_m \mapsto y_m\tau_2\lambda_2\}$ and $\lambda_2' = \lambda_2 \cup \{x_1 \mapsto x_1\tau_1\lambda_1, \ldots, x_n \mapsto x_n\tau_1\lambda_1\}$. We can define a new substitution $\lambda' = \lambda_1' \cup \lambda_2'$ and put it instead of λ_1 and λ_2 in the previous two equations. It results in the conclusion that λ' is a unifier of τ_1 and τ_2 because for every $x \in (dom(\tau_1) \cup dom(\tau_2))$ it holds that $x\tau_1\lambda' = x\tau_2\lambda' = x\theta$.
Let $\sigma = \mathrm{mgu}(\tau_1, \tau_2)$, then $\lambda' = \sigma\lambda$ for some substitution λ.
The following inference completes the proof of this case:

$$\frac{\mathrm{F}\, A_1 \cdot \tau_1 \quad \mathrm{F}\, A_2 \cdot \tau_2}{\mathrm{F}\, A_1 \wedge A_2 \cdot \tau_1\sigma}\ (\mathrm{F}\wedge).$$

CASE (F $\vee_l$). The last inference in the $\mathbf{C}_{inv}$-refutation is

$$\frac{\mathrm{F}\, A\theta_1}{\mathrm{F}\, (A_1 \vee A_2)\theta}\ (\mathrm{F}\vee_l)$$

where $\theta_1 = \theta\lceil_{free(A_1)}$.
By the induction hypothesis, there exist a sequent $\mathrm{F}\, A_1 \cdot \tau_1$ refutable in $\mathbf{C}^{\xi}_{inv}$ and a substitution λ' such that $\mathrm{F}\, A_1\theta_1 = \mathrm{F}\, A\tau_1\lambda'$. Let us show that the following inference completes the proof of this case:

$$\frac{\mathrm{F}\, A_1 \cdot \tau_1}{\mathrm{F}\, A_1 \vee A_2 \cdot \tau_1}\ (\mathrm{F}\vee_l).$$

Let $\{x_1, \ldots, x_n\} = dom(\theta) \setminus free(A)$. Define $\lambda = \lambda' \cup \{x_1 \mapsto x_1\theta, \ldots x_n \mapsto x_n\theta\}$. Using $vran(\tau_1) \cap var(A_2) = \emptyset$, it is not hard to argue that $(\mathrm{F}\, A_1 \vee A_2)\tau\lambda = (\mathrm{F}\, A_1 \vee A_2)\theta$.
CASE (F $\forall$). The last inference of the $\mathbf{C}_{inv}$-refutation is

$$\frac{\mathrm{F}\, A(y) \cdot \theta}{\mathrm{F}\, \forall x A(x) \cdot \theta}\ (\mathrm{F}\forall). \tag{3.3}$$