

Abstract Syntax

Types	$\tau ::= \tau_1 \rightarrow \tau_2 \mid \forall \alpha. \tau \mid \alpha \mid \text{bool} \mid \tau_1 \times \tau_2$	
Terms	$e ::= x \mid \lambda x:\tau. e \mid e_1 e_2$	$(\rightarrow)$
	$\mid \Lambda \alpha. e \mid e[\tau]$	$(\forall)$
	$\mid \text{true} \mid \text{false} \mid \text{if } e_1 e_2 e_3$	(bool)
	$\mid \langle e_1, e_2 \rangle \mid \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e')$	$(\times)$
Contexts	$\Gamma ::= \cdot \mid \Gamma, \alpha \text{ type} \mid \Gamma, x : \tau$	(all variables distinct)

Judgments

$\Gamma \text{ ctx}$	Γ is a valid context	
$\Gamma \vdash \tau \text{ type}$	τ is a valid type	presupposes $\Gamma \text{ ctx}$
$\Gamma \vdash e : \tau$	expression e has type τ	presupposes $\Gamma \text{ ctx}$, ensures $\Gamma \vdash \tau \text{ type}$
$e \text{ value}$	expression e is a value	presupposes $\cdot \vdash e : \tau$ for some τ
$e \mapsto e'$	expression e steps to e'	presupposes $\cdot \vdash e : \tau$ for some τ

Contexts Γ

$\frac{}{(\cdot) \text{ ctx}} \text{ ctx/emp}$	$\frac{\Gamma \text{ ctx}}{(\Gamma, \alpha \text{ type}) \text{ ctx}} \text{ ctx/tpvar}$	$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash \tau \text{ type}}{(\Gamma, x : \tau) \text{ ctx}} \text{ ctx/var}$
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Functions $\tau_1 \rightarrow \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \rightarrow \tau_2 \text{ type}} \text{ tp/arrow}$		
$\frac{\Gamma, x_1 : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \lambda x_1:\tau_1. e_2 : \tau_1 \rightarrow \tau_2} \text{ tp/lam}$	$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \text{ tp/var}$	$\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau_1} \text{ tp/app}$
$\frac{}{\lambda x. e \text{ value}} \text{ val/lam}$	$\frac{e_2 \text{ value}}{(\lambda x. e_1) e_2 \mapsto [e_2/x]e_1} \text{ step/app/lam}$	
$\frac{e_1 \mapsto e'_1}{e_1 e_2 \mapsto e'_1 e_2} \text{ step/app}_1$	$\frac{e_1 \text{ value} \quad e_2 \mapsto e'_2}{e_1 e_2 \mapsto e_1 e'_2} \text{ step/app}_2$	

Polymorphic Types $\forall\alpha. \tau$

$\frac{\alpha \text{ type} \in \Gamma}{\Gamma \vdash \alpha \text{ type}} \text{ tp/tpvar}$	$\frac{\Gamma, \alpha \text{ type} \vdash \tau \text{ type}}{\Gamma \vdash \forall \alpha. \tau \text{ type}} \text{ tp/forall}$
$\frac{\Gamma, \alpha \text{ type} \vdash e : \tau}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. \tau} \text{ tp/tplam}$	$\frac{\Gamma \vdash e : \forall \alpha. \tau \quad \Gamma \vdash \sigma \text{ type}}{\Gamma \vdash e[\sigma] : [\sigma/\alpha]\tau} \text{ tp/tpapp}$
$\frac{}{\Lambda \alpha. e \text{ value}} \text{ val/tplam}$	$\frac{}{(\Lambda \alpha. e)[\tau] \mapsto [\tau/\alpha]e} \text{ step/tpapp/tplam}$ $\frac{e \mapsto e'}{e[\tau] \mapsto e'[\tau]} \text{ step/tpapp}$

Booleans **bool**

$\frac{}{\Gamma \vdash \text{bool type}} \text{ tp/bool}$	
$\frac{}{\Gamma \vdash \text{true} : \text{bool}} \text{ tp/true}$	$\frac{}{\Gamma \vdash \text{false} : \text{bool}} \text{ tp/false}$
$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if } e_1 \text{ } e_2 \text{ } e_3 : \tau} \text{ tp/if}$	
$\frac{}{\text{true value}} \text{ val/true}$	$\frac{}{\text{false value}} \text{ val/false}$
$\frac{}{\text{if true } e_2 \text{ } e_3 \mapsto e_2} \text{ step/if/true}$	$\frac{}{\text{if false } e_2 \text{ } e_3 \mapsto e_3} \text{ step/if/false}$
$\frac{e_1 \mapsto e'_1}{\text{if } e_1 \text{ } e_2 \text{ } e_3 \mapsto \text{if } e'_1 \text{ } e_2 \text{ } e_3} \text{ step/if}$	

Pairs $\tau_1 \times \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \times \tau_2 \text{ type}} \text{ tp/times}$	
$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ tp/pair}$	$\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e' : \tau'}{\Gamma \vdash \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e') : \tau'} \text{ tp/casep}$
$\frac{e_1 \text{ value} \quad e_2 \text{ value}}{\langle e_1, e_2 \rangle \text{ value}} \text{ val/pair}$	$\frac{e_1 \mapsto e'_1}{\langle e_1, e_2 \rangle \mapsto \langle e'_1, e_2 \rangle} \text{ step/pair}_1 \quad \frac{v_1 \text{ value} \quad e_2 \mapsto e'_2}{\langle v_1, e_2 \rangle \mapsto \langle v_1, e'_2 \rangle} \text{ step/pair}_2$
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto \text{case } e'_0 (\langle x_1, x_2 \rangle \Rightarrow e_3)} \text{ step/casep}_0$	
$\frac{v_1 \text{ value} \quad v_2 \text{ value}}{\text{case } \langle v_1, v_2 \rangle (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto [v_1/x_2][v_2/x_2]e_3} \text{ step/casep/pair}$	

Theorems

Canonical Forms Assume $\cdot \vdash e : \tau$ and e *value*. Then

- (i) If $\tau = \tau_1 \rightarrow \tau_2$ then $v = \lambda x. e_2$ for some e_2 .
- (ii) If $\tau = \forall \alpha. \tau'$ then $e = \Lambda \alpha. e'$ for some e' .
- (iii) If $\tau = \text{bool}$ then $e = \text{true}$ or $e = \text{false}$.
- (iv) If $\tau = \tau_1 \times \tau_2$ then $e = \langle e_1, e_2 \rangle$ for some e_1 *value* and e_2 *value*.

Preservation. If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

Progress. If $\cdot \vdash e : \tau$ then either $e \mapsto e'$ for some e' or e *value*.

Finality of Values. There is no $\cdot \vdash e : \tau$ such that $e \mapsto e'$ for some e' and e *value*.

Determinacy. If $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.