Abstract Syntax

Types	τ	$::=$	$\tau_1 \rightarrow \tau_2 \mid \forall \alpha.\, \tau \mid \alpha \mid \tau_1 \times \tau_2 \mid 1 \mid \tau_1 + \tau_2 \mid 0 \mid \mu\alpha.\, \tau$	
Terms	e	$::=$	$x \mid \lambda x.\, e \mid e_1\, e_2$	$(\rightarrow)$
		$\mid$	$\Lambda\alpha.\, e \mid e\,[\tau]$	$(\forall)$
		$\mid$	$\langle e_1, e_2 \rangle \mid \mathsf{case}\ e\ (\langle x_1, x_2 \rangle \Rightarrow e')$	$(\times)$
		$\mid$	$\langle\,\rangle \mid \mathsf{case}\ e\ (\langle\,\rangle \Rightarrow e')$	(1)
		$\mid$	$\mathbf{l} \cdot e \mid \mathbf{r} \cdot e \mid \mathsf{case}\ e\ (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2)$	$(+)$
		$\mid$	$\mathsf{case}\ e\ (\,)$	(0)
		$\mid$	$\mathsf{fold}\ e \mid \mathsf{unfold}\ e$	(μ)
		$\mid$	$\mathsf{fix}\ f.\, e$	
Contexts	Γ	$::=$	$\cdot \mid \Gamma, \alpha\ \mathit{type} \mid \Gamma, x : \tau$	(all variables distinct)

Judgments

Γ *ctx*	Γ is a valid context	
$\Gamma \vdash \tau$ *type*	τ is a valid type	presupposes Γ *ctx*
$\Gamma \vdash e : \tau$	expression e has type τ	presupposes Γ *ctx*, ensures $\Gamma \vdash \tau$ *type*
e *value*	expression e is a value	presupposes $\cdot \vdash e : \tau$ for some τ
$e \mapsto e'$	expression e steps to e'	presupposes $\cdot \vdash e : \tau$ for some τ

Theorems

Preservation. If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

Progress. For every expression $\cdot \vdash e : \tau$ either $e \mapsto e'$ for some e' or e *value*.

Finality of Values. There is no $\cdot \vdash e : \tau$ such that $e \mapsto e'$ for some e' and e *value*.

Sequentiality. If $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.

Canonical Forms. Assume $\cdot \vdash v : \tau$ and v *value*.

(i) If $\tau = \tau_1 \rightarrow \tau_2$ then $v = \lambda x.\, e_2$ for some e_2

(ii) If $\tau = \forall\alpha.\, \tau'$ then $v = \Lambda\alpha.\, e'$ for some e'

(iii) If $\tau = \tau_1 \times \tau_2$ then $v = \langle v_1, v_2 \rangle$ for some v_1 *value* and v_2 *value*

(iv) If $\tau = 1$ then $v = \langle\,\rangle$

(v) If $\tau = \tau_1 + \tau_2$ then $v = \mathbf{l} \cdot v_1$ for some v_1 *value* or $v = \mathbf{r} \cdot v_2$ for some v_2 *value*

(vi) If $\tau = 0$ then we have a contradiction

(vii) If $\tau = \mu\alpha.\, \tau'$ then $v = \mathsf{fold}\ v'$ for some v' *value*

Contexts Γ

$$\frac{}{(\cdot)\ \mathit{ctx}}\ \mathsf{ctx/emp} \qquad \frac{\Gamma\ \mathit{ctx}}{(\Gamma, \alpha\ \mathit{type})\ \mathit{ctx}}\ \mathsf{ctx/tpvar} \qquad \frac{\Gamma\ \mathit{ctx} \quad \Gamma \vdash \tau\ \mathit{type}}{(\Gamma, x : \tau)\ \mathit{ctx}}\ \mathsf{ctx/var}$$

Functions $\tau_1 \to \tau_2$

$$\frac{\Gamma \vdash \tau_1\ \mathit{type} \quad \Gamma \vdash \tau_2\ \mathit{type}}{\Gamma \vdash \tau_1 \to \tau_2\ \mathit{type}}\ \mathsf{tp/arrow}$$

$$\frac{\Gamma \vdash \tau_1\ \mathit{type} \quad \Gamma, x_1 : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \lambda x_1{:}\tau_1.\, e_2 : \tau_1 \to \tau_2}\ \mathsf{tp/lam} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \mathsf{tp/var} \qquad \frac{\Gamma \vdash e_1 : \tau_2 \to \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\, e_2 : \tau_1}\ \mathsf{tp/app}$$

$$\frac{}{\lambda x.\, e\ \mathit{value}}\ \mathsf{val/lam} \qquad \frac{e_2\ \mathit{value}}{(\lambda x.\, e_1)\, e_2 \mapsto [e_2/x]e_1}\ \mathsf{step/app/lam}$$

$$\frac{e_1 \mapsto e_1'}{e_1\, e_2 \mapsto e_1'\, e_2}\ \mathsf{step/app_1} \qquad \frac{e_1\ \mathit{value} \quad e_2 \mapsto e_2'}{e_1\, e_2 \mapsto e_1\, e_2'}\ \mathsf{step/app_2}$$

Polymorphic Types $\forall \alpha.\, \tau$

$$\frac{\alpha\ \mathit{type} \in \Gamma}{\Gamma \vdash \alpha\ \mathit{type}}\ \mathsf{tp/tpvar} \qquad \frac{\Gamma, \alpha\ \mathit{type} \vdash \tau\ \mathit{type}}{\Gamma \vdash \forall \alpha.\, \tau\ \mathit{type}}\ \mathsf{tp/forall}$$

$$\frac{\Gamma, \alpha\ \mathit{type} \vdash e : \tau}{\Gamma \vdash \Lambda \alpha.\, e : \forall \alpha.\, \tau}\ \mathsf{tp/tplam} \qquad \frac{\Gamma \vdash e : \forall \alpha.\, \tau \quad \Gamma \vdash \sigma\ \mathit{type}}{\Gamma \vdash e\, [\sigma] : [\sigma/\alpha]\tau}\ \mathsf{tp/tpapp}$$

$$\frac{}{\Lambda \alpha.\, e\ \mathit{value}}\ \mathsf{val/tplam} \qquad \frac{}{(\Lambda \alpha.\, e)\, [\tau] \mapsto [\tau/\alpha]e}\ \mathsf{step/tpapp/tplam} \qquad \frac{e \mapsto e'}{e\, [\tau] \mapsto e'\, [\tau]}\ \mathsf{step/tpapp}$$

Pairs $\tau_1 \times \tau_2$

$$\frac{\Gamma \vdash \tau_1\ \mathit{type} \quad \Gamma \vdash \tau_2\ \mathit{type}}{\Gamma \vdash \tau_1 \times \tau_2\ \mathit{type}}\ \textsf{tp/prod}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}\ \textsf{tp/pair} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e' : \tau'}{\Gamma \vdash \textsf{case}\ e\ (\langle x_1, x_2 \rangle \Rightarrow e') : \tau'}\ \textsf{tp/casep}$$

$$\frac{e_1\ \mathit{value} \quad e_2\ \mathit{value}}{\langle e_1, e_2 \rangle\ \mathit{value}}\ \textsf{val/pair} \qquad \frac{v_1\ \mathit{value} \quad v_2\ \mathit{value}}{\textsf{case}\ \langle v_1, v_2 \rangle\ (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto [v_1/x_1][v_2/x_2]e_3}\ \textsf{step/casep/pair}$$

$$\frac{e_1 \mapsto e_1'}{\langle e_1, e_2 \rangle \mapsto \langle e_1', e_2 \rangle}\ \textsf{step/pair}_1 \qquad \frac{v_1\ \mathit{value} \quad e_2 \mapsto e_2'}{\langle v_1, e_2 \rangle \mapsto \langle v_1, e_2' \rangle}\ \textsf{step/pair}_2$$

$$\frac{e_0 \mapsto e_0'}{\textsf{case}\ e_0\ (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto \textsf{case}\ e_0'\ (\langle x_1, x_2 \rangle \Rightarrow e_3)}\ \textsf{step/casep}_0$$

Unit 1

$$\frac{}{\Gamma \vdash 1\ \mathit{type}}\ \textsf{tp/one} \qquad \frac{}{\Gamma \vdash \langle\,\rangle : 1}\ \textsf{tp/unit} \qquad \frac{\Gamma \vdash e : 1 \quad \Gamma \vdash e' : \tau'}{\Gamma \vdash \textsf{case}\ e\ (\langle\,\rangle \Rightarrow e') : \tau'}\ \textsf{tp/caseu}$$

$$\frac{}{\langle\,\rangle\ \mathit{value}}\ \textsf{val/unit} \qquad \frac{}{\textsf{case}\ \langle\,\rangle\ (\langle\,\rangle \Rightarrow e) \mapsto e}\ \textsf{step/caseu/unit}$$

$$\frac{e_0 \mapsto e_0'}{\textsf{case}\ e_0\ (\langle\,\rangle \Rightarrow e_1) \mapsto \textsf{case}\ e_0'\ (\langle\,\rangle \Rightarrow e_1)}\ \textsf{step/caseu}_0$$

Sums $\tau_1 + \tau_2$

$$\frac{\Gamma \vdash \tau_1 \; type \quad \Gamma \vdash \tau_2 \; type}{\Gamma \vdash \tau_1 + \tau_2 \; type} \; \mathsf{tp/sum}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash \tau_2 \; type}{\Gamma \vdash \mathbf{l} \cdot e_1 : \tau_1 + \tau_2} \; \mathsf{tp/left} \qquad \frac{\Gamma \vdash \tau_1 \; type \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \mathbf{r} \cdot e_2 : \tau_1 + \tau_2} \; \mathsf{tp/right}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \sigma \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \sigma}{\Gamma \vdash \mathsf{case}\; e \; (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) : \sigma} \; \mathsf{tp/cases}$$

$$\frac{e_1 \; value}{\mathbf{l} \cdot e_1 \; value} \; \mathsf{val/left} \qquad \frac{e_2 \; value}{\mathbf{r} \cdot e_2 \; value} \; \mathsf{val/right}$$

$$\frac{v_1 \; value}{\mathsf{case}\; \mathbf{l} \cdot v_1 \; (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_1/x_1]e_1} \; \mathsf{step/cases/left}$$

$$\frac{v_2 \; value}{\mathsf{case}\; \mathbf{r} \cdot v_2 \; (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_2/x_2]e_2} \; \mathsf{step/cases/right}$$

$$\frac{e_1 \mapsto e_1'}{\mathbf{l} \cdot e_1 \mapsto \mathbf{l} \cdot e_1'} \; \mathsf{step/left} \qquad \frac{e_2 \mapsto e_2'}{\mathbf{r} \cdot e_2 \mapsto \mathbf{r} \cdot e_2'} \; \mathsf{step/right}$$

$$\frac{e_0 \mapsto e_0'}{\mathsf{case}\; e_0 \; (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto \mathsf{case}\; e_0 \; (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2)} \; \mathsf{step/cases}_0$$

Empty Type 0

$$\frac{}{\Gamma \vdash 0 \; type} \; \mathsf{tp/zero}$$

(no constructor)

$$\frac{\Gamma \vdash e_0 : 0 \quad \Gamma \vdash \tau \; type}{\Gamma \vdash \mathsf{case}\; e_0 \; (\,) : \tau} \; \mathsf{tp/casez}$$

(no values)

$$\frac{e_0 \mapsto e_0'}{\mathsf{case}\; e_0 \; (\,) \mapsto \mathsf{case}\; e_0' \; (\,)} \; \mathsf{step/casez}_0$$

Recursive Types $\mu\alpha.\,\tau$

$$\frac{\Gamma, \alpha\ \mathit{type} \vdash \tau\ \mathit{type}}{\Gamma \vdash \mu\alpha.\,\tau\ \mathit{type}}\ \mathsf{tp/rho}$$

$$\frac{\Gamma \vdash e : [\mu\alpha.\,\tau/\alpha]\tau}{\Gamma \vdash \mathsf{fold}\ e : \mu\alpha.\,\tau}\ \mathsf{tp/fold} \qquad \frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash \mathsf{unfold}\ e : [\mu\alpha.\,\tau/\alpha]\tau}\ \mathsf{tp/unfold}$$

$$\frac{e\ \mathit{value}}{\mathsf{fold}\ e\ \mathit{value}}\ \mathsf{val/fold} \qquad \frac{v\ \mathit{value}}{\mathsf{unfold}\ (\mathsf{fold}\ v) \mapsto v}\ \mathsf{step/unfold/fold}$$

$$\frac{e \mapsto e'}{\mathsf{fold}\ e \mapsto \mathsf{fold}\ e'}\ \mathsf{step/fold} \qquad \frac{e \mapsto e'}{\mathsf{unfold}\ e \mapsto \mathsf{unfold}\ e'}\ \mathsf{step/unfold}_0$$

Recursion

$$\frac{\Gamma, f : \tau \vdash e : \tau}{\Gamma \vdash \mathsf{fix}\ f{:}\tau.\, e : \tau}\ \mathsf{tp/fix} \qquad \frac{}{\mathsf{fix}\ f.\, e \mapsto [\mathsf{fix}\ f.\, e/f]e}\ \mathsf{step/fix}$$