Constructive Logic (15-317), Fall 2009
Assignment 1: Natural Deduction

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Out: Thursday, September 3, 2009
Due: Thursday, September 10, 2009 (before class)

Welcome to 15-317, Fall 2009 edition! In this, your first homework assignment, you will practice essential logic skills like carrying out proofs in natural deduction and verifying the harmony of logical connectives.

The Tutch portion of your work (Section 2) should be submitted electronically using the command

```
$ /afs/andrew/course/15/317/bin/submit -r hw01 <files...>
```

from any Andrew server. You may check the status of your submission by running the command

```
$ /afs/andrew/course/15/317/bin/status hw01
```

If you have trouble running either of these commands, email William.

The written portion of your work (Sections 1 and 3) should be submitted at the beginning of class. If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand.

1 Bulletin Board (3 points)

The course has an Andrew BBoard, academic.cs.15-317, which you are expected to read. We may occasionally post announcements, clarifications, or corrections, so you should check the BBoard often.

You will find a current post on the BBoard entitled "Homework 1", and the contents of that post will reveal to you a password.

Task 1 (3 pts)**.** What is the password?

2 Tutch Proofs (15 points)

Task 2 (15 pts)**.** Prove the following theorems using Tutch:

```
proof K : A => B => A
proof S : (A => B => C) => (A => B) => A => C
proof orComm : A | B => B | A
proof distribOrImp : (A | B => C) => (A => C) & (B => C)
proof clue : (P => (C & K) | (D & L))
            => (~K => S)
            => (D | L)
            => (P => ~S) & (S => ~P)
            => (C => ~D) & (D => ~C)
            => (K => ~L) & (L => ~K)
            => ~P
```

On Andrew machines, you can check your progress against the requirements file `/afs/andrew/course/15/317/req/hw01.req` by running the command

```
$ /afs/andrew/course/15/317/bin/tutch -r hw01 <files...>
```

3 Harmony and Derivability (22 points)

3.1 Conjunction revisited

When we formulated elimination rules for conjunction, we said that given a proof of $A \wedge B$ *true*, you could conclude either A *true* or B *true*. We could equally well have specified an elimination rule that reasons toward a generic conclusion C *true* from hypotheses of A *true* and B *true*. We can illustrate this precisely by specifying a new connective $A \times B$ with the same introduction rule as $A \wedge B$ but a new elimination rule.

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \times B\ \mathit{true}} \times I \qquad\qquad \frac{A \times B\ \mathit{true} \qquad \begin{array}{c} \overline{A\ \mathit{true}}^{\,u}, \ \overline{B\ \mathit{true}}^{\,v} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}} \times E^{u,v}$$

In the elim rule's hypothetical proof of C *true*, the hypotheses A *true* and B *true* may each be used any number of times (including zero).

Task 3 (3 pts)**.** Show that the rule $\times E^{u,v}$ is locally sound by giving an appropriate local reduction.

Task 4 (3 pts)**.** Show that the rule $\times E^{u,v}$ is locally complete by giving an appropriate local expansion.

Although showing local soundness and completeness for a set of elimination rules gives us some confidence that they are sensibly harmonious with the introduction rules, it does not tell us everything about the behavior of those elimination rules. We can justify the assertion that the new elimination rule $\times E^{u,v}$ is equivalent to the original conjunction elimination rules $\wedge E_L$ and $\wedge E_R$ by showing that the two sets of rules are *interderivable*.

Task 5 (4 pts)**.** Show that the following rules are derivable by providing derivations in terms of the rule $\times E^{u,v}$:

$$\frac{A \times B\ \mathit{true}}{A\ \mathit{true}}\ \times E'_L \qquad\qquad \frac{A \times B\ \mathit{true}}{B\ \mathit{true}}\ \times E'_R$$

Task 6 (4 pts)**.** Show that the corresponding new rule $\wedge E'^{\,u,v}$ is derivable by showing how you could replace any subdeduction of the form

$$\dfrac{\begin{array}{c} \\ \mathcal{D} \\ A \wedge B\ \mathit{true} \end{array} \qquad \begin{array}{c} \overline{A\ \mathit{true}}^{\,u},\ \overline{B\ \mathit{true}}^{\,v} \\ \mathcal{E} \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}}\ \wedge E'^{\,u,v}$$

with one that uses only the rules $\wedge E_L$ and $\wedge E_R$. (**Hint**: think of the substitution principle!)

3.2 To call a spade a spade

Consider a new connective $\spadesuit(A, B, C)$ defined by the introduction rule:

$$\dfrac{\begin{array}{c} \overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ C\ \mathit{true} \end{array} \qquad \begin{array}{c} \overline{B\ \mathit{true}}^{\,v} \\ \vdots \\ C\ \mathit{true} \end{array}}{\spadesuit(A, B, C)\ \mathit{true}}\ \spadesuit I^{u,v}$$

with the following candidate elimination rule:

$$\frac{\spadesuit(A, B, C)\ \mathit{true} \quad A\ \mathit{true} \quad B\ \mathit{true}}{C\ \mathit{true}}\ \spadesuit E$$

Task 7 (4 pts)**.** Is this elimination rule locally sound? If so, give an appropriate local reduction; otherwise, explain (informally) why no such reduction exists.

Task 8 (4 pts)**.** Is this elimination rule locally complete? If so, give an appropriate local expansion; otherwise, explain (informally) why no such expansion exists.