Solutions 1:
Natural Deduction

15-317: Constructive Logic

Out: Thursday, September 4, 2008
Due: Thursday, September 11, 2008, before class

1 Local Soundness and Completeness (12 pts)

See the Lecture 3 (Harmony) notes for a discussion of local soundness and completeness.

1.1 Hearts

Consider a connective defined by the following rules:

$$\frac{A\ \mathsf{true}}{A \heartsuit B\ \mathsf{true}}\ \heartsuit I_L \qquad \frac{B\ \mathsf{true}}{A \heartsuit B\ \mathsf{true}}\ \heartsuit I_R \qquad \frac{A \heartsuit B\ \mathsf{true} \qquad \begin{array}{c}\overline{A\ \mathsf{true}}^{\,u}\ ,\ \overline{B\ \mathsf{true}}^{\,v} \\ \vdots \\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \heartsuit E^{u,v}$$

Task 3 (3 pts). Is this connective locally sound? If so, show the reduction; if not, explain (informally) why no such reduction exists.

Solution. It is not sound. The setup for one case that we must consider is as follows:

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathsf{true}\end{array}}{A \heartsuit B\ \mathsf{true}}\ \heartsuit I_L \qquad \begin{array}{c}\overline{A\ \mathsf{true}}^{\,u}\ ,\ \overline{B\ \mathsf{true}}^{\,v} \\ \vdots \\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \heartsuit E^{u,v}$$

We need a proof of C. Substituting $\mathcal{D}$ for the assumption u leaves a derivation

$$\begin{array}{c}\overline{B\ \mathsf{true}}^{\,v} \\ \vdots \\ C\ \mathsf{true}\end{array}$$

but we do not have a proof of B to substitute for the remaining hypothesis. The other case, where the introduction rule used is $\heartsuit I_R$, is unsound for symmetric reasons.

It should not be surprising that this connective is unsound, since it mixes the intro rules for $\vee$ with an elimination rule for $\wedge$: the elim rule takes out more than the intro rule puts in.

Task 4 (3 pts). Is this connective locally complete? If so, show the expansion; if not, explain (informally) why no such expansion exists.

Solution. The connective is locally complete. Here is one possible expansion:

$$\begin{array}{c}\mathcal{D}\\ A \heartsuit B\ \mathsf{true}\end{array} \;\Longrightarrow_E\; \dfrac{\begin{array}{c}\mathcal{D}\\ A \heartsuit B\ \mathsf{true}\end{array} \qquad \dfrac{\dfrac{}{A\ \mathsf{true}}\,u}{A \heartsuit B\ \mathsf{true}}\,\heartsuit I_L}{A \heartsuit B\ \mathsf{true}}\,\heartsuit E^{u,v}$$

It is also possible to use $\heartsuit I_R$ and the assumption v of B true. The fact that there are two possible expansions should make you suspicious about soundness: not all of the information that the elim rule produces is necessary to introduce the connective.

1.2 Clubs

Consider a connective defined by the following rules:

$$\dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ B\ \mathsf{true}\end{array}}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_L^u \qquad \dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ C\ \mathsf{true}\end{array}}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_R^u \qquad \dfrac{\clubsuit(A,B,C)\ \mathsf{true} \quad A\ \mathsf{true} \quad \begin{array}{c}\overline{B\ \mathsf{true}}\ u\\ \vdots\\ D\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathsf{true}}\ v\\ \vdots\\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\,\clubsuit E^{u,v}$$

Task 1 (3 pts). Is this connective locally sound? If so, show the reduction; if not, explain (informally) why no such reduction exists.

Solution. Yes, it is locally sound. Two reductions are necessary:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\ w\\ \mathcal{D}\\ B\ \mathsf{true}\end{array}}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_L^w \quad \begin{array}{c}\mathcal{E}\\ A\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathsf{true}}\ u\\ \mathcal{F}_1\\ D\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathsf{true}}\ v\\ \mathcal{F}_2\\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\,\clubsuit E^{u,v} \;\Longrightarrow_R\; \begin{array}{c}\mathcal{E}\\ \overline{A\ \mathsf{true}}\ w\\ \mathcal{D}\\ B\ \mathsf{true}\\ \overline{B\ \mathsf{true}}\ u\\ \mathcal{F}_1\\ D\ \mathsf{true}\end{array}$$

i.e. $[\,[\mathcal{E}/w]\mathcal{D}\,/\,u]\mathcal{F}_1$

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\ w\\ \mathcal{D}\\ C\ \mathsf{true}\end{array}}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_R^w \quad \begin{array}{c}\mathcal{E}\\ A\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathsf{true}}\ u\\ \mathcal{F}_1\\ D\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathsf{true}}\ v\\ \mathcal{F}_2\\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\,\clubsuit E^{u,v} \;\Longrightarrow_R\; \begin{array}{c}\mathcal{E}\\ \overline{A\ \mathsf{true}}\ w\\ \mathcal{D}\\ C\ \mathsf{true}\\ \overline{C\ \mathsf{true}}\ v\\ \mathcal{F}_2\\ D\ \mathsf{true}\end{array}$$

i.e. $[\,[\mathcal{E}/w]\mathcal{D}\,/\,u]\mathcal{F}_2$

Task 2 (3 pts). Is this connective locally complete? If so, show the expansion; if not, explain (informally) why no such expansion exists.

Solution. This connective is not locally complete.

Note that the intro rules are essentially the intros for $(A \supset B) \vee (A \supset C)$, but the elim rule is essentially the elim rule for $A \supset (B \vee C)$. The problem is that the intro rule forces the choice between B and C to be made too early for this elimination rule. For example, if we try to expand using $\clubsuit I_L$ first, then we get stuck in the second branch of the elim:

$$\begin{array}{c}\mathcal{D}\\ \clubsuit(A,B,C)\ \mathsf{true}\end{array} \Longrightarrow_E \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \clubsuit(A,B,C)\ \mathsf{true}\end{array} \quad \dfrac{}{A\ \mathsf{true}}\,u \quad \dfrac{}{B\ \mathsf{true}}\,v_1 \quad \begin{array}{c}\dfrac{}{C\ \mathsf{true}}\,v_2\\ \vdots\\ B\ \mathsf{true}\end{array}}{B\ \mathsf{true}}\,\clubsuit E^{v_1,v_2}}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_L^u$$

Symmetrically, if we tried $\clubsuit I_R$ first, then we'd get stuck in the first branch.

Finally, we cannot use $\clubsuit E$ at the outside (like you do for disjunction) because then we don't have a proof of A:

$$\begin{array}{c}\mathcal{D}\\ \clubsuit(A,B,C)\ \mathsf{true}\end{array} \Longrightarrow_E \dfrac{\begin{array}{c}\mathcal{D}\\ \clubsuit(A,B,C)\ \mathsf{true}\end{array} \quad \dfrac{???}{A\ \mathsf{true}} \quad \dfrac{\dfrac{}{B\ \mathsf{true}}\,v_1}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_L \quad \dfrac{\dfrac{}{C\ \mathsf{true}}\,v_2}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit I_R}{\clubsuit(A,B,C)\ \mathsf{true}}\,\clubsuit E^{v_1,v_2}$$