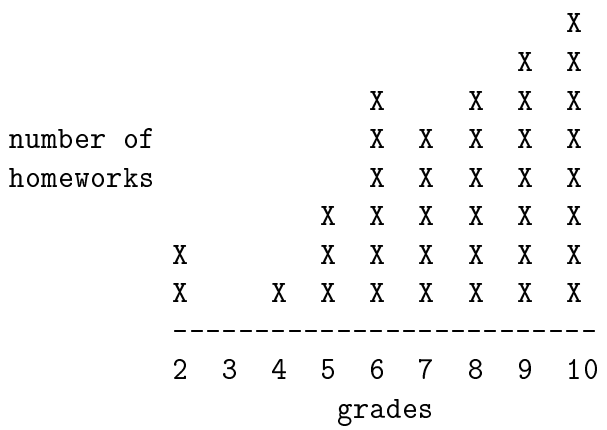
# Automata Theory: Solutions 1




The histogram shows the distribution of grades, from 0 to 10.

**Problem 1**
Consider the following sets of integer numbers:

$S_1 = \{4, 5, 6\}$
$S_2 = \{i : \ i \text{ is even}\}$
$S_3 = \{i : \ i \text{ is divisible by } 3\}$

For each set below, specify its elements and determine whether it is finite or infinite:

$S_4 = S_1 \times S_1 = \{(4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5), (6,6)\}$
$S_5 = 2^{S_1} = \{\emptyset, \{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}\}$
$S_6 = S_1 \cap S_2 = \{4, 6\}$
$S_7 = S_2 \cap S_3 = \{i : \ i \text{ is divisible by } 6\}$

The sets $S_4$, $S_5$, and $S_6$ are finite, whereas $S_7$ is infinite.

**Problem 2**
Prove that, if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.

By the definition of subsets, we need to show that every element $x$ of $\overline{S_2}$ is also an element of $\overline{S_1}$. The proof is as follows:

$$\begin{aligned} x \in \overline{S_2} &\Rightarrow x \notin S_2 \quad \text{(by definition of } \overline{S_2}\text{)} \\ &\Rightarrow x \notin S_1 \quad \text{(because } S_1 \subseteq S_2\text{)} \\ &\Rightarrow x \in \overline{S_1} \quad \text{(by definition of } \overline{S_1}\text{)} \end{aligned}$$

**Problem 3**

**(a)** $1 + 2 + 3 + 4 + ... + n = \frac{n \cdot (n+1)}{2}$.

We use a proof by induction. Clearly, the equality holds for $n = 1$; we now show that, if it holds for $n$, then it also holds for $n + 1$:

$$\begin{aligned} 1 + 2 + ... + n + (n+1) &= (1 + 2 + ... + n) + (n+1) \\ &= \frac{n \cdot (n+1)}{2} + (n+1) \\ &= \frac{n \cdot (n+1) + 2 \cdot (n+1)}{2} \\ &= \frac{(n+2) \cdot (n+1)}{2} \\ &= \frac{(n+1) \cdot ((n+1)+1)}{2} \end{aligned}$$

**(b)** $1 + x + x^2 + x^3 + ... + x^n = \frac{x^{n+1}-1}{x-1}$ (where $x \neq 1$).

We observe that the equality holds for $n = 1$, and apply induction to "step" from $n$ to $n+1$:

$$\begin{aligned} 1 + x + ... + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + x^{n+1} \cdot (x-1)}{x-1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x-1} \\ &= \frac{x^{(n+1)+1} - 1}{x-1} \end{aligned}$$