Strong Restricted-Orientation Convexity

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Abstract

Strong restricted-orientation convexity is a generalization of standard convexity. We explore the properties of strongly convex sets in multidimensional Euclidean space and identify major properties of standard convex sets that also hold for strong convexity.

We characterize strongly convex flats and halfspaces, and establish the strong convexity of the affine hull of a strongly convex set. We then show that, for every point in the boundary of a strongly convex set, there is a supporting strongly convex hyperplane through it. Finally, we show that a closed set with nonempty interior is strongly convex if and only if it is the intersection of strongly convex halfspaces; we state a condition under which this result extends to sets with empty interior.

Keywords: generalized convexity, restricted orientations, higher dimensions.

MSC 1991 classification: 52A20 (convex sets in d dimensions).

1 Introduction

Convex sets are a comparatively recent yet fruitful concept in geometry, which has applications in optimization, statistics, geometric number theory, functional analysis, and combinatorics [Klee, 1971; Preparata and Shamos, 1985], as well as in more practical areas, such as VLSI design, computer graphics, architectural databases, and geographic databases. For example, the convex hull of a geometric object is often used as an approximation of the object. As another example, decomposing a polygon into convex subpolygons makes polygonal processing easier to handle.

Researchers have studied many notions of nontraditional convexity, such as orthogonal convexity [Montuno and Fournier, 1982; Nicholl *et al.*, 1983; Ottmann *et al.*, 1984], finitely oriented convexity [Güting, 1983b; Widmayer *et al.*, 1987; Rawlins and Wood, 1987], restricted-orientation convexity [Rawlins, 1987; Rawlins and Wood, 1991; Schuierer, 1991], NESW convexity [Lipski and Papadimitriou, 1981; Soisalon-Soininen and Wood, 1984; Widmayer *et al.*, 1987], and link convexity [Bruckner and Bruckner, 1962; Valentine, 1965; Schuierer, 1991].

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Rawlins introduced the notion of strong restricted-orientation convexity in two dimensions, as part of his investigation of restricted-orientation visibility [Rawlins, 1987]. This notion is stronger than standard convexity, hence the name. Rawlins and Wood [Rawlins and Wood, 1988; Rawlins and Wood, 1991] studied the properties of strongly convex sets in two dimensions and demonstrated that strong convexity generalizes not only standard convexity but also the notions of ortho-rectangles (that is, rectangles whose edges are parallel to the coordinate axes) and convex C-oriented polygons [Güting, 1983a; Güting, 1984].

The research on nontraditional notions of convexity has so far been restricted to two dimensions. The work reported here is the first step in exploring nontraditional convexity in higher dimensions. In this first paper in a series [Fink and Wood, 1996; Fink and Wood, 1995], we extend the notion of strong convexity to higher dimensions and establish strong-convexity analogs of the following properties of standard convex sets:

Intersection The intersection of a collection of convex sets is a convex set.

Supporting planes For every point in the boundary of a convex set, there is a hyperplane through it that supports the set.

Halfspace intersection A closed convex set is the intersection of the halfspaces that contain it.

We also characterize strongly convex flats, halfspaces, and polytopes, and establish the strong convexity of the affine hull of a strongly convex set.

The article is organized as follows. In Section 2, we define strong restricted-orientation convexity and give basic properties of strongly convex sets. In Section 3, we explore properties of strongly convex flats. In Section 4, we describe strongly convex halfspaces and present analogs of the supporting-planes and halfspace-intersection properties. Finally, we conclude, in Section 5, with a summary of the results and a discussion of future work.

2 Strongly convex sets

We define the notion of strong restricted-orientation convexity, also called *strong $\mathcal{O}$-convexity,* in d-dimensional Euclidean space $\mathcal{R}^d$ and present some basic properties of strongly $\mathcal{O}$-convex sets. We assume that the space $\mathcal{R}^d$ is fixed; however, all the result are independent of the particular value of d.

Rawlins introduced planar strong $\mathcal{O}$-convexity in his doctoral dissertation [Rawlins, 1987], as part of his research on restricted-orientation visibility. He defined strong $\mathcal{O}$-convexity through a generalized visibility, by analogy with standard convexity. We use the same analogy to extend strong $\mathcal{O}$-convexity to higher dimensions.

We can describe standard convex sets in terms of visibility: a set is convex if every two points of the set are visible to each other. In other words, for every two points of a convex set, the straight segment joining them is wholly in the set. We introduce a new type of visibility by replacing straight segments with a different type of objects, called *blocks*, and define strong convexity in terms of this new visibility. The definition of blocks is based on the notion of an *orientation set,* which is a set of hyperplanes through a common point.

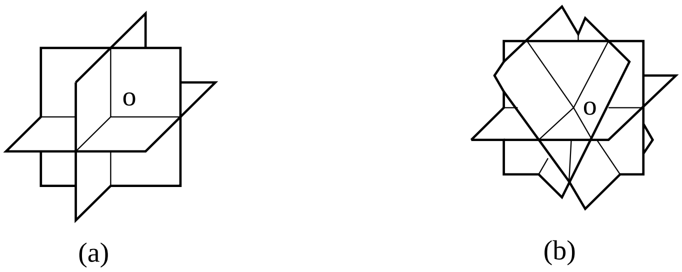

Figure 1: Orientation sets in three dimensions.

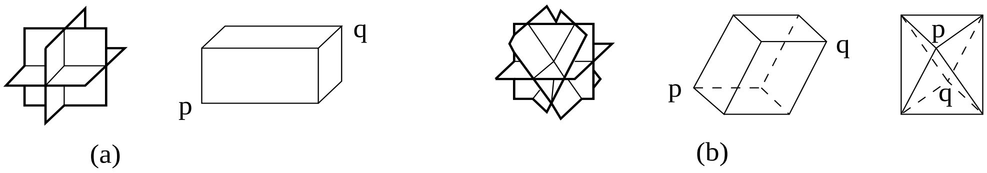

Figure 2: $\mathcal{O}$-blocks in three dimensions.

A *hyperplane* in d dimensions is a subset of $\mathcal{R}^d$ that is a $(d-1)$-dimensional space. For example, hyperplanes in three dimensions are usual planes. Two hyperplanes are *parallel* if they are translations of each other.

Definition 1 (Orientation sets and $\mathcal{O}$-oriented hyperplanes) *An* orientation set $\mathcal{O}$ *is a (finite or infinite) set of hyperplanes through a fixed point o. A hyperplane parallel to one of the elements of $\mathcal{O}$ is called an* $\mathcal{O}$-oriented hyperplane.

Note that every translation of an $\mathcal{O}$-oriented hyperplane is an $\mathcal{O}$-oriented hyperplane and a particular choice of the point o is not important. When we speak of several different orientation sets in $\mathcal{R}^d$, we always assume that the elements of all these sets are through the same common point o. In Figure 1, we give two examples of finite orientation sets in three dimensions.

We next define the notions of a *layer* and $\mathcal{O}$*-block* of two points, p and q. Let $\mathcal{H}$ be a hyperplane from the orientation set $\mathcal{O}$, $\mathcal{H}_p$ be the hyperplane through p parallel to $\mathcal{H}$, and $\mathcal{H}_q$ be the hyperplane through q parallel to $\mathcal{H}$. The closed "layer" of space formed by all points between the planes $\mathcal{H}_p$ and $\mathcal{H}_q$ is called the $\mathcal{H}$*-layer* of p and q. The $\mathcal{O}$*-block* of p and q is the intersection of all the $\mathcal{O}$-oriented layers of p and q:

$$\mathcal{O}\text{-block}(p,q) = \bigcap_{\mathcal{H}\in\mathcal{O}} \mathcal{H}\text{-layer}(p,q).$$

In other words, a point is in the $\mathcal{O}$-block of p and q if, for every $\mathcal{O}$-oriented hyperplane $\mathcal{H}$, the point is between $\mathcal{H}_p$ and $\mathcal{H}_q$.

We show some examples of three-dimensional $\mathcal{O}$-blocks in Figure 2. For the three-element orientation set in Figure 2(a), $\mathcal{O}$-blocks are parallelepipeds with $\mathcal{O}$-oriented facets. The orientation set in Figure 2(b) contains four planes and gives rise to more complex $\mathcal{O}$-blocks.

Definition 2 (Strong $\mathcal{O}$-convexity) A set in $\mathcal{R}^d$ is *strongly $\mathcal{O}$-convex* if, for every two points of the set, their $\mathcal{O}$-block is contained in the set.

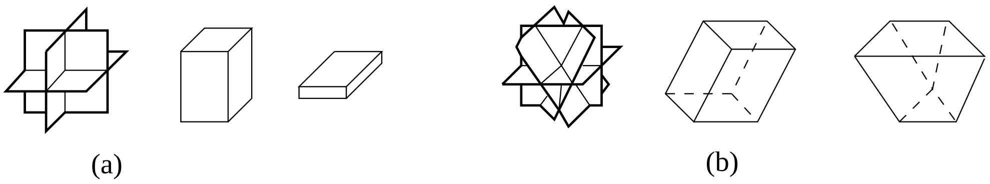

Figure 3: Strongly $\mathcal{O}$-convex sets.

We give examples of strongly $\mathcal{O}$-convex polytopes in Figure 3. For the orientation set in Figure 3(a), strongly $\mathcal{O}$-convex polytopes are parallelepipeds with $\mathcal{O}$-oriented facets. The four-element orientation set of Figure 1(b) gives rise to more complex strongly $\mathcal{O}$-convex objects (Figure 3b); the facets of these objects are also $\mathcal{O}$-oriented, as we show in Section 4 (see Corollary 16).

The following properties of strongly $\mathcal{O}$-convex sets readily follow from the definition (Properties 1 and 3–5 were stated by Rawlins for two dimensions [Rawlins, 1987]).

Observation 1

1. *Every translation of a strongly $\mathcal{O}$-convex set is strongly $\mathcal{O}$-convex.*
2. *The $\mathcal{O}$-block of every two points is strongly $\mathcal{O}$-convex.*
3. **(Intersection)** *If C is a collection of strongly $\mathcal{O}$-convex sets, then the intersection $\bigcap C$ of this collection is also strongly $\mathcal{O}$-convex.*
4. *For every orientation set $\mathcal{O}$, every strongly $\mathcal{O}$-convex set is standard convex.*
5. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every strongly $\mathcal{O}_1$-convex set is strongly $\mathcal{O}_2$-convex.*

Thus, strong $\mathcal{O}$-convexity is a stronger property than standard convexity. We now characterize the situations when strong and standard convexity are equivalent.

Lemma 2 *Every convex set is strongly $\mathcal{O}$-convex if and only if every straight line is strongly $\mathcal{O}$-convex.*

Proof. Suppose that every line is strongly $\mathcal{O}$-convex. Then, for every two points p and q, their $\mathcal{O}$-block is the straight segment joining them: if the $\mathcal{O}$-block were a superset of this segment, then the line through p and q would not be strongly $\mathcal{O}$-convex. Therefore, strong $\mathcal{O}$-convexity is equivalent to standard convexity. $\square$

We next show that strong convexity for a nonclosed orientation set is equivalent to strong convexity for its closure. According to this result, *we may restrict our attention to the study of strong $\mathcal{O}$-convexity for closed orientation sets.*

Lemma 3 *If $\mathcal{O}_2$ is the closure of $\mathcal{O}_1$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity; that is, a set is strongly $\mathcal{O}_1$-convex if and only if it is strongly $\mathcal{O}_2$-convex.*

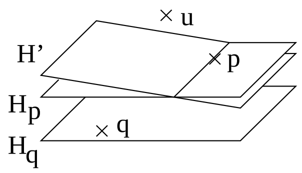

Figure 4: Proof of Lemma 3.

Proof. We prove the equivalence by demonstrating that, for every two points p and q, $\mathcal{O}_2$-block$(p, q) = \mathcal{O}_1$-block(p, q). Note that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and, hence, $\mathcal{O}_2$-block$(p, q) \subseteq \mathcal{O}_1$-block$(p, q)$. We prove the converse inclusion by showing that, for every $\mathcal{O}_2$-oriented layer of p and q, $\mathcal{O}_1$-block(p, q) is a subset of this layer; that is, if a point u is not in the layer, then it is not in the $\mathcal{O}_1$-block(p, q) either.

Let $\mathcal{H}$-layer(p, q) be an $\mathcal{O}_2$-oriented layer, with boundary hyperplanes $\mathcal{H}_p$ (through p) and $\mathcal{H}_q$ (through q), and let u be a point outside of $\mathcal{H}$-layer(p, q). Without loss of generality, we assume that either $\mathcal{H}_p$ is between u and $\mathcal{H}_q$ (see Figure 4) or $\mathcal{H}_p = \mathcal{H}_q$. If $\mathcal{H}_p$ is $\mathcal{O}_1$-oriented, then $\mathcal{O}_1$-block$(p, q) \subseteq \mathcal{H}$-layer$(p, q)$ and, hence, $u \notin \mathcal{O}_1$-block(p, q). If $\mathcal{H}_p$ is not $\mathcal{O}_1$-oriented, then there is a sequence of $\mathcal{O}_1$-oriented hyperplanes through p convergent to $\mathcal{H}_p$. For some element $\mathcal{H}'$ of this sequence, q and u are "on different sides" of $\mathcal{H}'$ (Figure 4). The layer of p and q parallel to $\mathcal{H}'$ is $\mathcal{O}_1$-oriented and u is outside of this layer; therefore, we again have $u \notin \mathcal{O}_1$-block(p, q). □

3 Strongly convex flats

We now explore the properties of strongly $\mathcal{O}$-convex flats (affine varieties). We first characterize strongly $\mathcal{O}$-convex flats in terms of the intersections of $\mathcal{O}$-oriented hyperplanes. We then use this result to derive a necessary and sufficient condition for the equivalence of strong convexity with respect to two different orientation sets. Finally, we establish the strong $\mathcal{O}$-convexity of the *affine hull* of a strongly $\mathcal{O}$-convex set, which is the minimal flat containing the set.

A *flat*, also known as an *affine variety*, is a subset of $\mathcal{R}^d$ that is itself a lower-dimensional space. For example, points, straight lines, two-dimensional planes, and hyperplanes are flats. The whole space $\mathcal{R}^d$ is also a flat. Two flats are *parallel* if they are translations of each other (note that parallel flats are of the same dimension). We use the following properties of flats.

Proposition 4 (Properties of flats)

1. *The intersection of a collection of flats is either empty or a flat.*

2. *The intersection of a k-dimensional flat η with a hyperplane is empty, or η, or a $(k-1)$-dimensional flat.*

We now define *$\mathcal{O}$-oriented flats.*

Definition 3 ($\mathcal{O}$-oriented flats) *A flat is* $\mathcal{O}$-oriented *if it is the intersection of several $\mathcal{O}$-oriented hyperplanes. $\mathcal{O}$-oriented hyperplanes themselves and the whole space $\mathcal{R}^d$ are also $\mathcal{O}$-oriented flats.*

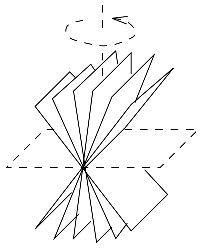

Figure 5: Construction of the orientation set $\mathcal{O}_{\text{sc}}$.

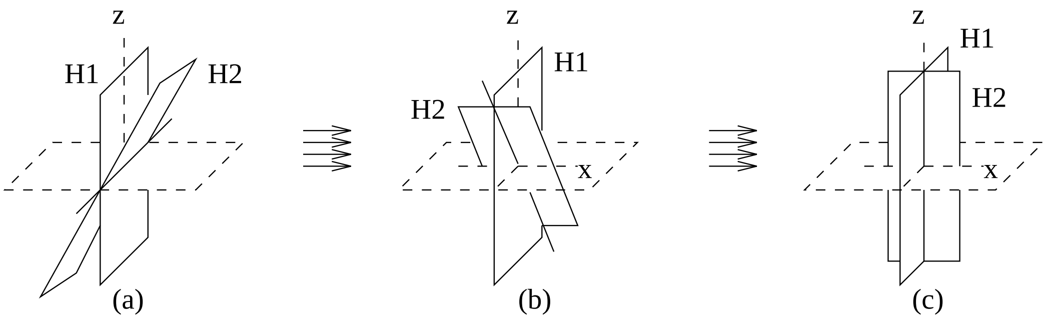

Figure 6: Demonstrating that all lines are $\mathcal{O}_{\text{sc}}$-lines.

Observation 5 *Every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex.*

Can a flat be strongly $\mathcal{O}$-convex if it is not $\mathcal{O}$-oriented? If $\mathcal{O}$ is a closed countable set, only $\mathcal{O}$-oriented flats are strongly $\mathcal{O}$-convex (see Theorem 7). If $\mathcal{O}$ is not closed, all hyperplanes in the closure of $\mathcal{O}$ are strongly $\mathcal{O}$-convex, even though some of them are not $\mathcal{O}$-oriented. For closed uncountable $\mathcal{O}$, flats may also be strongly $\mathcal{O}$-convex even if they are not $\mathcal{O}$-oriented, as we show in the following example.

Example 1: A strongly $\mathcal{O}$-convex flat may not be $\mathcal{O}$-oriented.
Let $\mathcal{O}_{\text{sc}}$ be the orientation set in three dimensions that includes all planes through o whose angle with the "horizontal" plane is at least $\pi/3$ (where any plane through o can serve as the horizontal plane). We illustrate the construction of $\mathcal{O}_{\text{sc}}$ in Figure 5, where the horizontal plane is shown by dashed lines. The set contains the (uncountably many) planes shown by solid lines and all the rotations of these planes around the vertical axis. The index "sc" stands for "standard convexity," as we show that strong $\mathcal{O}_{\text{sc}}$-convexity is equivalent to standard convexity.

We now demonstrate that every line through o is the intersection of two elements of $\mathcal{O}_{\text{sc}}$. An informal proof of this claim is illustrated in Figure 6, where H_1 and H_2 are elements of $\mathcal{O}_{\text{sc}}$. In Figure 6(a), the intersection of H_1 and H_2 is a horizontal line. Now suppose that we rotate H_2 around the vertical axis z, until it reaches the position shown in Figure 6(b). We then rotate H_2 around the horizontal axis x, until it becomes as shown in Figure 6(c). At all times H_2 remains an element of $\mathcal{O}$. The intersection of H_1 and H_2 is always a line, whose position continuously changes from horizontal to vertical. Since every rotation around the vertical axis z maps $\mathcal{O}_{\text{sc}}$ into itself, we conclude that every line through o is formed by the intersection of two elements of $\mathcal{O}_{\text{sc}}$.

Since translations of elements of $\mathcal{O}_{\rm sc}$ are $\mathcal{O}_{\rm sc}$-oriented planes, every line is the intersection of two $\mathcal{O}_{\rm sc}$-oriented planes; therefore, every line is strongly $\mathcal{O}_{\rm sc}$-convex, which implies that strong $\mathcal{O}_{\rm sc}$-convexity is equivalent to standard convexity (Lemma 2). Thus, all planes are strongly $\mathcal{O}_{\rm sc}$-convex, even though some of them are not $\mathcal{O}_{\rm sc}$-oriented. $\square$

We next characterize strongly $\mathcal{O}$-convex flats in terms of $\mathcal{O}$-oriented flats.

Theorem 6 *For a closed orientation set $\mathcal{O}$, a flat η is strongly $\mathcal{O}$-convex if and only if, for every two points of η, there is an $\mathcal{O}$-oriented flat through them that is contained in η.*

Proof. Suppose that, for every $p, q \in \eta$, there is an $\mathcal{O}$-oriented flat $H \subseteq \eta$ through p and q. Since H is strongly $\mathcal{O}$-convex, $\mathcal{O}\text{-block}(p,q) \subseteq H \subseteq \eta$. Thus, for every two points of η, their $\mathcal{O}$-block is in η; therefore, η is strongly $\mathcal{O}$-convex.

Suppose, conversely, that η is strongly $\mathcal{O}$-convex and consider two points, p and q, of η. Let H be the intersection of all $\mathcal{O}$-oriented hyperplanes through p and q; then, H is an $\mathcal{O}$-oriented flat. We show, by contradiction, that $H \subseteq \eta$.

Suppose that H is *not* in η. Then, $H \cap \eta$ is a strongly $\mathcal{O}$-convex flat whose dimension is less than the dimension of H. Let u be the middle point of the straight segment joining p and q. Since $\mathcal{O}\text{-block}(p,q) \subseteq H \cap \eta$ and the dimension of $H \cap \eta$ is less than the dimension of H, we conclude that, for every ball S_u centered at u, $H \cap S_u \not\subseteq \mathcal{O}\text{-block}(p,q)$ and, hence, there is an $\mathcal{O}$-oriented layer of p and q that does not contain $H \cap S_u$.

If a layer of p and q does not contain $H \cap S_u$, then each boundary hyperplane of this layer intersects S_u and does not contain H. Thus, we can select a sequence of $\mathcal{O}$-oriented hyperplanes through p that do not contain H such that the distances from these hyperplanes to u converge to zero. Selecting a convergent subsequence of this sequence and taking its limit, we get an $\mathcal{O}$-oriented hyperplane through p and q that does not contain H, which contradicts the definition of H. (Recall that we have defined H as the intersection of *all* $\mathcal{O}$-oriented hyperplanes through p and q.) $\square$

We have observed that every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex. We now show that, for finite and closed countably infinite orientation sets, only $\mathcal{O}$-oriented flats are strongly $\mathcal{O}$-convex.

Theorem 7 *If $\mathcal{O}$ is a closed countable set, a flat is strongly $\mathcal{O}$-convex if and only if it is $\mathcal{O}$-oriented.*

Proof. We suppose that $\mathcal{O}$ is countable and demonstrate that only $\mathcal{O}$-oriented flats are $\mathcal{O}$-convex. Consider a flat η that is *not* $\mathcal{O}$-oriented. We denote the dimension of η by k. For every $\mathcal{O}$-oriented flat contained in η, its dimension is at most $(k-1)$.

Let p be some point of η. The set of $\mathcal{O}$-oriented hyperplanes through p is countable. The intersections of these hyperplanes form countably many $\mathcal{O}$-oriented flats. Therefore, there are only countably many $\mathcal{O}$-oriented flats through p contained in η. Since the dimension of these flats is at most $(k-1)$, they do not cover η. Thus, there is a point q in η such that no $\mathcal{O}$-oriented flat through p and q is contained in η. Therefore, by Theorem 6, η is not strongly

$\mathcal{O}$-convex. $\square$

For lines and points, the analog of Theorem 7 holds even when an orientation set is uncountable.

Theorem 8 *If $\mathcal{O}$ is a closed orientation set, then a line or point is strongly $\mathcal{O}$-convex if and only if it is $\mathcal{O}$-oriented.*

Proof. Every $\mathcal{O}$-oriented flat is strongly $\mathcal{O}$-convex; it remains to prove the "only if" part. We first prove it for a point and then for a line.

Suppose that a point p is strongly $\mathcal{O}$-convex. If $p = q$, then the $\mathcal{H}$-oriented layer of p and q is just the hyperplane through p parallel to $\mathcal{H}$. Therefore, the $\mathcal{O}$-block of p and q is the intersection of all $\mathcal{O}$-oriented hyperplanes through p. Since p is strongly $\mathcal{O}$-convex, this $\mathcal{O}$-block is contained in p. Therefore, p is the intersection of $\mathcal{O}$-oriented hyperplanes and, hence, it is $\mathcal{O}$-oriented.

Now suppose that a line l is strongly $\mathcal{O}$-convex and let p and q be two distinct points of l. By Theorem 6, there is an $\mathcal{O}$-oriented flat through p and q contained in l. Since the only flat through p and q contained in l is l itself, we conclude that l is $\mathcal{O}$-oriented. $\square$

For a given orientation set $\mathcal{O}$, we define $\widetilde{\mathcal{O}}$ as the set of all strongly $\mathcal{O}$-convex hyperplanes through o. For example, consider the three-dimensional orientation set $\mathcal{O}_{\mathrm{sc}}$, described in Example 1. We have shown that all planes are strongly convex for $\mathcal{O}_{\mathrm{sc}}$; thus, $\widetilde{\mathcal{O}}_{\mathrm{sc}}$ contains all planes through o.

We consider the notion of strong $\widetilde{\mathcal{O}}$-convexity, which is strong convexity with respect to the orientation set $\widetilde{\mathcal{O}}$. Observe that $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$ and, hence, every strongly $\mathcal{O}$-convex set is strongly $\widetilde{\mathcal{O}}$-convex. We next show that the converse also holds: every strongly $\widetilde{\mathcal{O}}$-convex set is strongly $\mathcal{O}$-convex.

Theorem 9

1. *Strong $\mathcal{O}$-convexity is equivalent to strong $\widetilde{\mathcal{O}}$-convexity. Moreover, for every orientation set $\mathcal{O}_1$, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}$-convexity, then $\mathcal{O}_1 \subseteq \widetilde{\mathcal{O}}$.*

2. *Strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity if and only if $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$.*

Proof.
(1) We prove the equivalence by demonstrating that, for every two points p and q, we have $\mathcal{O}$-block$(p, q) = \widetilde{\mathcal{O}}$-block$(p, q)$. Without loss of generality, we assume that $\mathcal{O}$ is closed (Lemma 3).

Since $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$, we immediately conclude that $\widetilde{\mathcal{O}}$-block$(p, q) \subseteq \mathcal{O}$-block$(p, q)$. To prove the converse inclusion, we show that, for every $\widetilde{\mathcal{O}}$-oriented layer of p and q, $\mathcal{O}$-block(p, q) is a subset of this layer; that is, if a point u is not in the layer, then it is not in the $\mathcal{O}$-block(p, q) either.

Let $\mathcal{H}$-layer(p, q) be an $\widetilde{\mathcal{O}}$-oriented layer, with boundary hyperplanes $\mathcal{H}_p$ (through p) and $\mathcal{H}_q$ (through q), and let u be a point outside of $\mathcal{H}$-layer(p, q) (see Figure 7). Since $\mathcal{H}_p$ and $\mathcal{H}_q$ are $\widetilde{\mathcal{O}}$-oriented, they are strongly $\mathcal{O}$-convex.

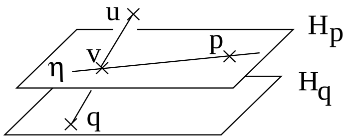

Figure 7: Proof of Theorem 9.

First, suppose that $\mathcal{H}_p = \mathcal{H}_q$; that is, q is in $\mathcal{H}_p$. Then, the $\mathcal{O}$-block of p and q is a subset of $\mathcal{H}_p$, because $\mathcal{H}_p$ is strongly $\mathcal{O}$-convex; therefore, u is not in $\mathcal{O}$-block(p, q).

Next, suppose that $\mathcal{H}_p$ and $\mathcal{H}_q$ are distinct hyperplanes. Without loss of generality, we assume that $\mathcal{H}_p$ is between u and $\mathcal{H}_q$ (Figure 7). Then, the segment joining q and u intersects $\mathcal{H}_p$; we denote the point of their intersection by v. Since $\mathcal{O}$ is closed, we conclude, by Theorem 6, that there is an $\mathcal{O}$-oriented flat η through p and v that is contained in $\mathcal{H}_p$. By the definition of $\mathcal{O}$-oriented flats, η is the intersection of several $\mathcal{O}$-oriented hyperplanes; since $u \notin \eta$, one of these hyperplanes, say $\mathcal{H}_1$, does not contain u. Since v is in the segment joining u and q, the point q is not contained in $\mathcal{H}_1$ either; we conclude that $\mathcal{H}_1$ separates u and q. Therefore, u is not in the $\mathcal{H}_1$-layer of p and q. Since the $\mathcal{H}_1$-layer is $\mathcal{O}$-oriented, we conclude that $u \notin \mathcal{O}$-block(u, v).

Finally, we have to show that, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}$-convexity, then $\mathcal{O}_1 \subseteq \widetilde{\mathcal{O}}$. If the two convexities are equivalent, then $\widetilde{\mathcal{O}}$ contains all the strongly $\mathcal{O}_1$-convex hyperplanes through o. Since every $\mathcal{O}_1$-oriented hyperplane is strongly $\mathcal{O}_1$-convex, we conclude that $\mathcal{O}_1 \subseteq \widetilde{\mathcal{O}}$.

(2) Since strong $\mathcal{O}_1$-convexity is equivalent to strong $\widetilde{\mathcal{O}}_1$-convexity and the same holds for $\mathcal{O}_2$, we conclude that, if $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity. On the other hand, if strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity, then every hyperplane is strongly $\mathcal{O}_1$-convex if and only if it is strongly $\mathcal{O}_2$-convex; therefore, by definition, $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$. □

We conclude that $\widetilde{\mathcal{O}}$ is the maximal orientation set for which strong convexity is equivalent to strong $\mathcal{O}$-convexity. We thus have shown that, for every orientation set $\mathcal{O}$, there is a unique maximal set for which strong convexity is equivalent to strong $\mathcal{O}$-convexity.

Since strong convexity for every orientation set is equivalent to strong convexity for its closure, Theorems 7 and 9 readily give us the following result.

Corollary 10

1. *If $\mathcal{O}$ is a closed countable set, then $\widetilde{\mathcal{O}} = \mathcal{O}$.*
2. *For every $\mathcal{O}$, the set $\widetilde{\mathcal{O}}$ is closed.*

We now establish the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set. The *affine hull* η of a set P is the minimal flat that contains P. In other words, it is the intersection of all flats that contain P (recall that the intersection of flats is a flat). For example, the affine hull of a straight segment is a line, the affine hull of a triangle is a two-dimensional plane, and the affine hull of a ball is the whole space.

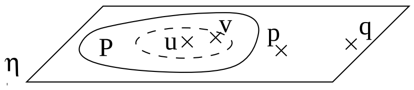

Figure 8: Proof of Lemma 12.

Next, we define the *relative interior* of a set P in its affine hull η. Since η is a lower-dimensional space, we can speak of the interior of P within this space; this interior is called the relative interior of P. We use the following property of relative interiors [Grünbaum *et al.*, 1967].

Proposition 11 *If P is a convex set and η is its affine hull, then the relative interior of P in η is nonempty.*

The next result gives an important property of the affine hulls of strongly $\mathcal{O}$-convex sets, which we use to characterize strongly $\mathcal{O}$-convex sets in terms of halfspace intersections (see Theorem 17).

Lemma 12 *The affine hull of a strongly $\mathcal{O}$-convex set is strongly $\mathcal{O}$-convex.*

Proof. Let P be a strongly $\mathcal{O}$-convex set and η be the affine hull of P (see Figure 8). Since P is convex, the relative interior of P in η is nonempty. Therefore, we can choose an interior point u in P and a ball $S_u \subseteq P$ centered at u. (Note that S_u is a ball in the space η rather than in $\mathcal{R}^d$; this ball is shown by the dashed circle in Figure 8.)

We have to show that, for every two points p and q of η, the $\mathcal{O}$-block of these two points is in η. Let v be a point in S_u such that the line through u and v is parallel to the line through p and q (Figure 8). The $\mathcal{O}$-block of u and v is in P; therefore, it is in η. The $\mathcal{O}$-block of p and q is a scaled version of $\mathcal{O}$-block(u, v); therefore, it is also in η. $\square$

4 Strongly convex halfspaces

We now study the properties of strongly $\mathcal{O}$-convex halfspaces and show that their role in strong $\mathcal{O}$-convexity is similar to the role of halfspaces in standard convexity. We characterize strongly $\mathcal{O}$-convex sets in terms of supporting hyperplanes and in terms of halfspace intersections.

We begin by characterizing strongly $\mathcal{O}$-convex halfspaces through their boundaries.

Theorem 13 *A halfspace is strongly $\mathcal{O}$-convex if and only if its boundary is a strongly $\mathcal{O}$-convex hyperplane.*

Proof. Let P be a halfspace and $\mathcal{H}$ be its boundary hyperplane. Suppose that $\mathcal{H}$ is strongly $\mathcal{O}$-convex. We show that P is strongly $\mathcal{O}$-convex by demonstrating that it is strongly $\widetilde{\mathcal{O}}$-convex. (Recall that, by Theorem 9, strong $\mathcal{O}$-convexity is equivalent to strong $\widetilde{\mathcal{O}}$-convexity). Thus, we have to show that, for every two points p and q of P, their $\widetilde{\mathcal{O}}$-block is in P. Since $\mathcal{H}$ is strongly $\mathcal{O}$-convex, it is $\widetilde{\mathcal{O}}$-oriented; therefore, $\widetilde{\mathcal{O}}$-block(p, q) is a subset of the $\mathcal{H}$-layer of p and q, which is contained in P.

Now suppose, conversely, that the boundary $\mathcal{H}$ of a halfspace P is *not* strongly $\mathcal{O}$-convex. Then, there are points p and q in $\mathcal{H}$ such that $\mathcal{O}$-block(p, q) is not in $\mathcal{H}$. The $\mathcal{O}$-block is centrally symmetric with respect to the middle point of the straight segment joining p and q; therefore, it is not in P, which implies that P is *not* strongly $\mathcal{O}$-convex. □

We next describe supporting hyperplanes of strongly $\mathcal{O}$-convex sets. A hyperplane *supports* a set if it "touches" the set in some of its boundary points and does not cut the set in two parts. For example, if we put a three-dimensional object on a table, then the surface of the table is a plane that supports the object. To put it more formally, a hyperplane $\mathcal{H}$ supports a set P if the intersection of $\mathcal{H}$ and the boundary of P is nonempty and P is contained in one of the two halfspaces whose boundary is $\mathcal{H}$.

We can describe standard convex sets in terms of supporting hyperplanes: a closed set with a nonempty interior is convex if and only if, for every point of its boundary, there is a supporting hyperplane through it. We generalize this property to strongly $\mathcal{O}$-convex sets.

Theorem 14 *A closed set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if, for every point in the boundary of the set, there is a strongly $\mathcal{O}$-convex hyperplane through this point that supports the set.*

Proof. Let P be a closed set with a nonempty interior. Suppose that, for every point r of P's boundary, there is a strongly $\mathcal{O}$-convex hyperplane through r that supports the set. Then, for every boundary point r, there is a strongly $\mathcal{O}$-convex halfspace with boundary through r that contains P. Clearly, the intersection of all such halfspaces is the set P. By Theorem 13, these halfspaces are strongly $\mathcal{O}$-convex; therefore, their intersection P is also strongly $\mathcal{O}$-convex.

Suppose, conversely, that P is strongly $\mathcal{O}$-convex and let r be a point in the boundary of P. Since P is convex, its boundary in some neighborhood of r can be viewed as a graph of some convex function f.

First, we consider the case when r corresponds to a *regular point* of the function f, which means that the function is differentiable at this point. Then, there is exactly one supporting hyperplane $\mathcal{H}$ through r. We have to prove that this hyperplane is strongly $\mathcal{O}$-convex. For convenience, we view $\mathcal{H}$ as a horizontal hyperplane and P as being below $\mathcal{H}$ (see Figure 9a).

Suppose that $\mathcal{H}$ is *not* strongly $\mathcal{O}$-convex. Then, the halfspace with boundary $\mathcal{H}$ that contains P is not strongly $\mathcal{O}$-convex either (Theorem 13). Therefore, there are points p and q in this halfspace such that $\mathcal{O}$-block(p, q) is not in the halfspace (Figure 9a). Without loss of generality, we assume that p and q are *not* in $\mathcal{H}$ (if p or q is in $\mathcal{H}$, we can move these points down "a little bit," in such a way that a part of $\mathcal{O}$-block(p, q) remains above $\mathcal{H}$).

Let us choose some point $r' \in \mathcal{O}\text{-block}(p, q) \cap \mathcal{H}$ and translate $\mathcal{O}$-block(p, q) in such a way that r' becomes identical to r (Figure 9b). Next, we scale $\mathcal{O}$-block(p, q) in such a way that the point r' of the $\mathcal{O}$-block remains identical to the point r of the set P (Figure 9c). Since the function f is differentiable at r, for a sufficiently small scaled version the $\mathcal{O}$-block, the points p and q are below the graph of the function; that is, they are in P (Figure 9c). On the other hand, a part of the scaled version of $\mathcal{O}$-block(p, q) is above $\mathcal{H}$ and, hence, outside P. Since a translation and a scaled version of an $\mathcal{O}$-block is an $\mathcal{O}$-block, we conclude that

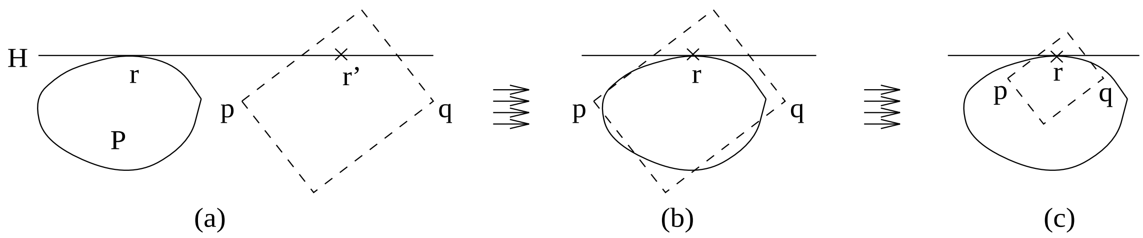

Figure 9: Proof of Theorem 14.

there are two points of P such that their $\mathcal{O}$-block is not in P, contradicting the assumption that P is strongly $\mathcal{O}$-convex.

Next, we consider the case when r is *not* a regular point; that is, f is not differentiable at r. Then, there may be more than one supporting hyperplane through r. We have to show that at least one of these hyperplanes is strongly $\mathcal{O}$-convex.

Since f is a convex function, it is a function of locally bounded variation. Functions of bounded variation are differentiable "almost everywhere," which means that the set of nonregular points is of measure zero. Therefore, there is a sequence of regular points in the graph of f convergent to r. The supporting hyperplane through each of these points is strongly $\mathcal{O}$-convex.

We can select a convergent subsequence from this sequence of supporting hyperplanes; let $\mathcal{H}$ be the limit of this subsequence. Then, $r \in \mathcal{H}$ and, since the set $\widetilde{\mathcal{O}}$ of strongly $\mathcal{O}$-convex hyperplanes is closed (Corollary 10), $\mathcal{H}$ is strongly $\mathcal{O}$-convex. It remains to show that $\mathcal{H}$ supports P. If $\mathcal{H}$ does *not* support P, then, since P is convex, $\mathcal{H}$ intersects the interior of P. Let u be an interior point of P that belongs to $\mathcal{H}$ and $S_u \subseteq P$ be an open ball centered at u. Then, some hyperplane of the convergent subsequence intersects S_u and, hence, this hyperplane does not support P, yielding a contradiction. $\square$

To see that the analogous result does not hold for sets with empty interior, consider an $\mathcal{O}$-oriented plane H (say, in three dimensions) and a nonconvex set P contained in H. Then, for every point in P's boundary, H is a supporting plane through this point; however, P is not strongly $\mathcal{O}$-convex.

Our next goal is to generalize the halfspace-intersection property of convex sets: every closed convex set is the intersection of the halfspaces that contain it. We first show that an analogous result holds for strongly $\mathcal{O}$-convex sets with a *nonempty interior*.

Theorem 15 *A closed set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is the intersection of strongly $\mathcal{O}$-convex halfspaces that contain it.*

Proof. Clearly, the intersection of strongly $\mathcal{O}$-convex halfspaces is strongly $\mathcal{O}$-convex. Now suppose that P is a strongly $\mathcal{O}$-convex set with a nonempty interior. To demonstrate that P is the intersection of strongly $\mathcal{O}$-convex halfspaces, we show that, for every point $p \notin P$, there is a strongly $\mathcal{O}$-convex halfspace that contains P and does not contain p.

Let q be an interior point of P and r be a point of the intersection of the straight segment joining p and q with P's boundary (see Figure 10); since P is closed, $r \neq p$. By Theorem 14,

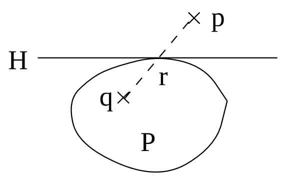

Figure 10: Proof of Theorem 15.

there is a strongly $\mathcal{O}$-convex hyperplane $\mathcal{H}$ through r that supports P. (We show this hyperplane by a solid line in Figure 10.) Since q is an interior point of P, we conclude that $q \notin \mathcal{H}$; therefore, $p \notin \mathcal{H}$. Thus, P and p are "on different sides" of $\mathcal{H}$, which means that the halfspace with boundary $\mathcal{H}$ that contains P does not contain p. $\square$

This result can be readily generalized to nonclosed sets if we use *open halfspaces*, that is, halfspaces that do not contain their boundaries. A set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is the intersection of strongly $\mathcal{O}$-convex open halfspaces.

We next observe that halfspaces are maximal convex sets. We conclude from Theorem 15 that this observation generalizes to strongly $\mathcal{O}$-convex halfspaces: they are maximal among strongly $\mathcal{O}$-convex sets with nonempty interior. In other words, every strongly $\mathcal{O}$-convex set with nonempty interior is a subset of some strongly $\mathcal{O}$-convex halfspace.

For finite $\mathcal{O}$, the boundaries of $\mathcal{O}$-convex halfspaces are $\mathcal{O}$-oriented and the intersection of such halfspaces is a convex polytope with $\mathcal{O}$-oriented facets. Thus, the following result describes strongly $\mathcal{O}$-convex sets for finite $\mathcal{O}$.

Corollary 16 *For a finite orientation set $\mathcal{O}$, a set with a nonempty interior is strongly $\mathcal{O}$-convex if and only if it is a convex polytope whose facets are $\mathcal{O}$-oriented.*

If $\mathcal{O}$ is an infinite orientation set, a polytope may be strongly $\mathcal{O}$-convex even if its facets are not $\mathcal{O}$-oriented. For example, if $\mathcal{O}$ is a (countable or uncountable) set whose closure contains all hyperplanes through o, then strong $\mathcal{O}$-convexity is equivalent to standard convexity and, hence, every convex polytope is strongly $\mathcal{O}$-convex.

We now describe a condition under which all strongly $\mathcal{O}$-convex sets, even those with an empty interior, are formed by the intersections of strongly $\mathcal{O}$-convex halfspaces.

Theorem 17 *Every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces if and only if every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes.*

Proof. Suppose that every closed strongly $\mathcal{O}$-convex set is the intersection of strongly $\mathcal{O}$-convex halfspaces and consider a strongly $\mathcal{O}$-convex flat η. We note that, if a halfspace contains η, then η is either wholly in the interior of the halfspace or wholly in its boundary. We consider the collection C of all the strongly $\mathcal{O}$-convex halfspaces whose boundaries contain η.

Clearly, the intersection of this collection C is equal to the intersection of the collection of *all* the strongly $\mathcal{O}$-convex halfspaces that contain η; this intersection is exactly η. Since η

is wholly contained in the boundary of every halfspace in C, we conclude that η is the intersection of the boundaries of the halfspaces in C. By Theorem 13, the boundaries of strongly $\mathcal{O}$-convex halfspaces are strongly $\mathcal{O}$-convex hyperplanes; therefore, η is the intersection of strongly $\mathcal{O}$-convex hyperplanes.

Now suppose, conversely, that every strongly $\mathcal{O}$-convex flat is the intersection of strongly $\mathcal{O}$-convex hyperplanes and consider a strongly $\mathcal{O}$-convex set P. Let η be the affine hull of P and k be the dimension of η. Since η is a lower-dimensional space, we can speak of halfspaces within this space; we call them η-halfflats. We prove that P is the intersection of strongly $\mathcal{O}$-convex halfspaces in two steps: we first demonstrate that P is the intersection of strongly $\mathcal{O}$-convex η-halfflats and then that every strongly $\mathcal{O}$-convex η-halfflat is the intersection of strongly $\mathcal{O}$-convex halfspaces.

We treat η as an independent k-dimensional space and define the orientation set $\mathcal{O}_\eta$ in this space as follows: a $(k-1)$-dimensional flat $H \subseteq \eta$ is $\mathcal{O}_\eta$-oriented if it is the intersection of η with some $\mathcal{O}$-oriented hyperplane. Every $\mathcal{O}$-oriented hyperplane that intersects and does not contain η gives rise to an $\mathcal{O}_\eta$-oriented $(k-1)$-dimensional flat (Proposition 4).

Note that, for every two points p and q of η, a set is an $\mathcal{O}_\eta$-oriented layer of p and q if and only if it is the intersection of an $\mathcal{O}$-oriented layer of p and q with η; therefore, $\mathcal{O}_\eta$-block$(p,q) = \mathcal{O}$-block$(p,q) \cap \eta$. Since η is strongly $\mathcal{O}$-convex (Lemma 12), $\mathcal{O}$-block(p,q) is in η; therefore, $\mathcal{O}_\eta$-block$(p,q) = \mathcal{O}$-block(p,q). We conclude from this equality that a set contained in η is strongly $\mathcal{O}_\eta$-convex if and only if it is strongly $\mathcal{O}$-convex; therefore, P is strongly $\mathcal{O}_\eta$-convex. The relative interior of P in η is nonempty (Proposition 11). Therefore, by Theorem 15, P is the intersection of strongly $\mathcal{O}$-convex η-halfflats.

We next demonstrate that every strongly $\mathcal{O}$-convex η-halfflat Q is the intersection of strongly $\mathcal{O}$-convex halfspaces. Let H be the boundary of Q in η (H is a $(k-1)$-dimensional flat). Since Q is strongly $\mathcal{O}$-convex, its boundary H is also strongly $\mathcal{O}$-convex (Theorem 6). Therefore, H is the intersection of strongly $\mathcal{O}$-convex hyperplanes. At least one of these hyperplanes, say $\mathcal{H}$, does not contain η; the η-halfflat Q is the intersection of η and a halfspace with boundary $\mathcal{H}$.

Finally, we note that, since η is strongly $\mathcal{O}$-convex, it is the intersection of strongly $\mathcal{O}$-convex hyperplanes and every strongly $\mathcal{O}$-convex hyperplane is the intersection of two strongly $\mathcal{O}$-convex halfspaces. Thus, η is the intersection of strongly $\mathcal{O}$-convex halfspaces and, hence, Q is also the intersection of strongly $\mathcal{O}$-convex halfspaces. □

We readily conclude from Theorem 8 that, in two and three dimensions, all strongly $\mathcal{O}$-convex flats are the intersections of strongly $\mathcal{O}$-convex hyperplanes. By Theorem 17, this observation implies that, *in two and three dimensions, every strongly $\mathcal{O}$-convex set is the intersection of the strongly $\mathcal{O}$-convex halfspaces that contain it.*

If $\mathcal{O}$ is a finite or closed countably infinite orientation set in higher dimensions, then every $\mathcal{O}$-convex flat is $\mathcal{O}$-oriented (Theorem 7) and, hence, formed by the intersection of strongly $\mathcal{O}$-convex hyperplanes. Therefore, *for closed countable $\mathcal{O}$, all strongly $\mathcal{O}$-convex sets are also the intersections of strongly $\mathcal{O}$-convex halfspaces.*

We have observed that strongly $\mathcal{O}$-convex halfspaces are maximal among strongly $\mathcal{O}$-convex sets with nonempty interior. This observation does not generalize to sets with empty interior: a strongly $\mathcal{O}$-convex set with empty interior may not be a subset of any strongly

$\mathcal{O}$-convex halfspace. In particular, we readily see from the proof of Theorem 17 that, if a strongly $\mathcal{O}$-convex flat is not formed by the intersection of strongly $\mathcal{O}$-convex hyperplanes, then it is not contained in a strongly $\mathcal{O}$-convex halfspace.

5 Concluding Remarks

We described a generalization of standard convexity in higher dimensions, called strong $\mathcal{O}$-convexity, and demonstrated that properties of strongly $\mathcal{O}$-convex sets are similar to the properties of standard convex sets.

We showed that, for every point in the boundary of an $\mathcal{O}$-convex set, there is a supporting strongly $\mathcal{O}$-convex hyperplane through it (Theorem 14) and that every strongly $\mathcal{O}$-convex set with nonempty interior is formed by the intersection of strongly $\mathcal{O}$-convex halfspaces (Theorem 15). We then presented a condition under which strongly $\mathcal{O}$-convex sets with empty interior are also the intersections of strongly $\mathcal{O}$-convex halfspaces (Theorem 17).

In addition, we characterized strongly $\mathcal{O}$-convex flats in terms of $\mathcal{O}$-oriented flats (Theorem 6), established the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set (Lemma 12), and derived a condition for the equivalence of strong convexity for two different orientation sets (Theorem 9).

The work leaves many unanswered questions, which we are currently trying to address. In particular, we are exploring an alternative generalization of convexity, called *restricted-orientation convexity* [Rawlins, 1987], in higher dimensions [Fink and Wood, 1996; Fink and Wood, 1995]. We also plan to study the computational aspects of strong convexity, such as determining the complexity of finding strongly $\mathcal{O}$-convex hulls.

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