GENERALIZED HALFSPACES IN RESTRICTED-ORIENTATION CONVEXITY

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Restricted-orientation convexity, also called *$\mathcal{O}$-convexity,* is the study of geometric objects whose intersection with lines from some fixed set is empty or connected. The notion of $\mathcal{O}$-convexity generalizes standard convexity and several types of nontraditional convexity.

We introduce *$\mathcal{O}$-halfspaces,* which are an analog of halfspaces in the theory of $\mathcal{O}$-convexity. We show that this notion generalizes standard halfspaces, explore properties of these generalized halfspaces, and demonstrate their relationship to $\mathcal{O}$-convex sets. We also describe directed $\mathcal{O}$-halfspaces, which are a subclass of $\mathcal{O}$-halfspaces that has some special properties.

We first present some basic properties of $\mathcal{O}$-halfspaces and compare them with the properties of standard halfspaces. We show that $\mathcal{O}$-halfspaces may be disconnected, characterize an $\mathcal{O}$-halfspace in terms of its connected components, and derive the upper bound on the number of components. We then study properties of the boundaries of $\mathcal{O}$-halfspaces. Finally, we describe the complements of $\mathcal{O}$-halfspaces and give a necessary and sufficient condition under which the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.

1 INTRODUCTION

The study of convex sets is a branch of geometry, analysis, and linear algebra [5, 7] that has applications in many practical areas of computer science, including VLSI design, computer graphics, and motion planning [12]. Researchers have studied many notions of nontraditional convexity, such as orthogonal convexity [9, 10, 11], finitely oriented convexity [6, 20, 14], strong convexity [13, 2], NESW convexity [8, 18, 20], and link convexity [1, 19, 17].

Rawlins introduced **restricted-orientation convexity,** also called **$\mathcal{O}$-convexity,** in his doctoral dissertation [13], as a generalization of standard convexity and orthogonal convexity. Rawlins and Wood investigated planar $\mathcal{O}$-convex sets and demonstrated that their properties are similar to the properties of standard convex sets [15, 16]. Schuierer continued their exploration and presented an extensive study of geometric and computational properties of $\mathcal{O}$-convex sets in his doctoral thesis [17].

The purpose of our research is to investigate nontraditional convexities in multidimensional space. We generalized strong convexity [13] to higher dimensions and explored the properties of strongly convex sets [2, 4]. We then demonstrated that $\mathcal{O}$-convexity can also be extended to multidimensional space [3]. We presented several major properties of $\mathcal{O}$-convex sets in higher dimensions and showed that their properties are much richer than that of planar $\mathcal{O}$-convex sets.

We now further develop the theory of $\mathcal{O}$-convexity. We present an $\mathcal{O}$-convexity analog of halfspaces, explore their properties, and describe their relationship to $\mathcal{O}$-convex sets.

We restrict our attention to the exploration of closed sets. We conjecture that most results hold for nonclosed sets as well; however, some of our proofs work only for closed sets.

> **CLOSED-SET ASSUMPTION:** We consider only closed geometric objects. An object is **closed** if, for every convergent sequence of its points, the limit of the sequence belongs to the object.

The paper is organized as follows. In Section 2, we describe $\mathcal{O}$-convexity and introduce **$\mathcal{O}$-halfspaces**, which are an $\mathcal{O}$-convexity analog of halfspaces. In Section 3, we present basic properties of $\mathcal{O}$-halfspaces. In Section 4, we define **directed $\mathcal{O}$-halfspaces,** which are a subclass of $\mathcal{O}$-halfspaces that has several special properties. In Section 5, we characterize $\mathcal{O}$-halfspaces in terms of their boundaries. In Section 6, we explore the properties of the complement of an $\mathcal{O}$-halfspace and state a condition under which the closure of the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace. Finally, we conclude in Section 7, with a summary of the results and a discussion of open problems.

2 $\mathcal{O}$-CONVEXITY AND $\mathcal{O}$-HALFSPACES

We present the notion of restricted-orientation convexity, also called $\mathcal{O}$-convexity, in two and higher dimensions. We then introduce an $\mathcal{O}$-convexity analog of halfspaces.

Rawlins defined planar $\mathcal{O}$-convex sets in terms of their intersection with straight lines, by analogy with one of the definitions of standard convexity [13]. We used the same analogy to extend $\mathcal{O}$-convexity to higher dimensions [3].

We can describe convex sets through their intersection with straight lines: a set of points is convex if its intersection with every line is connected. Note that *we consider the empty set to be connected.*

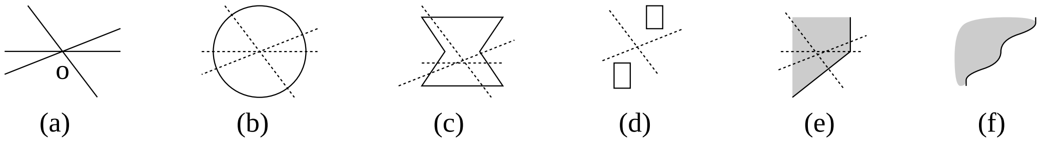

Figure 1: Planar $\mathcal{O}$-convex sets (b–d) and $\mathcal{O}$-halfplanes (e,f).

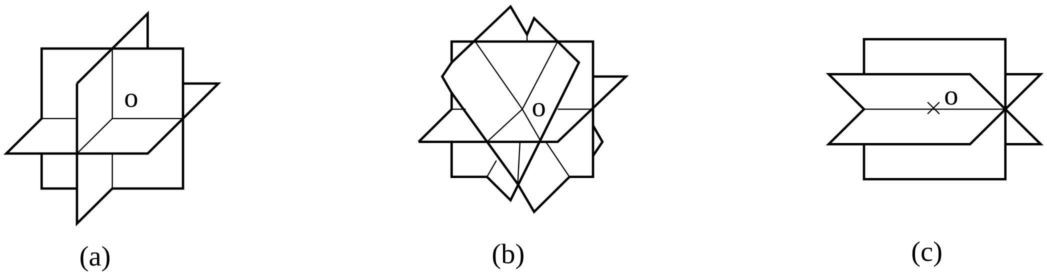

Figure 2: Examples of finite orientation sets.

$\mathcal{O}$-convex sets are defined through their intersection with lines from a *certain set* (rather than all lines). The definition of this restricted collection of lines is based on the notion of an orientation set. An **orientation set** $\mathcal{O}$ in two dimensions is a (finite or infinite) set of lines through some fixed point o. An example of a finite orientation set is shown in Figure 1(a). A line parallel to one of the lines of $\mathcal{O}$ is called an **$\mathcal{O}$-line**. For example, the dotted lines in Figure 1 are $\mathcal{O}$-lines. A set is **$\mathcal{O}$-convex** if its intersection with every $\mathcal{O}$-line is connected. For example, sets in Figures 1(b)–(d) are $\mathcal{O}$-convex for the orientation set in Figure 1(a). Note that, unlike standard convex sets, $\mathcal{O}$-convex sets may be disconnected (see Figure 1d).

We now extend the notion of $\mathcal{O}$-convexity to d-dimensional space $\mathcal{R}^d$. We assume that the space $\mathcal{R}^d$ is fixed; however, the results are independent of the particular value of d. We introduce a set $\mathcal{O}$ of hyperplanes through a fixed point o, show how this set gives rise to $\mathcal{O}$-lines, and define $\mathcal{O}$-convex sets in terms of their intersection with $\mathcal{O}$-lines.

A **hyperplane** in d dimensions is a subset of $\mathcal{R}^d$ that is a $(d-1)$-dimensional space. For example, hyperplanes in three dimensions are the usual planes. Analytically, a hyperplane is a set of points satisfying a linear equation, $a_0 + a_1x_1 + a_2x_2 + \cdots + a_dx_d = 0$, in Cartesian coordinates. Two hyperplanes are **parallel** if they are translates of each other; that is, their equations differ only by the value of a_0.

DEFINITION 2.1 (Orientation set and $\mathcal{O}$-hyperplanes) *An* **orientation set** $\mathcal{O}$ *is a set of hyperplanes through a fixed point o. A hyperplane parallel to one of the elements of $\mathcal{O}$ is an* **$\mathcal{O}$-hyperplane.**

Note that every translate of an $\mathcal{O}$-hyperplane is an $\mathcal{O}$-hyperplane and a particular choice of the point o is not important. In Figure 2, we show examples of finite orientation sets in three dimensions. The set in Figure 2(a) contains three mutually orthogonal planes; we call it an **orthogonal-orientation set.**

$\mathcal{O}$-lines in $\mathcal{R}^d$ are formed by the intersections of $\mathcal{O}$-hyperplanes. In other words, a line is an $\mathcal{O}$-line if it is the intersection of several $\mathcal{O}$-hyperplanes. Note that every translate of an $\mathcal{O}$-line is an $\mathcal{O}$-line.

When $\mathcal{O}$ is nonempty and the intersection of the elements of $\mathcal{O}$ is the point o, rather than a superset of o, we say that $\mathcal{O}$ has the **point-intersection property.** For example, the orientation sets in Figures 2(a) and 2(b) have this property, whereas the set in Figure 2(c) does not. Some of our results hold only for orientation sets with the point-intersection property. We will use the following properties of point-intersection sets to prove these results.

LEMMA 2.1 *If the orientation set $\mathcal{O}$ has the point-intersection property, then:*

1. *There is at least one $\mathcal{O}$-line.*
2. *For every line, there is an $\mathcal{O}$-hyperplane that intersects it and does not contain it.*

Proof. **(1)** We consider a minimal set of $\mathcal{O}$-hyperplanes whose intersection forms o. We denote these hyperplanes by $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$. Then, $\mathcal{H}_2 \cap \cdots \cap \mathcal{H}_n$ is a flat (affine variety) different from the point o and the intersection of this flat with $\mathcal{H}_1$ is o, which may happen only if $\mathcal{H}_2 \cap \cdots \cap \mathcal{H}_n$ is a line. This line is an $\mathcal{O}$-line.

(2) We consider a line l and assume, without loss of generality, that this line is through o. The intersection of all elements of $\mathcal{O}$ is o; therefore, for some $\mathcal{O}$-hyperplane $\mathcal{H} \in \mathcal{O}$, the line l is not contained in $\mathcal{H}$. $\square$

If the orientation set $\mathcal{O}$ does not have the point-intersection property, then the intersection of the elements of $\mathcal{O}$ is either a line or a higher-dimensional flat (affine variety). If this intersection is a line (see Figure 2c), then there is exactly one $\mathcal{O}$-line through o and all other $\mathcal{O}$-lines are parallel to this $\mathcal{O}$-line. If the intersection is neither a point nor a line, then there are no $\mathcal{O}$-lines at all.

We define $\mathcal{O}$-convex sets in higher dimensions in the same way as in two dimensions.

DEFINITION 2.2 ($\mathcal{O}$-Convexity) *A closed set is* **$\mathcal{O}$-convex** *if its intersection with every $\mathcal{O}$-line is connected.*

Figures 3(b)–(e) provide examples of $\mathcal{O}$-convex sets for the orthogonal-orientation set shown in Figure 3(a).

Standard halfspaces can also be characterized in terms of their intersection with straight lines: a set is a halfspace if and only if its intersection with every line is empty, a ray, or a line. We use this observation to define an $\mathcal{O}$-convexity analog of halfspaces.

DEFINITION 2.3 ($\mathcal{O}$-halfspaces) *A closed set is an* **$\mathcal{O}$-halfspace** *if its intersection with every $\mathcal{O}$-line is empty, a ray, or a line.*

Note that the *empty set* and the *whole space* $\mathcal{R}^d$ are considered to be $\mathcal{O}$-halfspaces. This convention simplifies some of the results. When we consider a planar case, we will use the term **$\mathcal{O}$-halfplanes** to refer to two-dimensional $\mathcal{O}$-halfspaces. For example, the sets in

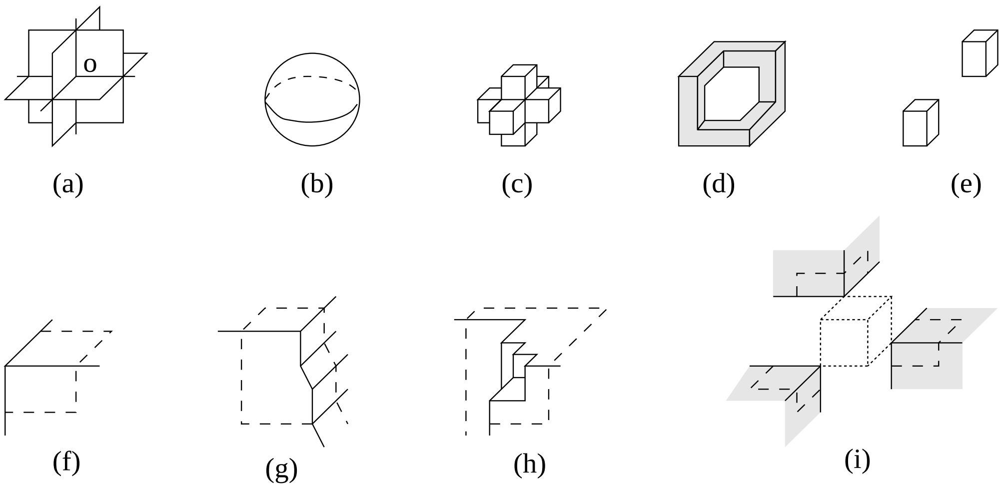

Figure 3: $\mathcal{O}$-convex sets (b–e) and $\mathcal{O}$-halfspaces (f–i) in three dimensions.

Figures 1(e) and 1(f) are $\mathcal{O}$-halfplanes, and the sets in Figures 3(f)–(h) are $\mathcal{O}$-halfspaces (we use dashed lines to show infinite planar regions in the boundaries of these $\mathcal{O}$-halfspaces). Unlike standard halfspaces, $\mathcal{O}$-halfspaces may be disconnected. In Figure 3(i), we show an $\mathcal{O}$-halfspace that consists of three components, located around the dotted cube.

3 BASIC PROPERTIES OF $\mathcal{O}$-HALFSPACES

We present some simple properties of $\mathcal{O}$-halfspaces and compare them with properties of standard halfspaces.

OBSERVATION 3.1

1. *Every translate of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.*
2. *Every standard halfspace is an $\mathcal{O}$-halfspace.*
3. *Every $\mathcal{O}$-halfspace is $\mathcal{O}$-convex.*

We give necessary and sufficient conditions under which an $\mathcal{O}$-convex set is an $\mathcal{O}$-halfspace.

OBSERVATION 3.2 *A set P is an $\mathcal{O}$-halfspace if and only if the following conditions hold:*

1. *P is an $\mathcal{O}$-convex set.*
2. *For every point p in P and every $\mathcal{O}$-line l, one of the two parallel-to-l rays with endpoint p is wholly contained in P.*

We now show that every component of a disconnected $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace and that, if the orientation set has the point-intersection property, the number of components of an $\mathcal{O}$-halfspace is bounded.

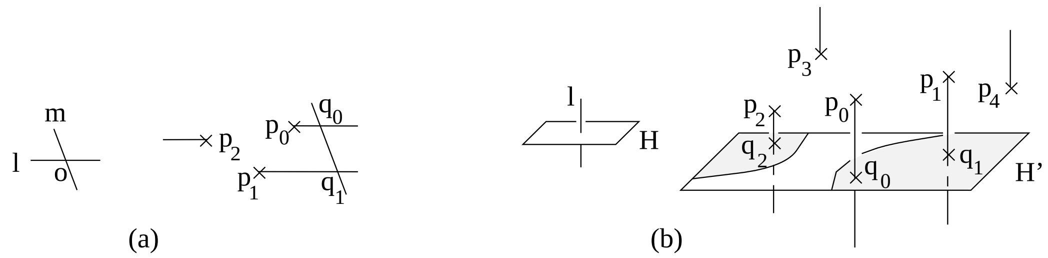

Figure 4: Proof of Theorem 3.3.

THEOREM 3.3

1. *A disconnected set is an $\mathcal{O}$-halfspace if and only if every connected component of the set is an $\mathcal{O}$-halfspace and no $\mathcal{O}$-line intersects two components.*

2. *In d dimensions, if the orientation set $\mathcal{O}$ has the point-intersection property, then the number of connected components of an $\mathcal{O}$-halfspace is at most 2^{d-1}.*

Proof. **(1)** If P is the union of $\mathcal{O}$-halfspaces and no $\mathcal{O}$-line intersects two of them, then, clearly, P is an $\mathcal{O}$-halfspace. If some connected component of P is not an $\mathcal{O}$-halfspace, then the intersection of this component with some $\mathcal{O}$-line is not empty, not a ray, and not a line; therefore, the intersection of P with this $\mathcal{O}$-line is not empty, not a ray, and not a line. Finally, if some $\mathcal{O}$-line intersects two components, the intersection of P with this line is disconnected.

(2) We show that, for every $2^{d-1}+1$ points of an $\mathcal{O}$-halfspace P, two of these points are in the same connected component. We use induction on the dimension d.

We first consider the planar case, which provides an induction base. We show that, for every three points p_0, p_1, p_2 of an $\mathcal{O}$-halfplane P, two of them are in the same component. We pick two lines, l and m, of the orientation set $\mathcal{O}$. One of the two parallel-to-l rays with endpoint p_0 is contained in P (we show this ray in Figure 4a). Similarly, we may choose a parallel-to-l ray with endpoint p_1 and a parallel-to-l ray with endpoint p_2 contained in P.

We select two of these rays that point in the same direction. Without loss of generality, we assume that these two rays correspond to p_0 and p_1. We choose a parallel-to-m line that intersects the two selected rays and denote the intersection points by q_0 and q_1, respectively (see Figure 4a). The polygonal line (p_0, q_0, q_1, p_1) is wholly in P; therefore, p_0 and p_1 are in the same connected component.

The induction step is based on Observation 4.5, which we will present in Section 4: the intersection of an $\mathcal{O}$-halfspace P with an $\mathcal{O}$-hyperplane $\mathcal{H}$ is an $\mathcal{O}$-halfspace in the $(d-1)$-dimensional space $\mathcal{H}$. That is, the intersection of $P \cap \mathcal{H}$ with every $\mathcal{O}$-line contained in $\mathcal{H}$ is empty, a ray, or a line.

We denote the $2^{d-1}+1$ points in P by $p_0, p_1, \ldots, p_{2^{d-1}}$ (in Figure 4(b), we illustrate the proof for $d = 3$). By Lemma 2.1, since $\mathcal{O}$ has the point-intersection property, we may choose some

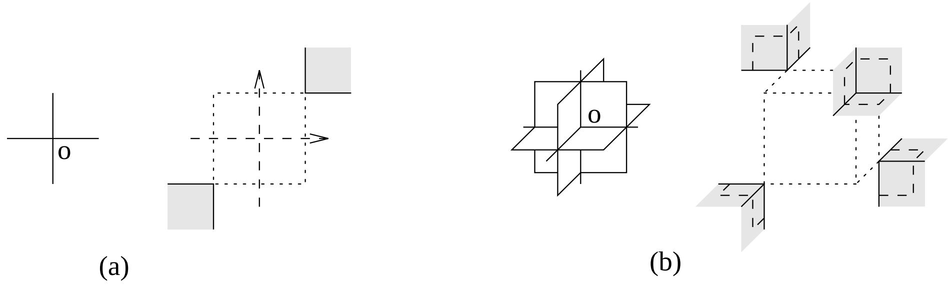

Figure 5: $\mathcal{O}$-halfspace with 2^{d-1} connected components in (a) two and (b) three dimensions.

$\mathcal{O}$-line l and an $\mathcal{O}$-hyperplane $\mathcal{H}$ that intersects l and does not contain it. For every point p_k, one of the two parallel-to-l rays with endpoint p_k is contained in P. Thus, we get $2^{d-1}+1$ parallel rays in P, and at least $2^{d-2}+1$ of them point in the same direction. We assume that these $2^{d-2}+1$ same-direction rays correspond to the points $p_0, p_1, \ldots, p_{2^{d-2}}$. We select an $\mathcal{O}$-plane $\mathcal{H}'$, parallel to $\mathcal{H}$, that intersects these $2^{d-2}+1$ rays and denote the intersection points by $q_0, q_1, \ldots, q_{2^{d-2}}$, respectively.

$P \cap \mathcal{H}'$ (shaded in Figure 4b) is an $\mathcal{O}$-halfspace in the $(d-1)$-dimensional space $\mathcal{H}$. Therefore, by the induction hypothesis, some points q_k and q_m belong to the same connected component of $P \cap \mathcal{H}'$, which implies that p_k and p_m belong to the same component of P. $\square$

If the orientation set $\mathcal{O}$ does not have the point-intersection property, then an $\mathcal{O}$-halfspace may have infinitely many components. For example, if there is only one $\mathcal{O}$-line through o, then any collection of lines parallel to this $\mathcal{O}$-line forms an $\mathcal{O}$-halfspace.

We show that the bound on the number of components given in Theorem 3.3 is tight.

EXAMPLE: An $\mathcal{O}$-halfspace that has 2^{d-1} connected components.
We consider the orthogonal-orientation set in d dimensions, which comprises d mutually orthogonal hyperplanes, and construct an $\mathcal{O}$-halfspace P whose components are rectangular polyhedral angles (quadrants). Note that this orientation set has the point-intersection property. In Figure 5, we illustrate the construction for two and three dimensions.

We choose a cube (shown by dotted lines) whose facets are parallel to elements of $\mathcal{O}$. The cube has 2^d vertices; we select 2^{d-1} vertices no two of which are adjacent. We define P as the union of the 2^{d-1} polyhedral angles vertical to the selected angles of the cube.

We may describe P analytically, using $\mathcal{O}$-lines through the cube's center (shown by dashed lines in Figure 5a) as coordinate axes and taking one of the cube's vertices for $(1, 1, \ldots, 1)$. The equation for P is as follows:

$$|x_1|, |x_2|, \ldots, |x_d| \geq 1 \text{ and } x_1 \times x_2 \times \cdots \times x_d > 0.$$

To show that P is indeed an $\mathcal{O}$-halfspace, we observe that each $\mathcal{O}$-line either intersects one component, in which case the intersection is a ray, or does not intersect any component. $\square$

Every standard convex set is the intersection of halfspaces. We demonstrate that an anal-

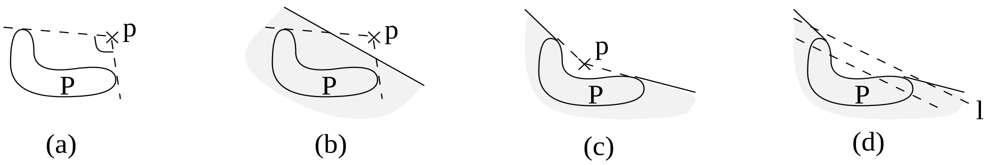

Figure 6: Proof of Lemma 3.4.

ogous result holds for $\mathcal{O}$-convex sets in two dimensions, but it does not generalize to three and higher dimensions.

LEMMA 3.4 *A connected set in two dimensions is $\mathcal{O}$-convex if and only if it is the intersection of $\mathcal{O}$-halfplanes.*

Proof. Suppose that a set P is the intersection of $\mathcal{O}$-halfplanes. For every $\mathcal{O}$-line l, the intersection of each $\mathcal{O}$-halfplane with l is connected and, hence, the intersection of P with l is also connected. Therefore, P is $\mathcal{O}$-convex.

Suppose, conversely, that P is $\mathcal{O}$-convex. We show that P is the intersection of $\mathcal{O}$-halfplanes by demonstrating that, for every point p *not* in P, some $\mathcal{O}$-halfplane contains P and does not contain p.

We draw the two lines through p that support P (see Figure 6a). If the marked angle between them is less than π, then there is a standard halfplane that contains P and does not contain p (Figure 6b). This halfplane is the desired $\mathcal{O}$-halfplane.

If the marked angle is greater than or equal to π, we consider the set Q shown by shading in Figure 6(c). The boundary of Q consists of the segment of P's boundary between the supporting lines and the parts of the supporting lines that extend this segment. We show that Q is an $\mathcal{O}$-halfplane.

If the intersection of Q with some $\mathcal{O}$-line l is disconnected, then there is an $\mathcal{O}$-line parallel to l whose intersection with P is disconnected (see Figure 6d), contradicting the assumption that P is $\mathcal{O}$-convex. On the other hand, there is no line whose intersection with Q is a point or segment. Therefore, the intersection of Q with every $\mathcal{O}$-line is empty, a ray, or a line. $\square$

In three and higher dimensions, some $\mathcal{O}$-convex sets are not formed by the intersection of $\mathcal{O}$-halfspaces.

EXAMPLE: A connected $\mathcal{O}$-convex set that is not the intersection of $\mathcal{O}$-halfspaces. We consider the orthogonal-orientation set $\mathcal{O}$ and the set P shown in Figure 7(b). This set is a horizontal disk on eight vertical, equally spaced "pillars." The pillars are rays located in such a way that no $\mathcal{O}$-line intersects two of them. Clearly, P is $\mathcal{O}$-convex. We readily see that every $\mathcal{O}$-halfspace containing P also contains the point $p \notin P$, located under the center of the disk. Thus, the intersection of all $\mathcal{O}$-halfspaces containing P is *not* P. $\square$

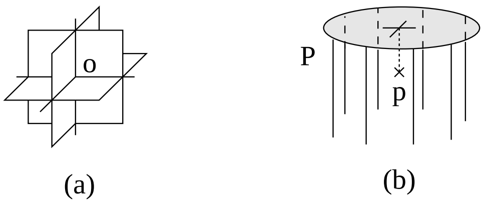

Figure 7: Every $\mathcal{O}$-halfspace containing the $\mathcal{O}$-convex set P also contains the point $p \notin P$.

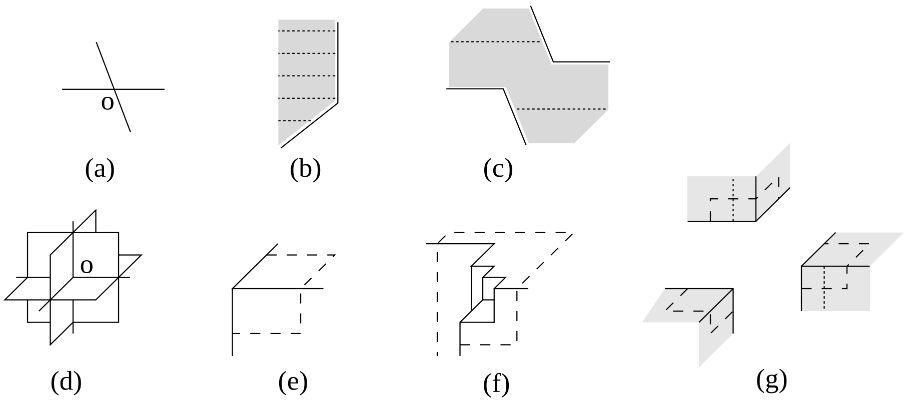

Figure 8: $\mathcal{O}$-halfplane (b), and $\mathcal{O}$-halfspaces (e) and (f) are directed. $\mathcal{O}$-halfplane (c) and $\mathcal{O}$-halfspace (g) are *not* directed.

4 DIRECTED $\mathcal{O}$-HALFSPACES

We have seen that $\mathcal{O}$-halfspaces may be disconnected, whereas standard halfspaces are always connected. We now describe a subclass of $\mathcal{O}$-halfspaces that has the connectedness property: all $\mathcal{O}$-halfspaces of this subclass are connected.

DEFINITION 4.1 (Directed $\mathcal{O}$-halfspaces) *An $\mathcal{O}$-halfspace is* **directed** *if, for every two parallel $\mathcal{O}$-lines whose intersection with the $\mathcal{O}$-halfspace forms rays, these rays point in the same direction (rather than in opposite directions).*

For example, the $\mathcal{O}$-halfplane in Figure 8(b) is directed for the orientation set shown in Figure 8(a); the intersection of this $\mathcal{O}$-halfplane with several parallel $\mathcal{O}$-lines is shown by dotted lines. On the other hand, the $\mathcal{O}$-halfplane in Figure 8(c) is not directed, because the dotted $\mathcal{O}$-rays point in opposite directions. Similarly, the $\mathcal{O}$-halfspaces in Figures 8(e) and 8(f) are directed for the orthogonal-orientation set in Figure 8(d), whereas the $\mathcal{O}$-halfspace in Figure 8(g) is not, since the dotted $\mathcal{O}$-rays are not in the same direction.

OBSERVATION 4.1

1. *Every translate of a directed $\mathcal{O}$-halfspace is a directed $\mathcal{O}$-halfspace.*

2. *Every standard halfspace is a directed $\mathcal{O}$-halfspace.*

We next introduce the notion of **$\mathcal{O}$-flats,** which are formed by the intersections of $\mathcal{O}$-

hyperplanes, and demonstrate that, for the orientation set with the point-intersection property, the intersection of a directed $\mathcal{O}$-halfspace with every $\mathcal{O}$-flat is connected. This property of directed $\mathcal{O}$-halfspaces is one of the main tools in our exploration.

A **flat,** also known as an **affine variety,** is a subset of $\mathcal{R}^d$ that is itself a lower-dimensional space. For example, points, lines, two-dimensional planes, and hyperplanes are flats. The whole space $\mathcal{R}^d$ is also a flat. Analytically, a k-dimensional flat is represented in Cartesian coordinates as a system of $d-k$ independent linear equations. We will use the following properties of flats.

PROPOSITION 4.2 (Properties of flats)

1. *The intersection of a collection of flats is either empty or a flat.*
2. *The intersection of a k-dimensional flat η and a hyperplane is empty, η, or a $(k-1)$-dimensional flat.*

We now define $\mathcal{O}$-flats.

DEFINITION 4.2 ($\mathcal{O}$-flats) *An* **$\mathcal{O}$-flat** *is a flat formed by the intersection of several $\mathcal{O}$-hyperplanes. $\mathcal{O}$-hyperplanes themselves and the whole space $\mathcal{R}^d$ are also $\mathcal{O}$-flats.*

OBSERVATION 4.3

1. *Every translate of an $\mathcal{O}$-flat is an $\mathcal{O}$-flat.*
2. *The intersection of a collection of $\mathcal{O}$-flats is either empty or an $\mathcal{O}$-flat.*

We next describe lower-dimensional orientation sets contained in $\mathcal{O}$-flats. We consider an $\mathcal{O}$-flat η and denote the dimension of η by k. We treat η as an independent k-dimensional space and define the orientation set $\mathcal{O}_\eta$ and $\mathcal{O}_\eta$-flats in this lower-dimensional space.

An **$\mathcal{O}_\eta$-flat** is an $\mathcal{O}$-flat contained in η. The $(k-1)$-dimensional $\mathcal{O}_\eta$-flats ("$\mathcal{O}_\eta$-hyperplanes") through some fixed point o_η form the lower-dimensional orientation set $\mathcal{O}_\eta$.

For example, consider the orthogonal-orientation set in Figure 9(a). The $\mathcal{O}$-plane η contains vertical and horizontal $\mathcal{O}$-lines. Therefore, the lower-dimensional orientation set $\mathcal{O}_\eta$ contains the vertical and horizontal line through o_η. In Figure 9(b), we show another example of a lower-dimensional orientation set.

We demonstrate that $\mathcal{O}_\eta$-flats have all necessary properties of $\mathcal{O}$-flats and that, if $\mathcal{O}$ has the point-intersection property, then $\mathcal{O}_\eta$ also has this property.

LEMMA 4.4 *Let η be an $\mathcal{O}$-flat and k be the dimension of η.*

1. *Every translate of an $\mathcal{O}_\eta$-flat within the space η is an $\mathcal{O}_\eta$-flat.*
2. *A set $H \subseteq \eta$ is an $\mathcal{O}_\eta$-flat if and only if it is either η itself or the intersection of several $(k-1)$-dimensional $\mathcal{O}_\eta$-flats.*

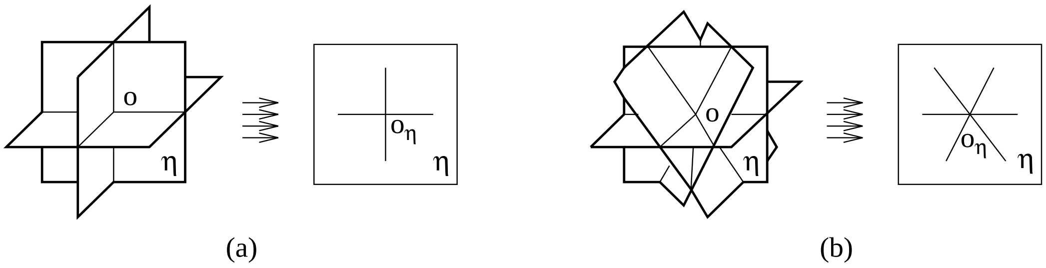

Figure 9: Lower-dimensional orientation sets.

3. *If the orientation set $\mathcal{O}$ has the point-intersection property, then $\mathcal{O}_\eta$ also has the point-intersection property; that is, the orientation set $\mathcal{O}_\eta$ is nonempty and $\bigcap \mathcal{O}_\eta = o_\eta$.*

Proof. **(1)** Every $\mathcal{O}_\eta$-flat is an $\mathcal{O}$-flat; therefore, a translate of an $\mathcal{O}_\eta$-flat is also an $\mathcal{O}$-flat. If the translate is in η, then it is an $\mathcal{O}_\eta$-flat.

(2) Since $\mathcal{O}_\eta$-flats are $\mathcal{O}$-flats, the intersection of $(k-1)$-dimensional $\mathcal{O}_\eta$-flats is an $\mathcal{O}$-flat. This $\mathcal{O}$-flat is in η and, hence, it is an $\mathcal{O}_\eta$-flat.

We next show, conversely, that every $\mathcal{O}_\eta$-flat H distinct from η is the intersection of $(k-1)$-dimensional $\mathcal{O}_\eta$-flats. Since H is an $\mathcal{O}$-flat, it is formed by the intersection of several $\mathcal{O}$-hyperplanes, say $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$ be those hyperplanes among them that do *not* contain η. Then, $\mathcal{H}_1 \cap \eta, \mathcal{H}_2 \cap \eta, \ldots, \mathcal{H}_k \cap \eta$ are $(k-1)$-dimensional flats (see Proposition 4.2). These flats are $\mathcal{O}_\eta$-flats and their intersection forms H.

(3) We assume, without loss of generality, that $o_\eta = o$. If the intersection of the elements of $\mathcal{O}$ is the point o, then, for some $\mathcal{O}$-hyperplane $\mathcal{H}$ through o, we have $\eta \not\subseteq \mathcal{H}$. Then, $\mathcal{H} \cap \eta$ is a $(k-1)$-dimensional $\mathcal{O}_\eta$-flat, which implies that $\mathcal{O}_\eta$ is nonempty.

We next show that $\bigcap \mathcal{O}_\eta = o$. Let $\mathcal{O}'$ be the set of $\mathcal{O}$-hyperplanes through o that do *not* contain η and $\{\mathcal{H} \cap \eta \mid \mathcal{H} \in \mathcal{O}'\}$ be the set of their intersections with η. Then, all elements of the latter set are $(k-1)$-dimensional $\mathcal{O}_\eta$-planes through o and their intersection is o. We conclude that the intersection of all $(k-1)$-dimensional $\mathcal{O}_\eta$-planes through o is exactly o; that is, $\bigcap \mathcal{O}_\eta = o$. $\square$

The orientation set $\mathcal{O}_\eta$ gives rise to lower-dimensional $\mathcal{O}_\eta$-halfspaces in the space η. These lower-dimensional $\mathcal{O}_\eta$-halfspaces are defined in the same way as $\mathcal{O}$-halfspaces, in terms of their intersection with $\mathcal{O}_\eta$-lines.

OBSERVATION 4.5 *The intersection of a (directed) $\mathcal{O}$-halfspace with an $\mathcal{O}$-flat η is a (directed) $\mathcal{O}_\eta$-halfspace.*

We use this observation, illustrated in Figure 10, to prove that, for orientation sets with the point-intersection property, the intersection of a directed $\mathcal{O}$-halfspace with an $\mathcal{O}$-flat is always connected.

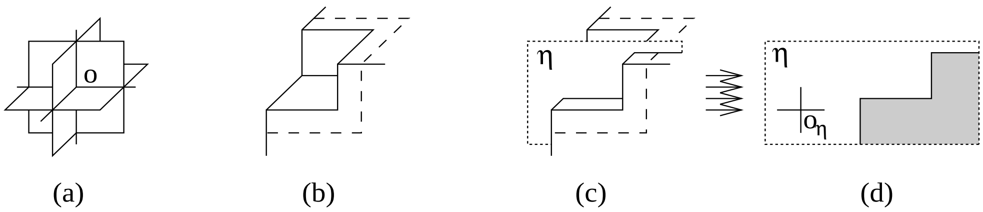

Figure 10: For (a) any orientation set, the intersection of (b) an $\mathcal{O}$-halfspace with (c) an $\mathcal{O}$-flat η is (d) an $\mathcal{O}_\eta$-halfspace.

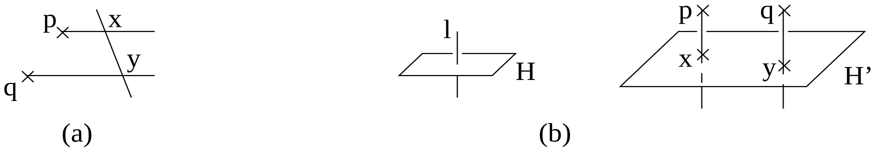

Figure 11: Proof of Theorem 4.6.

THEOREM 4.6 *If the orientation set $\mathcal{O}$ has the point-intersection property, then the intersection of a directed $\mathcal{O}$-halfspace with every $\mathcal{O}$-flat is connected.*

Proof. We first prove, by induction on the dimension d, that all directed $\mathcal{O}$-halfspaces are connected. To establish an induction base, we consider the planar case. We show that every two points p and q of a directed $\mathcal{O}$-halfplane P can be connected by a polygonal line in P. We choose two parallel $\mathcal{O}$-rays, with endpoints p and q, that are contained in P and point in the same direction (see Figure 11a). We next choose an $\mathcal{O}$-line that intersects these two rays and denote the intersection points by x and y, respectively. The polygonal line (p, x, y, q) is wholly in P.

We now consider a directed $\mathcal{O}$-halfspace P in d dimensions and show that every two points $p, q \in P$ can be connected by a path in P. By Lemma 2.1, since $\mathcal{O}$ has the point-intersection property, we may select some $\mathcal{O}$-line l and an $\mathcal{O}$-hyperplane $\mathcal{H}$ that intersects l and does not contain it (see Figure 11b). Since P is a directed $\mathcal{O}$-halfspace, there are two parallel-to-l rays, with endpoints p and q, that are contained in P and point in the same direction.

We select an $\mathcal{O}$-plane $\mathcal{H}'$, parallel to $\mathcal{H}$, that intersects the two rays with endpoints p and q, and denote the intersection points by x and y, respectively. $P \cap \mathcal{H}'$ is a directed $\mathcal{O}_{\mathcal{H}'}$-halfspace. By Lemma 4.4, the lower-dimensional orientation set $\mathcal{O}_{\mathcal{H}'}$ has the point-intersection property. Therefore, by the induction hypothesis, x and y can be connected by a path in $P \cap \mathcal{H}'$, which implies that p and q can be connected by a path in P.

Finally, we note that the intersection of a directed $\mathcal{O}$-halfspace with every $\mathcal{O}$-flat η is a directed $\mathcal{O}_\eta$-halfspace and the lower-dimensional orientation set $\mathcal{O}_\eta$ has the point-intersection property (Lemma 4.4). Therefore, the intersection of P with η is connected. $\square$

Theorem 4.6 cannot be extended to orientation sets without the point-intersection property. For example, if there is only one $\mathcal{O}$-line through o, then the union of several lines parallel to this $\mathcal{O}$-line is a disconnected directed $\mathcal{O}$-halfspace.

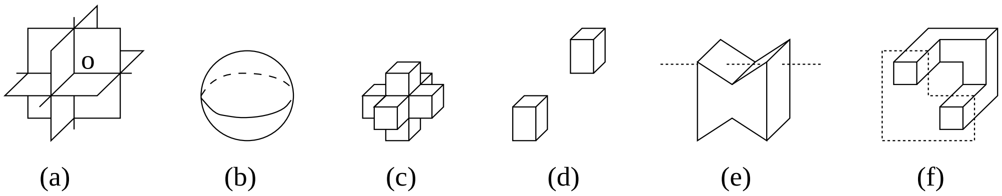

Figure 12: For the orthogonal-orientation set (a), the sets (b) and (c) are $\mathcal{O}$-connected, whereas the sets (d–f) are *not* $\mathcal{O}$-connected.

The property of directed $\mathcal{O}$-halfspaces stated in Theorem 4.6 is called $\mathcal{O}$-connectedness.

DEFINITION 4.3 ($\mathcal{O}$-Connectedness) *A closed set is* $\mathcal{O}$**-connected** *if its intersection with every $\mathcal{O}$-flat is connected.*

Note that $\mathcal{O}$-connected sets are connected, since the whole space $\mathcal{R}^d$ is an $\mathcal{O}$-flat. In particular, this observation implies that, *if the orientation set $\mathcal{O}$ has the point-intersection property, then every directed $\mathcal{O}$-halfspace is connected.*

Directed $\mathcal{O}$-halfspaces are not the only $\mathcal{O}$-connected sets. For example, the set in Figure 12(b) is $\mathcal{O}$-connected, even though it is not an $\mathcal{O}$-halfspace. In fact, every standard convex set is $\mathcal{O}$-connected. As another example, the set in Figure 12(c) is $\mathcal{O}$-connected for the orthogonal-orientation set in Figure 12(a). On the other hand, the set in Figure 12(d) is not $\mathcal{O}$-connected because it is disconnected, the set in Figure 12(e) is not $\mathcal{O}$-connected because its intersection with the dotted $\mathcal{O}$-line is disconnected, and the set in Figure 12(f) is not $\mathcal{O}$-connected because its intersection with the dotted $\mathcal{O}$-plane is disconnected. We have studied $\mathcal{O}$-connected sets as a part of our exploration of $\mathcal{O}$-convexity [3].

5 BOUNDARY CONVEXITY

We explore the properties of the boundaries of $\mathcal{O}$-halfspaces and characterize an $\mathcal{O}$-halfspace in terms of its boundary.

We begin by showing that, if an orientation set has the point-intersection property, then all points in the boundary of an $\mathcal{O}$-halfspace are "infinitely close" to its interior; that is, an $\mathcal{O}$-halfspace is the closure of its interior.

LEMMA 5.1 *Let P be an $\mathcal{O}$-halfspace and P_{int} be the interior of P. If the orientation set $\mathcal{O}$ has the point-intersection property, then* $\text{Closure}(P_{\text{int}}) = P$.

Proof. We show that, for every point p in the boundary of P and every distance ϵ, there is an interior point within distance ϵ of p. We use induction on the dimension d.

By Lemma 2.1, we may select some $\mathcal{O}$-line l and an $\mathcal{O}$-hyperplane $\mathcal{H}$ that intersects l and does not contain it. Let $\mathcal{H}'$ be the $\mathcal{O}$-hyperplane through p parallel to $\mathcal{H}$, let Q be the intersection

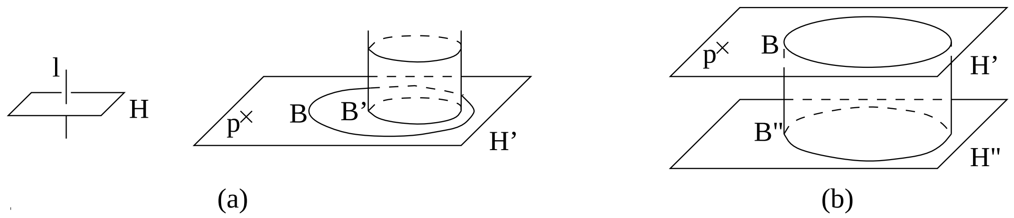

Figure 13: Proof of Lemma 5.1.

of P and $\mathcal{H}'$, and let Q_{int} be the interior of Q in the $(d-1)$-dimensional space $\mathcal{H}'$ (rather than in $\mathcal{R}^d$). If $d > 2$, the set Q is a lower-dimensional $\mathcal{O}$-halfspace and, by the induction hypothesis, $\text{Closure}(Q_{\text{int}}) = Q$. On the other hand, if $d = 2$, then Q is empty, a ray, or a line; therefore, $\text{Closure}(Q_{\text{int}}) = Q$, which establishes an induction base.

We select an interior point of Q within distance $\epsilon/2$ of p and a $(d-1)$-dimensional ball $B \subseteq Q$ centered at this point such that the radius of B is at most $\epsilon/2$ (see Figure 13a).

We assume that the line l is vertical and divide the rays parallel to l into "upward" and "downward" rays. For every point $q \in B$, the upward or downward ray with endpoint q is contained in P. Let U be the set of all points of B such that the upward rays from them are in P and D be the set of all points of B such that the downward rays from them are in P.

We consider two cases. First, suppose that the interior of U in the $(d-1)$-dimensional space $\mathcal{H}'$ is nonempty, which means that there is a $(d-1)$-dimensional ball $B' \subseteq U$ (see Figure 13a). The union of the upward rays from all points of B' forms a "cylinder," wholly contained in P. Some of the interior points of this cylinder are within distance ϵ of p. Clearly, the interior of the cylinder is in the interior of P; therefore, some of P's interior points are within ϵ of p.

Now suppose that the interior of U is empty. Then, every $(d-1)$-dimensional ball $B' \subseteq B$ contains a point of the set D, which implies that $\text{Closure}(D) = B$. Let $\mathcal{H}''$ be a plane parallel to $\mathcal{H}'$ and located below $\mathcal{H}'$ (see Figure 13b). The intersection of $\mathcal{H}''$ with the downward rays from all points of D forms a translate of D. This translate of D is in P and its closure is a $(d-1)$-dimensional ball, say B'', which is a translate of B.

By the closed-set assumption (see Section 1), P is closed; therefore, B'' is in P. The union of the vertical segments joining B and B'' forms a cylinder, wholly contained in P. The interior of this cylinder is in P's interior and some of the cylinder's interior points are within distance ϵ of p. $\square$

We call sets satisfying the property stated in Lemma 5.1 **interior-closed sets**: A set P is interior-closed if $\text{Closure}(P_{\text{int}}) = P$.

Our purpose in the rest of the section is to present $\mathcal{O}$-convexity analogs of the following boundary-convexity characterization of standard halfspaces.

PROPOSITION 5.2 *An interior-closed set is a halfspace if and only if its boundary is a nonempty convex set.*

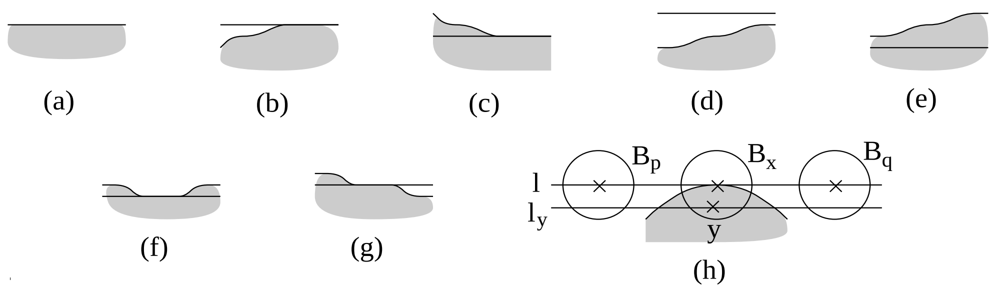

Figure 14: The eight cases considered in the proof of Lemma 5.3.

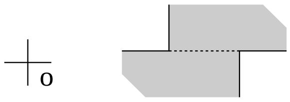

Figure 15: An $\mathcal{O}$-halfplane whose boundary is not $\mathcal{O}$-convex.

We first generalize the "if" part of this boundary-convexity characterization.

LEMMA 5.3 *An interior-closed set with an $\mathcal{O}$-convex boundary is an $\mathcal{O}$-halfspace.*

Proof. Let P be an interior-closed set with an $\mathcal{O}$-convex boundary. We show that the intersection of P with every $\mathcal{O}$-line l is empty, a ray, or a line. The intersection of l with P's boundary may be a line, a ray, a segment, a point, or empty. If this intersection is a line, then $P \cap l$ is also a line (see Figure 14a). If the intersection of l with P's boundary is a ray, then $P \cap l$ is a ray or a line (Figures 14b and 14c). If l does not intersect P's boundary, then $P \cap l$ is empty or a line (Figures 14d and 14e).

Finally, suppose that the intersection of l with P's boundary is a segment or point. We show, by contradiction, that $P \cap l$ is a line or ray (Figures 14f and 14g). If not, then $P \cap l$ is a segment or point (see Figure 14h). We select points $p, q \in l$ that are outside of P, on different sides of the intersection. We consider equal-sized balls, B_p and B_q, that are centered at these points and do *not* intersect P. We next select a point x in the intersection of l and P's boundary and consider the ball B_x, centered at x, of the same size as B_p and B_q. Since P is interior-closed, we may select a point $y \in B_x$ that is in the interior of P. We consider the line l_y through y parallel to l. This line intersects the balls B_p and B_q, which are outside of P. Since x is in the interior of P, we conclude that the intersection of the $\mathcal{O}$-line l_y with the boundary of P is not connected, contradicting the $\mathcal{O}$-convexity of P's boundary. $\square$

The converse of Lemma 5.3 does not hold: the boundary of an $\mathcal{O}$-halfspace may not be $\mathcal{O}$-convex. In Figure 15, we show an $\mathcal{O}$-halfplane whose boundary is not $\mathcal{O}$-convex: the intersection of the boundary with the dotted $\mathcal{O}$-line is disconnected. We now present a necessary and sufficient characterization of $\mathcal{O}$-halfspaces in terms of their boundary.

THEOREM 5.4 (Boundary of $\mathcal{O}$-halfspaces) *An interior-closed set P is an $\mathcal{O}$-halfspace if and only if, for every $\mathcal{O}$-line l, one of the following two conditions holds:*

1. *The intersection of l with the boundary of P is connected.*

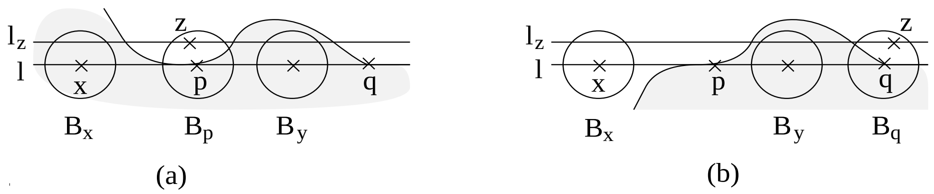

Figure 16: Proof of Theorem 5.4.

2. *The intersection of l with the boundary of P consists of two disconnected rays and the segment of l between these rays is in P.*

Proof. Suppose that the two conditions hold. We demonstrated in the proof of Lemma 5.3 that, if the intersection of l with the boundary of P is connected (Condition 1), then $P \cap l$ is empty, a ray, or a line. On the other hand, if the intersection of l with P's boundary satisfies Condition 2, then $P \cap l$ is a line. We conclude that the intersection of P with every $\mathcal{O}$-line is empty, a ray, or a line; thus, P is an $\mathcal{O}$-halfspace.

To prove the converse, suppose that P is an $\mathcal{O}$-halfspace and the intersection of an $\mathcal{O}$-line l with the boundary of P is *not* connected. We show that this intersection satisfies Condition 2.

Since the boundary is closed, we can select points $p, q \in l$ in P's boundary such that all points of l between p and q are *not* in the boundary. Since the intersection of l with P is connected, the segment of l between p and q is in P.

We next show, by contradiction, that all points of l outside of this segment are in P's boundary. Suppose that some point x is *not* in P's boundary. Without loss of generality, we assume that x is to the left of p and that q is to the right of p (see Figure 16a).

If x is an interior point of P, then there is a ball $B_x \subseteq P$ centered at x (Figure 16a). Let $y \in l$ be some point between p and q, $B_y \subseteq P$ be a ball centered at y, and B_p be a ball, centered at p, that is no larger than B_x and B_y. Since p is in the boundary of P, we may choose a point $z \in B_p$ that is *not* in P. We consider the $\mathcal{O}$-line l_z through z parallel to l. This line intersects the balls B_x and B_y, which are in P. Since z is not in P, the intersection of l_z with P is disconnected, contradicting the assumption that P is an $\mathcal{O}$-halfspace.

If x is an exterior point of P, we may select a ball B_x, centered at x, that does not intersect P (see Figure 16b). Let $y \in l$ be some point between p and q, $B_y \subseteq P$ be a ball centered at y, and B_q be a ball, centered at q, that is no larger than B_x and B_y. Since q is in the boundary of P, we may select a point $z \in B_q$ that is *not* in P. We again consider the $\mathcal{O}$-line l_z through z parallel to l. This line intersects the balls B_x and B_y, which implies that the intersection of l_z with P is not empty, not a ray, and not a line, contradicting the assumption that P is an $\mathcal{O}$-halfspace. □

Observe that, if the intersection of P's boundary with a line l consists of two rays and the segment of the line between these rays is in P, then l is wholly contained in P. We use this

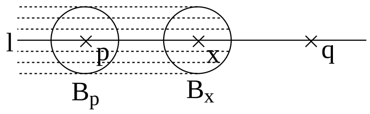

Figure 17: Proof of Lemma 5.6.

observation to simplify Condition 2 in Theorem 5.4.

COROLLARY 5.5 *An interior-closed set P is an $\mathcal{O}$-halfspace if and only if, for every $\mathcal{O}$-line l, one of the following two conditions holds:*

1. *The intersection of l with the boundary of P is connected.*
2. *l is wholly contained in P.*

We next present a boundary-convexity result for directed $\mathcal{O}$-halfspaces.

LEMMA 5.6 *The boundary of a directed $\mathcal{O}$-halfspace is $\mathcal{O}$-convex.*

Proof. Suppose that the boundary of P is *not* $\mathcal{O}$-convex. Then, the intersection of some $\mathcal{O}$-line l with P's boundary is disconnected and we may select points $p, q \in l$ that are in the boundary and a point $x \in l$ between them that is *not* in the boundary (see Figure 17). We assume, for convenience, that p is to the left of x.

Since the intersection of P with l is connected, x is in the interior of P and we can choose a ball $B_x \subseteq P$ centered at x. Either all left-pointed or all right-pointed rays with endpoints in B_x are contained in P. We assume that the left-pointed rays (shown by dotted lines) are in P. Then, some ball B_p centered at p is wholly contained in P; therefore, p is in P's interior, which yields a contradiction. $\square$

If the orientation set has the point-intersection property, then the *boundary of a directed $\mathcal{O}$-halfspace is $\mathcal{O}$-connected,* which is a stronger result. We will prove this result in Section 6, using properties of the complement of a directed $\mathcal{O}$-halfspace. The converse of this result does not hold, as we demonstrate in the following example.

EXAMPLE: $\mathcal{O}$-halfspace, with an $\mathcal{O}$-connected boundary, that is not directed. We show such an $\mathcal{O}$-halfspace in Figure 18(b). This $\mathcal{O}$-halfspace consists of two rectangular polyhedral angles (quadrants), which touch each other along one of their facets. The boundary of the $\mathcal{O}$-halfspace is $\mathcal{O}$-connected for the orthogonal-orientation set in Figure 18(a). The $\mathcal{O}$-halfspace, however, is *not* directed, since the dotted rays (Figure 18c), formed by the intersection of this $\mathcal{O}$-halfspace with vertical $\mathcal{O}$-lines, point in opposite directions. $\square$

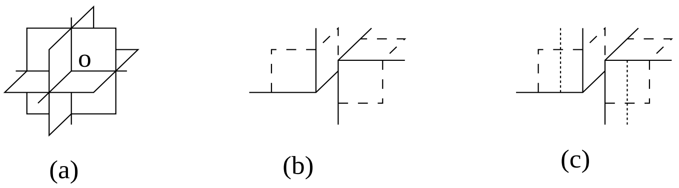

Figure 18: An $\mathcal{O}$-halfspace that is *not* directed, even though its boundary is $\mathcal{O}$-connected.

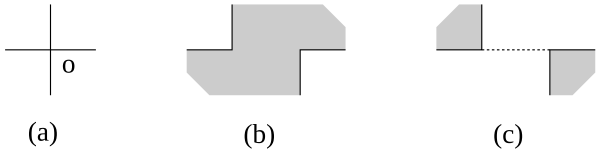

Figure 19: For (a) the orthogonal-orientation set, the set (b) is an $\mathcal{O}$-halfplane, whereas (c) its closed complement is *not* an $\mathcal{O}$-halfplane.

6 COMPLEMENTATION

We present a condition under which the closure of the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace. We then show that the closure of the complement of a directed $\mathcal{O}$-halfspace is always a directed $\mathcal{O}$-halfspace.

We call the closure of the complement of a set the **closed complement.** In general, the closed complement of an $\mathcal{O}$-halfspace may *not* be an $\mathcal{O}$-halfspace. For example, the closed complement of the $\mathcal{O}$-halfplane in Figure 19(b) is not an $\mathcal{O}$-halfplane, since its intersection with the dashed $\mathcal{O}$-line (Figure 19c) is disconnected. We state a necessary and sufficient condition under which the closed complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.

THEOREM 6.1 (Complements of $\mathcal{O}$-halfspaces) *The closed complement of an $\mathcal{O}$-halfspace P is an $\mathcal{O}$-halfspace if and only if the boundary of P is $\mathcal{O}$-convex.*

Proof. We denote the closed complement of P by Q. Note that Q is interior-closed and the boundary of Q is the same as the boundary of P. If the boundary of P is $\mathcal{O}$-convex, then Q is an interior-closed set with an $\mathcal{O}$-convex boundary, which implies that Q is an $\mathcal{O}$-halfspace (Lemma 5.3).

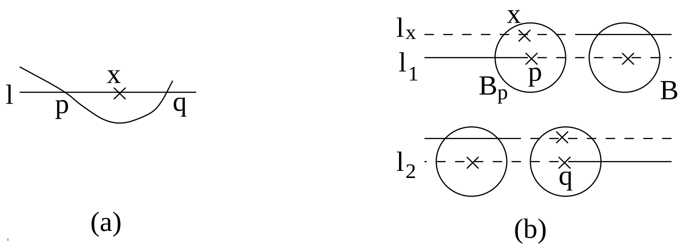

Figure 20: Proofs of (a) Theorem 6.1 and (b) Theorem 6.2.

Suppose, conversely, that Q is an $\mathcal{O}$-halfspace. We show, by contradiction, that the boundary of P is $\mathcal{O}$-convex. If the boundary is not $\mathcal{O}$-convex, then there are points p, x, and q on some $\mathcal{O}$-line l, such that p and q are in the boundary, whereas x, located between p and q, is *not* in the boundary (see Figure 20a). Note that p and q belong both to P and to Q, whereas x is either in P's interior or in Q's interior. If x is in P, then the intersection of P with the $\mathcal{O}$-line l is disconnected and, hence, P is not an $\mathcal{O}$-halfspace. Similarly, if x is in Q, then Q is not an $\mathcal{O}$-halfspace. □

We next characterize the closed complement of a directed $\mathcal{O}$-halfspace.

THEOREM 6.2 (Complements of directed $\mathcal{O}$-halfspaces) *The closed complement of a directed $\mathcal{O}$-halfspace is a directed $\mathcal{O}$-halfspace.*

Proof. The boundary of a directed $\mathcal{O}$-halfspace P is $\mathcal{O}$-convex (Lemma 5.6). Therefore, by Theorem 6.1, the closed complement of P is an $\mathcal{O}$-halfspace.

We show, by contradiction, that the closed complement of P is directed. Suppose that the intersection of the closed complement with two parallel $\mathcal{O}$-lines, l_1 and l_2, forms rays that point in opposite directions. We denote the endpoints of these rays by p and q, respectively. In Figure 20(b), we show the rays by solid lines. Note that p and q are in the boundary of P and the dashed parts of the lines are in the interior of P.

We select a point in the dashed part of l_1 and a ball $B \subseteq P$ centered at this point. Let B_p be the same-sized ball centered at p. Since p is in the boundary of P, we may select a point $x \in B_p$ that is *not* in P. We consider the $\mathcal{O}$-line l_x through x parallel to l_1. The intersection of l_x with P is empty, a ray, or a line. Since x is not in P and, on the other hand, l_x intersects the ball $B \subseteq P$, we conclude that the intersection of l_x with P is a ray pointing "to the right."

Using a similar construction with l_2, we get an $\mathcal{O}$-line whose intersection with P is a ray pointing "to the left" (see Figure 20b), which means that the $\mathcal{O}$-halfspace P is *not* directed, giving a contradiction. □

We use this result to demonstrate that, for orientation sets with the point-intersection property, the boundary of a directed $\mathcal{O}$-halfspace is $\mathcal{O}$-connected.

THEOREM 6.3 (Boundaries of directed $\mathcal{O}$-halfspaces) *If the orientation set $\mathcal{O}$ has the point-intersection property, then the boundary of every directed $\mathcal{O}$-halfspace is $\mathcal{O}$-connected.*

Proof. To demonstrate that the boundary of a directed $\mathcal{O}$-halfspace P is $\mathcal{O}$-connected, we first show that the boundary is connected and then use this result to demonstrate that the intersection of the boundary with every $\mathcal{O}$-flat η is also connected.

We establish the connectedness of P's boundary by contradiction. Since the orientation set $\mathcal{O}$ has the point-intersection property, P is connected (Theorem 4.6). Therefore, if the boundary of P is disconnected, then the closed complement of P is also disconnected.

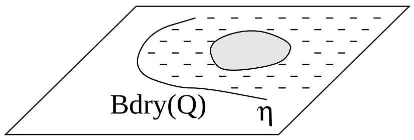

Figure 21: Proof of Theorem 6.3.

On the other hand, the closed complement of P is a directed $\mathcal{O}$-halfspace (Theorem 6.2), contradicting the connectedness of directed $\mathcal{O}$-halfspaces.

We next show that the intersection of P's boundary, denoted by Bdry(P), with every $\mathcal{O}$-flat η is connected. Let Q be the intersection of P with η and Bdry(Q) be the boundary of Q in the lower-dimensional space η (rather than in the whole space $\mathcal{R}^d$). Clearly, Bdry(Q) $\subseteq$ Bdry(P) $\cap\, \eta$. The set Q is a directed $\mathcal{O}_\eta$-halfspace and the orientation set $\mathcal{O}_\eta$ has the point-intersection property (Lemma 4.4). Therefore, by the first part of the proof, Bdry(Q) is connected.

Suppose that Bdry(P) $\cap\, \eta$ is *not* connected. Then, Bdry(P) $\cap\, \eta$ contains a component disconnected from the boundary of Q (see Figure 21, where this component is shown by the shaded region). This component is contained in Q; therefore, it is surrounded in the flat η by interior points of P (the interior points are shown by the dashed region in Figure 21).

We now consider the intersection of the closed complement of P with η. This intersection contains Bdry(Q) and the component of the Bdry(P) $\cap\, \eta$ disconnected from Bdry(Q). The intersection does *not* contain any interior points of P. Therefore, the intersection of the closed complement of P with η is disconnected. On the other hand, the closed complement of P is a directed $\mathcal{O}$-halfspace (Theorem 6.2), which implies that the intersection of the closed complement of P with the $\mathcal{O}$-flat η is connected (Theorem 4.6), yielding a contradiction. $\square$

7 CONCLUDING REMARKS

We described generalized halfspaces in the theory of $\mathcal{O}$-convexity and demonstrated that their properties are similar to the properties of standard halfspaces. These results extend our previous exploration of $\mathcal{O}$-convex and $\mathcal{O}$-connected sets [3].

The work presented here leaves some unanswered questions. For example, we have not established the contractability of $\mathcal{O}$-halfspaces. A set is *contractable* if it can be continuously transformed (contracted) to a point in such a way that all intermediate stages of the transformation are contained in the original set. We conjecture that, *if the orientation set $\mathcal{O}$ has the point-intersection property, then every connected $\mathcal{O}$-halfspace is contractable.*

As another example of an open problem, we conjecture that *every $\mathcal{O}$-connected set is the intersection of the directed $\mathcal{O}$-halfspaces that contain it.* We also conjecture that a similar property holds of connected $\mathcal{O}$-halfspaces: *every connected $\mathcal{O}$-halfspace is formed by the intersection of directed $\mathcal{O}$-halfspaces.*

ACKNOWLEDGEMENTS

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