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Restricted-Orientation Convexity

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Preface

Restricted-orientation convexity, also called $\mathcal{O}$-convexity, is the study of geometric objects whose intersections with lines from some fixed set are connected. This notion generalizes standard convexity and several types of nontraditional convexity. We explore this generalized convexity in multidimensional Euclidean space and identify the properties of standard convex sets that also hold for restricted-orientation convexity.

The purpose of the book is to present the current results on restricted-orientation convexity to the research community and discuss related open problems. The book requires only basic knowledge in geometry; the reader should be familiar with the notion of higher-dimensional Euclidean space and with basic objects in this space, such as lines, balls, and hyperplanes. We use geometric techniques in most proofs, which are accessible to all mathematics and computer-science researchers and graduate students.

$\mathcal{O}$-convexity: We begin with basic properties of $\mathcal{O}$-convex sets, and then introduce $\mathcal{O}$-connected sets, which are a subclass of $\mathcal{O}$-convex sets. We study restricted-orientation analogs of lines, flats and hyperplanes, and characterize $\mathcal{O}$-convex and $\mathcal{O}$-connected sets in terms of their intersections with hyperplanes. We also explore properties of $\mathcal{O}$-connected curves; in particular, we determine when the replacement of a segment of an $\mathcal{O}$-connected curve gives a new $\mathcal{O}$-connected curve, and when the catenation of several curvilinear segments gives an $\mathcal{O}$-connected segment. We use these results to characterize an $\mathcal{O}$-convex set in terms of $\mathcal{O}$-convex segments joining its points, and an $\mathcal{O}$-connected set in terms of $\mathcal{O}$-connected segments.

$\mathcal{O}$-halfspaces: We introduce $\mathcal{O}$-halfspaces, which are a generalization of standard halfspaces, defined as geometric objects whose intersection with every line from some fixed set is empty, a ray or a line. We give basic properties of $\mathcal{O}$-halfspaces and compare them with standard halfspaces; in particular, we show that $\mathcal{O}$-halfspaces may be disconnected and characterize them through their connected components. We also characterize $\mathcal{O}$-halfspaces in terms of $\mathcal{O}$-convexity of their boundaries, and give a condition under which the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.

Strong $\mathcal{O}$-convexity: We also introduce the notion of strong $\mathcal{O}$-convexity, which is an alternative generalization of convexity. We describe properties of strongly $\mathcal{O}$-convex flats and halfspaces, and establish the strong $\mathcal{O}$-convexity of the affine hull of a strongly $\mathcal{O}$-convex set. We then show that, for every point in the boundary of a strongly $\mathcal{O}$-convex set, there is a supporting strongly $\mathcal{O}$-convex hyperplane through it. Finally, we characterize strongly $\mathcal{O}$-convex sets in terms of the intersections of strongly $\mathcal{O}$-convex halfspaces.

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Introduction

The study of convex sets is a branch of geometry, analysis and linear algebra, which has numerous connections with other areas of mathematics, including topology, number theory and combinatorics [6, 14, 21]. Researchers have explored not only mathematical properties of convex sets, but also related computational problems [5, 13, 34], and applied the resulting algorithms in many practical areas, such as graphics, finite-element analysis, VLSI design and motion planning. They have also studied several types of nontraditional convexity, such as ortho-convexity [28, 30], restricted-orientation convexity [35], NESW convexity [25, 49, 50] and link convexity [2, 52].

The notion of restricted orientations has stemmed from the study of ortho-polygons, which are polygons with edges parallel to the coordinate axes [19]. Researchers have extensively investigated ortho-polygons [1,3,4,11,33,58,59], and used them in geometric models based on vertical and horizontal lines, such as VLSI wiring and architectural floor plans. They have also studied ortho-convex sets, which are sets whose intersection with every vertical and every horizontal line is connected [28, 30, 32, 39, 42].

Güting introduced restricted orientations as a generalization of ortho-polygons [16]; he explored computational properties of polygons whose edges were parallel to the elements of some fixed set of lines [16–18]. Widmayer, Wu, Schlag and Wong also studied computational problems related to restricted orientations [55–57]. Nilsson, Ottmann, Schuierer and Icking reviewed and extended the earlier results in restricted-orientation geometry [31].

Rawlins and Wood used restricted orientations to define the notion of $\mathcal{O}$-convexity, which generalized standard convexity and ortho-convexity [35, 37–41]. Schuierer continued their exploration and presented an extensive study of geometric and computational properties of $\mathcal{O}$-convex sets [43]. Rawlins introduced an alternative generalization of convexity based on restricted orientations, called strong $\mathcal{O}$-convexity [35]. We considered computational problems in strong $\mathcal{O}$-convexity and developed a suite of related algorithms [10].

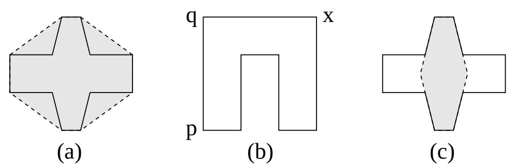

Fig. 1.1. Standard convex hull **(a)** and standard kernels **(b,c)**

Although researchers have extensively studied nontraditional convexity in the plane, they have not extended it to higher dimensions. The purpose of our work is to develop a theory of restricted-orientation convexity in multidimensional space [7–9].

We begin with a review of standard convexity, and define the related notions of convex hulls and kernels (Sect. 1.1). We also review ortho-convexity and strong ortho-convexity, which are special cases of restricted-orientation convexity (Sects. 1.2 and 1.3), and define a topological generalization of convex sets (Sect. 1.4). We then outline the organization of the book and dependencies between its chapters (Sect. 1.5).

1.1 Standard Convexity

We review basic properties of convex sets in the plane; a much more extended review is available in several texts on convexity, including *Convex Polytopes* by Grünbaum, Klee, Perles and Shephard [15], *Geometry and Convexity* by Kelly and Weiss [20], *Convex Sets* by Valentine [51] and *Convexity* by Webster [54].

We define convex sets through their intersections with lines; specifically, a set is **convex** if its intersection with every line is connected.

Proposition 1.1 (Properties of standard convex sets).

1. *The intersection of convex sets is a convex set.*
2. *Every convex set is simply connected.*
3. *A closed set is convex if and only if it is either the entire plane or the intersection of halfplanes.*

The **convex hull** of a geometric object is the intersection of all convex sets that contain the object; for example, the shaded region in Fig. 1.1a is the convex hull of the polygon shown by solid lines.

Proposition 1.2 (Properties of standard convex hulls).

1. *The convex hull of a geometric object is the minimal convex set that contains the object.*

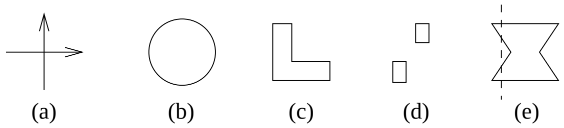

Fig. 1.2. Ortho-convexity

2. *An object is convex if and only if it is identical to its convex hull.*

Two points of a geometric object are **visible** to each other if the line segment joining them is wholly in the object; for example, the points p and q of the polygon in Fig. 1.1b are visible to each other, whereas p and x are not. Note that an object is convex if and only if every two of its points are visible to each other. The **kernel** of a geometric object is the set of points that are visible from all points of the object; for example, the kernel of the polygon in Fig. 1.1b is empty, whereas the kernel of the polygon in Fig. 1.1c is the nonempty shaded region.

Proposition 1.3 (Properties of standard kernels).

1. *The kernel of any geometric object is convex.*
2. *An object is convex if and only if it is identical to its kernel.*

1.2 Ortho-Convexity

We now consider ortho-convexity, which is weaker than standard convexity. A set is **ortho-convex** if its intersection with every vertical line and every horizontal line is connected. For example, the sets in Fig. 1.2b–d are ortho-convex, whereas the set in Fig. 1.2e is not ortho-convex, since its intersection with the dashed vertical line is disconnected. Note that ortho-convex sets may be disconnected; for instance, the set in Fig. 1.2d consists of two components.

Proposition 1.4 (Properties of ortho-convex sets).

1. *The intersection of ortho-convex sets is an ortho-convex set.*
2. *Every standard convex set is ortho-convex.*
3. *A disconnected set is ortho-convex if and only if every connected component of the set is ortho-convex and no vertical or horizontal line intersects two components.*
4. *Every connected ortho-convex set is simply connected.*

The **ortho-hull** of a geometric object is the intersection of all ortho-convex sets that contain the object; we give four examples of ortho-hulls in Fig. 1.3.

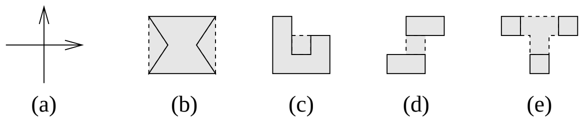

Fig. 1.3. Ortho-hulls of two connected sets **(b,c)** and two disconnected sets **(d,e)**

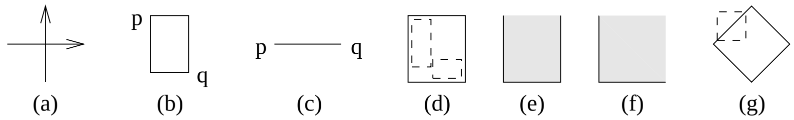

Fig. 1.4. Strong ortho-convexity

Proposition 1.5 (Properties of ortho-hulls).

1. *The ortho-hull of a geometric object is the minimal ortho-convex set that contains the object.*
2. *An object is ortho-convex if and only if it is identical to its ortho-hull.*
3. *The ortho-hull of an object is a subset of the standard convex hull of the object.*

1.3 Strong Ortho-Convexity

We next review a different type of nontraditional convexity, which is also defined through vertical and horizontal lines, and consider the related notions of ortho-rectangles and ortho-blocks. An **ortho-rectangle** is a rectangle whose sides are parallel to the coordinate axes. An **ortho-block** of two points p and q is the minimal ortho-rectangle that contains them; note that p and q are opposite vertices of this rectangle, as shown in Fig. 1.4b. In particular, if p and q are on the same vertical or horizontal line, their ortho-block is the line segment joining them, as shown in Fig. 1.4c.

A set is **strongly ortho-convex** if, for every two of its points, their ortho-block is wholly in the set. For example, the rectangle in Fig. 1.4d is strongly ortho-convex; two ortho-blocks contained in this rectangle are shown by dashed lines. As another example, the unbounded sets in Fig. 1.4e,f are also strongly ortho-convex. On the other hand, the square in Fig. 1.4g is not strongly ortho-convex, because the dashed ortho-block is not in this square.

Proposition 1.6 (Properties of strongly ortho-convex sets).

1. *The intersection of strongly ortho-convex sets is a strongly ortho-convex set.*
2. *Every strongly ortho-convex set is standard convex.*
3. *Every strongly ortho-convex set is simply connected.*

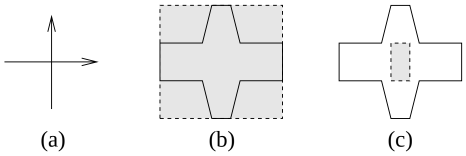

Fig. 1.5. Strong ortho-hull **(b)** and strong ortho-kernel **(c)**

4. *A halfplane is strongly ortho-convex if and only if its boundary line is vertical or horizontal.*
5. *A closed set is strongly ortho-convex if and only if it is either the entire plane or the intersection of strongly ortho-convex halfplanes.*
6. *A closed bounded set is strongly ortho-convex if and only if it is an ortho-rectangle.*

The **strong ortho-hull** of a geometric object is the intersection of all strongly ortho-convex sets that contain the object, as illustrated in Fig. 1.5b.

Proposition 1.7 (Properties of strong ortho-hulls).

1. *The strong ortho-hull of a geometric object is the minimal strongly ortho-convex set that contains the object.*
2. *An object is strongly ortho-convex if and only if it is identical to its strong ortho-hull.*
3. *The standard convex hull of an object is a subset of the strong ortho-hull of the object.*

We can define strong ortho-visibility in terms of ortho-blocks; that is, two points of a geometric object are strongly ortho-visible to each other if their ortho-block is wholly in the object. Note that an object is strongly ortho-convex if and only if every two of its points are strongly ortho-visible to each other. The **strong ortho-kernel** of a geometric object is the set of points that are strongly ortho-visible from every point of the object; we give an example of a strong ortho-kernel in Fig. 1.5c.

Proposition 1.8 (Properties of strong ortho-kernels).

1. *The strong ortho-kernel of any geometric object is strongly ortho-convex.*
2. *An object is strongly ortho-convex if and only if it is identical to its strong ortho-kernel.*
3. *The strong ortho-kernel of an object is a subset of the standard kernel of the object.*

Table 1.1. Comparison of different convexities

Intersection	
Standard convexity:	The intersection of convex sets is a convex set.
Ortho-convexity:	The intersection of ortho-convex sets is an ortho-convex set.
Strong ortho-convexity:	The intersection of strongly ortho-convex sets is a strongly ortho-convex set.
Line intersection	
Standard convexity:	A set is convex if and only if its intersection with every line is connected.
Ortho-convexity:	A set is ortho-convex if and only if its intersection with every vertical line and every horizontal line is connected.
Connectedness	
Standard convexity:	Every convex set is simply connected.
Ortho-convexity:	Every connected ortho-convex set is simply connected.
Strong ortho-convexity:	Every strongly ortho-convex set is simply connected.
Visibility	
Standard convexity:	A set is convex if and only if, for every two of its points, the line segment joining them is wholly in the set.
Strong ortho-convexity:	A set is strongly ortho-convex if and only if, for every two of its points, their ortho-block is wholly in the set.
Kernel convexity	
Standard convexity:	The standard kernel of any set is convex.
Ortho-convexity:	The ortho-kernel of any set is ortho-convex.
Strong ortho-convexity:	The strong ortho-kernel of any set is strongly ortho-convex.
Halfspace intersection	
Standard convexity:	A closed set is convex if and only if it is either the entire plane or the intersection of halfplanes.
Strong ortho-convexity:	A closed set is strongly ortho-convex if and only if it is either the entire plane or the intersection of strongly ortho-convex halfplanes.

1.4 Convexity Spaces

The properties of ortho-convexity and strong ortho-convexity are similar to those of standard convexity, as shown in Table 1.1. The basic results for other types of nontraditional convexity are also analogous to those for standard convexity. This similarity has led to the notion of convexity spaces [24], which is a topological generalization of convex sets; a review of the related results is available in the book by van de Vel [53].

A **convexity space** is defined by two sets, X and $\mathcal{C}$, where X is an arbitrary set, and $\mathcal{C}$ is a collection of subsets of X that satisfies two conditions:

1. The empty set and the entire set X are elements of $\mathcal{C}$.
2. For every subset C of $\mathcal{C}$, the intersection $\cap C$ of its elements is in $\mathcal{C}$.

Informally, X is an analog of the plane in standard convexity, and the elements of $\mathcal{C}$ are analogs of convex sets, which are called $\mathcal{C}$-convex sets. The two conditions generalize the observation that the empty set and the entire plane are convex, and the intersection of convex sets is a convex set.

The related definition of a hull is the same as in standard convexity; that is, for every subset Y of X, the **$\mathcal{C}$-hull** of Y is the intersection of all $\mathcal{C}$-convex sets that contain Y.

Proposition 1.9 (Properties of $\mathcal{C}$-hulls).

1. *The $\mathcal{C}$-hull of a subset Y of X is the minimal $\mathcal{C}$-convex set that contains Y.*
2. *A subset Y of X is $\mathcal{C}$-convex if and only if it is identical to its $\mathcal{C}$-hull.*

Schuierer, Rawlins and Wood defined visibility in convexity spaces and studied its properties [35,36,43,44,48,60]. Two elements p and q of a subset Y of X are visible to each other if the $\mathcal{C}$-hull of the two-element set $\{p, q\}$ is wholly in Y. Note that the hull of two points in standard convexity is the line segment joining them, and the strong ortho-hull of two points is their ortho-block, which means that visibility in convexity spaces generalizes standard visibility and strong ortho-visibility.

1.5 Book Outline

We present the results of exploring two notions of nontraditional convexity in multidimensional space, called $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity, which also satisfy the general conditions of convexity spaces. These two notions generalize standard convexity, ortho-convexity, and strong ortho-convexity. In Fig. 1.6, we summarize the organization of the book.

We first describe the properties of $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity in two dimensions (Chap. 2) and consider related computational problems (Chap. 3). We then generalize $\mathcal{O}$-convexity to higher dimensions, and show that the properties of the resulting generalization are much richer than those in two dimensions (Chap. 4). We also consider $\mathcal{O}$-convexity analogs of halfspaces and study their relationship to $\mathcal{O}$-convex sets (Chap. 5). Finally, we extend strong $\mathcal{O}$-convexity to higher dimensions, describe the main properties of strongly $\mathcal{O}$-convex sets, and give additional properties of strongly $\mathcal{O}$-convex flats and halfspaces (Chap. 6). We conclude with a summary of results and related open problems (Chap. 7).

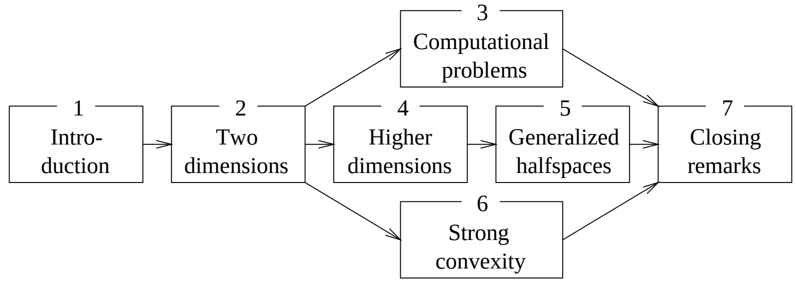

Fig. 1.6. Dependencies among chapters

2

Two Dimensions

We begin with two planar generalizations of convexity, called $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity. We first define $\mathcal{O}$-convex sets and present their basic properties (Sect. 2.1), then introduce a restricted-orientation analog of halfplanes (Sect. 2.2), and finally describe strong $\mathcal{O}$-convexity (Sect. 2.3).

2.1 $\mathcal{O}$-Convex Sets

Rawlins introduced the notion of planar $\mathcal{O}$-convexity in 1987, as a generalization of ortho-convexity [32] and standard convexity. He defined $\mathcal{O}$-convex sets in terms of their intersections with lines by analogy with one of the definitions of standard convex sets. Rawlins, Schuierer and Wood explored properties of $\mathcal{O}$-convex sets in two dimensions and demonstrated their similarity to standard convex sets [39, 41, 43].

Recall that convex sets can be described through their intersections with lines; specifically, a set is convex if its intersection with every line is connected. We define $\mathcal{O}$-convex sets through their intersections with lines in a given set rather than with all lines. To define such a restricted collection of lines, we first introduce the notion of an **orientation set** $\mathcal{O}$, which is a (possibly infinite) set of lines through some fixed point o; we give an example of a finite orientation set in Fig. 2.1a. A line that is a translate of an element of $\mathcal{O}$ is called an **$\mathcal{O}$-line;** for example, the dashed lines in Fig. 2.1b–e are $\mathcal{O}$-lines. We use the collection of all translates of all lines in a given $\mathcal{O}$ to define $\mathcal{O}$-convex sets.

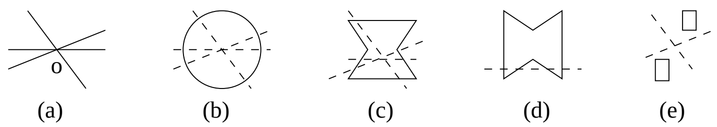

Fig. 2.1. Planar $\mathcal{O}$-convexity

Definition 2.1 ($\mathcal{O}$-convexity). *A set is* **$\mathcal{O}$-convex** *if its intersection with every $\mathcal{O}$-line is connected.*

For the orientation set in Fig. 2.1a, the objects in Fig. 2.1b,c are $\mathcal{O}$-convex; some $\mathcal{O}$-lines intersecting them are shown by dashed lines. On the other hand, the object in Fig. 2.1d is not $\mathcal{O}$-convex, since its intersection with the dashed $\mathcal{O}$-line is disconnected. Note that the object in Fig. 2.1d is a rotation of that in Fig. 2.1c, which shows that rotations may not preserve $\mathcal{O}$-convexity. Unlike standard convex sets, $\mathcal{O}$-convex sets may be disconnected; for example, the two rectangles in Fig. 2.1e form a disconnected $\mathcal{O}$-convex set. We now give some basic properties of planar $\mathcal{O}$-convex sets [41].

Lemma 2.1.

1. *Every translate of an $\mathcal{O}$-convex set is $\mathcal{O}$-convex.*
2. *If C is a collection of $\mathcal{O}$-convex sets, then the intersection $\bigcap C$ of these sets is also an $\mathcal{O}$-convex set.*
3. *Every standard convex set is $\mathcal{O}$-convex.*
4. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every $\mathcal{O}_2$-convex set is $\mathcal{O}_1$-convex.*
5. *A disconnected set is $\mathcal{O}$-convex if and only if every connected component of the set is $\mathcal{O}$-convex and no $\mathcal{O}$-line intersects two components.*
6. *If $\mathcal{O}$ is nonempty, then every connected $\mathcal{O}$-convex set is simply connected.*

Proof.

(1) By definition, every translate of an $\mathcal{O}$-line is an $\mathcal{O}$-line. Therefore, if the intersection of a set with every $\mathcal{O}$-line is connected, then the same holds for every translate of the set.

(2) If C is a collection of $\mathcal{O}$-convex sets, then, for every $\mathcal{O}$-line l, the intersection of each element of C with l is connected; hence, the intersection of $\bigcap C$ with l is also connected. We conclude that the intersection of $\bigcap C$ with every $\mathcal{O}$-line is connected, which implies that $\bigcap C$ is $\mathcal{O}$-convex.

(3) The intersection of a convex set with every line is connected. In particular, its intersection with every $\mathcal{O}$-line is connected, which implies that it is $\mathcal{O}$-convex.

(4) If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every $\mathcal{O}_1$-line is an $\mathcal{O}_2$-line. The intersection of an $\mathcal{O}_2$-convex set with every $\mathcal{O}_2$-line is connected, which implies that its intersection with every $\mathcal{O}_1$-line is connected; thus, it is $\mathcal{O}_1$-convex.

(5) If a set P is the union of disjoint $\mathcal{O}$-convex components and no $\mathcal{O}$-line intersects two components, then the intersection of P with every $\mathcal{O}$-line is connected; therefore, P is $\mathcal{O}$-convex. If one of P's components is not $\mathcal{O}$-convex, then the intersection of this component with some $\mathcal{O}$-line is disconnected. The intersection of P with this $\mathcal{O}$-line is also disconnected; hence, P is not $\mathcal{O}$-convex. Finally, if some $\mathcal{O}$-line intersects two or more components, then the intersection of P with this $\mathcal{O}$-line is disconnected; therefore, we again conclude that P is not $\mathcal{O}$-convex.

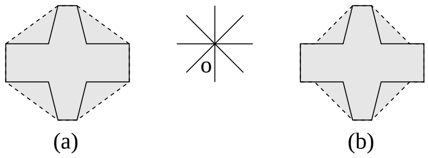

Fig. 2.2. Standard convex hull **(a)** and $\mathcal{O}$-hull **(b)**

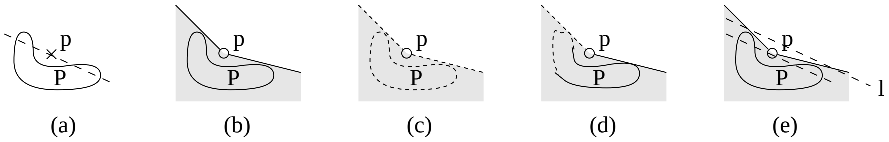

Fig. 2.3. Proof of Theorem 2.3

(6) If a set P is connected but not simply connected, then P has a hole, and there is an $\mathcal{O}$-line that cuts through the hole. The intersection of P with this $\mathcal{O}$-line is disconnected; thus, P is not $\mathcal{O}$-convex. $\square$

We now introduce the notion of an $\mathcal{O}$-hull. Recall that the standard convex hull of a geometric object is the intersection of all convex sets containing the object. Similarly, the **$\mathcal{O}$-hull** of an object is the intersection of all $\mathcal{O}$-convex sets that contain the object. We show a standard convex hull in Fig. 2.2a and an $\mathcal{O}$-hull in Fig. 2.2b. We list basic properties of $\mathcal{O}$-hulls, which immediately follow from the definition.

Lemma 2.2.

1. *The $\mathcal{O}$-hull of a geometric object contains the object.*
2. *A geometric object is $\mathcal{O}$-convex if and only if it is identical to its $\mathcal{O}$-hull.*
3. *The $\mathcal{O}$-hull of a geometric object is a subset of the standard convex hull of the object.*
4. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then the $\mathcal{O}_1$-hull of a geometric object is a subset of the $\mathcal{O}_2$-hull of the object.*

We now establish separation and decomposition properties of $\mathcal{O}$-hulls [40,41].

Theorem 2.3 (Separation). *Suppose that P is a connected set and p is a point outside of P. Then, $p \in \mathcal{O}\text{-hull}(P)$ if and only if there is an $\mathcal{O}$-line through p that intersects P on both sides of p.*

Proof. If an $\mathcal{O}$-line through p intersects P on both sides of p, as shown in Fig. 2.3a, then every $\mathcal{O}$-convex set that contains P also includes p, which implies that $p \in \mathcal{O}\text{-hull}(P)$.

To show the converse, suppose that $p \in \mathcal{O}\text{-hull}(P)$. We draw the two rays from p that support P, as shown in Fig. 2.3b, and consider the shaded angle, which does not include its vertex p. The exact definition of this angle depends on whether P is open or closed. If both rays intersect P, they belong to the angle. If the rays do not intersect P, as shown in Fig. 2.3c, the angle is an open set that does not include its sides. Finally, if only one of the two rays intersects P, as shown in Fig. 2.3d, the angle includes one of its sides.

In all cases, the angle contains P and does not include p. Since $p \in \mathcal{O}$-hull(P), the angle is not $\mathcal{O}$-convex, which means that its intersection with some $\mathcal{O}$-line l is disconnected, as shown in Fig. 2.3e. The parallel-to-l line through p is an $\mathcal{O}$-line that intersects P on both sides of p. $\square$

Theorem 2.4 (Decomposition). *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two orientation sets through the same point o, then, for every connected set P,*

$$(\mathcal{O}_1 \cup \mathcal{O}_2)\text{-hull}(P) = \mathcal{O}_1\text{-hull}(\mathcal{O}_2\text{-hull}(P)) = \mathcal{O}_1\text{-hull}(P) \cup \mathcal{O}_2\text{-hull}(P).$$

Proof. We readily conclude from Lemma 2.2 that

$$\mathcal{O}_1\text{-hull}(P) \cup \mathcal{O}_2\text{-hull}(P) \subseteq \mathcal{O}_1\text{-hull}(\mathcal{O}_2\text{-hull}(P)) \subseteq (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-hull}(P).$$

We now show that, if a point p is in $(\mathcal{O}_1 \cup \mathcal{O}_2)$-hull$(P)$, then it is also in $\mathcal{O}_1$-hull$(P) \cup \mathcal{O}_2$-hull(P). By Theorem 2.3, if p is in $(\mathcal{O}_1 \cup \mathcal{O}_2)$-hull$(P)$, then some $\mathcal{O}_1$-line or $\mathcal{O}_2$-line through p intersects P on both sides of p, which implies that p is in $\mathcal{O}_1$-hull(P) or in $\mathcal{O}_2$-hull(P). $\square$

2.2 $\mathcal{O}$-Halfplanes

Standard halfplanes can also be characterized through their intersections with lines; specifically, a closed set is a halfplane only if its intersection with every line is empty, a ray or a line. We use this observation to define an $\mathcal{O}$-convexity analog of halfplanes.

Definition 2.2 ($\mathcal{O}$-halfplanes). *An* **$\mathcal{O}$-halfplane** *is a closed set whose intersection with every $\mathcal{O}$-line is empty, a ray or a line.*

Note that the *empty set* and the *whole plane* are considered $\mathcal{O}$-halfplanes, which simplifies some definitions and results. For example, the objects in Fig. 2.4b–f are $\mathcal{O}$-halfplanes for the orientation set in Fig. 2.4a; note that the $\mathcal{O}$-halfplane in Fig. 2.4f is disconnected. As another example, the objects in Fig. 2.4h,i are $\mathcal{O}$-halfplanes for the orientation set in Fig. 2.4g.

This notion of $\mathcal{O}$-halfplanes is different from the $\mathcal{O}$-convexity analogs of halfplanes in the work of Rawlins, who defined an **$\mathcal{O}$-stairhalfplane** as a region of the plane bounded by an $\mathcal{O}$-convex curve [35]. $\mathcal{O}$-stairhalfplanes are a proper subclass of $\mathcal{O}$-halfplanes as follows from Lemma 5.8 (page 60).

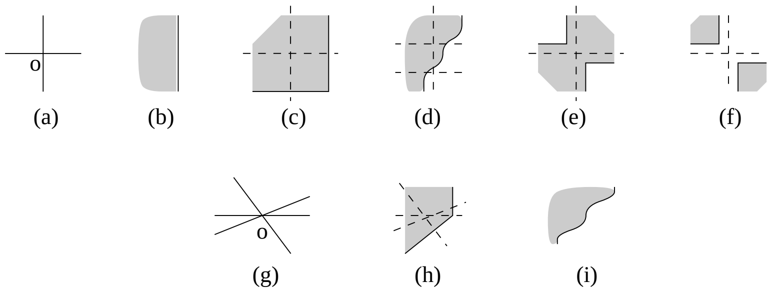

Fig. 2.4. $\mathcal{O}$-halfplanes

Lemma 2.5.

1. *Every translate of an $\mathcal{O}$-halfplane is an $\mathcal{O}$-halfplane.*
2. *Every standard closed halfplane is an $\mathcal{O}$-halfplane.*
3. *Every $\mathcal{O}$-halfplane is $\mathcal{O}$-convex.*
4. *A disconnected set is an $\mathcal{O}$-halfplane if and only if each of its connected components is an $\mathcal{O}$-halfplane and no $\mathcal{O}$-line intersects two components.*

Proof.

(1) If the intersection of a set with every $\mathcal{O}$-line is empty, a ray or a line, then the same holds for every translate of the set.

(2) The intersection of a standard halfplane with every line is empty, a ray or a line; hence, it is an $\mathcal{O}$-halfplane.

(3) The intersection of an $\mathcal{O}$-halfplane with every $\mathcal{O}$-line is connected; therefore, every $\mathcal{O}$-halfplane is $\mathcal{O}$-convex.

(4) If P is the union of disjoint $\mathcal{O}$-halfplanes and no $\mathcal{O}$-line intersects two of them, then the intersection of P with every $\mathcal{O}$-line is empty, a ray or a line; hence, P is an $\mathcal{O}$-halfplane. If one of P's components is not an $\mathcal{O}$-halfplane, the intersection of this component with some $\mathcal{O}$-line is not empty, not a ray and not a line. Then, the intersection of P with this line is not empty, not a ray and not a line; thus, P is not an $\mathcal{O}$-halfplane. Finally, if some $\mathcal{O}$-line intersects two components, then its intersection with P is disconnected, which implies that P is not an $\mathcal{O}$-halfplane. $\square$

We characterize closed $\mathcal{O}$-convex sets in terms of the intersections of $\mathcal{O}$-halfplanes [41].

Lemma 2.6. *A closed connected set is $\mathcal{O}$-convex if and only if it is the intersection of $\mathcal{O}$-halfplanes.*

Proof. Suppose that a set P is the intersection of $\mathcal{O}$-halfplanes. Since every $\mathcal{O}$-halfplane is $\mathcal{O}$-convex by Lemma 2.5, their intersection P is also $\mathcal{O}$-convex by Lemma 2.1.

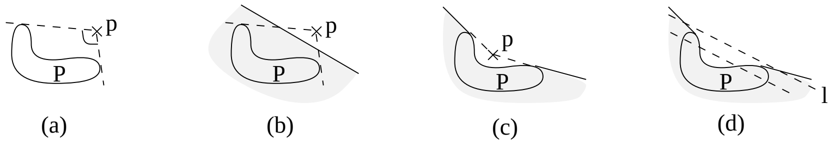

Fig. 2.5. Proof of Lemma 2.6

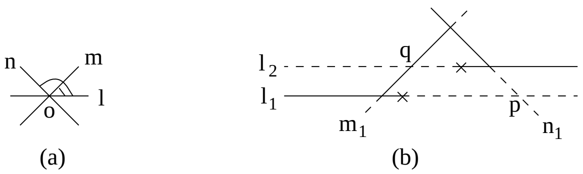

Fig. 2.6. Proof of Lemma 2.7

Now suppose, conversely, that P is $\mathcal{O}$-convex. We show that P is the intersection of $\mathcal{O}$-halfplanes by demonstrating that, for every point p outside of P, some $\mathcal{O}$-halfplane contains P and does not contain p.

We draw the two lines through p that support P, as shown in Fig. 2.5a. If the marked angle between these lines is less than π, there is a standard halfplane that contains P and does not contain p, as shown in Fig. 2.5b.

If the marked angle is at least π, we consider the set Q shown by shading in Fig. 2.5c. The boundary of Q consists of the segment of P's boundary between the supporting lines and the parts of the supporting lines that extend this segment. We show that Q is an $\mathcal{O}$-halfplane.

If the intersection of Q with some $\mathcal{O}$-line l is disconnected, then there is an $\mathcal{O}$-line parallel to l whose intersection with P is disconnected, as shown in Fig. 2.5d, contradicting the assumption that P is $\mathcal{O}$-convex. Furthermore, there is no line whose intersection with Q is a point or segment. Therefore, the intersection of Q with every $\mathcal{O}$-line is empty, a ray or a line. $\square$

If $\mathcal{O}$ contains at least *three distinct lines,* then $\mathcal{O}$-halfplanes have additional basic properties. To derive these properties, we use the notion of the **direction** of a ray; specifically, two rays have the same direction if they are translates of each other.

Lemma 2.7. *Suppose that the orientation set $\mathcal{O}$ contains at least three distinct lines. If the intersection of an $\mathcal{O}$-halfplane with two parallel $\mathcal{O}$-lines forms two rays, these rays have the same direction, rather than opposite directions.*

Proof. Suppose that the intersection of an $\mathcal{O}$-halfplane P with parallel $\mathcal{O}$-lines l_1 and l_2 gives rays of opposite directions; we show these two rays by solid lines in Fig. 2.6b. For convenience, we assume that l_1 is below l_2 and the lower ray's direction is to the left.

(a) (b)

Fig. 2.7. Directed $\mathcal{O}$-halfplane **(a)** and nondirected $\mathcal{O}$-halfplane **(b)**

Let l be the element of $\mathcal{O}$ parallel to l_1 and l_2, and let m and n be two other elements of $\mathcal{O}$, as shown in Fig. 2.6a. We assume that the marked angle between l and m is smaller than the marked angle between l and n.

We choose a point $p \in l_1$ and draw a line n_1 through p parallel to n. We select this point p in such a way that p is not in P and n_1 intersects the upper ray, as shown in Fig. 2.6b. Since n_1 is an $\mathcal{O}$-line, its intersection with P must be empty, a ray or a line; hence, the part of n_1 above l_2 (shown by a solid line) is in P.

We next choose a point $q \in l_2$ and draw a line m_1 through q parallel to m. We pick q is such a way that q is not in P and m_1 intersects the lower ray. Note that m_1 intersects n_1 *above* l_2, which implies that m_1 intersects the part of n_1 contained in P. Since m_1 is an $\mathcal{O}$-line, its intersection with P is connected; therefore, the segment of m_1 between l_1 and n_1 is in P, contradicting the assumption that q is not in P. $\square$

We illustrate the directed-ray property of $\mathcal{O}$-halfplanes in Fig. 2.7a, where the intersection of an $\mathcal{O}$-halfplane with several parallel $\mathcal{O}$-lines is shown by dashed rays. The $\mathcal{O}$-halfplanes that satisfy this property are called **directed $\mathcal{O}$-halfplanes.** If $\mathcal{O}$ contains two lines, an $\mathcal{O}$-halfplane may not be directed; for example, the $\mathcal{O}$-halfplane in Fig. 2.7b is not directed, since the dashed $\mathcal{O}$-rays have opposite directions.

Lemma 2.8. *Suppose $\mathcal{O}$ contains at least two distinct lines. Then:*

1. *Every $\mathcal{O}$-halfplane is either connected or consists of two components.*
2. *Every directed $\mathcal{O}$-halfplane is connected.*
3. *The boundary of every directed $\mathcal{O}$-halfplane is connected and $\mathcal{O}$-convex.*

Proof.

(1) We prove that every $\mathcal{O}$-halfplane P has at most two components by showing that, for every three points $p, q, a \in P$, two of them are in the same components.

Let l and m be two elements of $\mathcal{O}$, and suppose for convenience that l is horizontal, as shown in Fig. 2.8a. Since P is an $\mathcal{O}$-halfplane, one of the two horizontal rays with endpoint p is contained in P; we show this ray in Fig. 2.8b. Similarly, we can choose a horizontal ray with endpoint q and a horizontal ray with endpoint a contained in P.

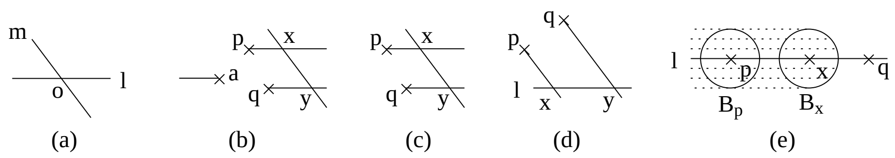

Fig. 2.8. Proof of Lemma 2.8

We select two of these three rays that have the same direction; without loss of generality, assume that the endpoints of the selected rays are p and q. We choose a parallel-to-m line that intersects these two rays, and denote the respective intersection points by x and y, as shown in Fig. 2.8b. The polygonal line (p, x, y, q) is wholly in P, which implies that p and q are in the same connected component.

(2) We show that every two points p and q of a directed $\mathcal{O}$-halfplane P can be connected by a polygonal line in P. We pick two parallel $\mathcal{O}$-rays, with endpoints p and q, that are contained in P and have the same direction, and consider an $\mathcal{O}$-line that intersects these rays. We illustrate this construction in Fig. 2.8c, where the respective intersection points are denoted by x and y. The polygonal line (p, x, y, q) is a path from p to q within P.

(3) Suppose that the boundary of a directed $\mathcal{O}$-halfplane P is not connected. Since P is connected, the complement of P is disconnected and we can choose points p and q in different connected components of P's complement. Next, we pick two parallel $\mathcal{O}$-rays, with endpoints p and q, that do not intersect P and have the same direction, as shown in Fig. 2.8d. Finally, we select an $\mathcal{O}$-line l that intersects these two rays, and denote the respective intersection points by x and y. The segment of l between x and y does not intersect P, because, if some point z of this segment were in P, then one of the two contained-in-l rays with endpoint z would be in P, contradicting the assumption that x and y are not in P. Therefore, the polygonal line (p, x, y, q) is wholly in P's complement, contradicting the assumption that p and q are in different components of P's complement.

Now suppose that the boundary of P is not $\mathcal{O}$-convex. Then, the intersection of some $\mathcal{O}$-line l with P's boundary is disconnected, and we can select points $p, q \in l$ that are in the boundary and a point $x \in l$ between them that is not in the boundary; we assume that p is to the left of x, as shown in Fig. 2.8e. Since the intersection of P with l is connected, x is in the interior of P, and we can choose a circle $B_x \subseteq P$ centered at x. Either all left-directed or all right-directed rays with endpoints in B_x are contained in P; we assume that the left-directed rays are in P. Then, some circle B_p centered at p is wholly in P; therefore, p is in P's interior, which yields a contradiction. □

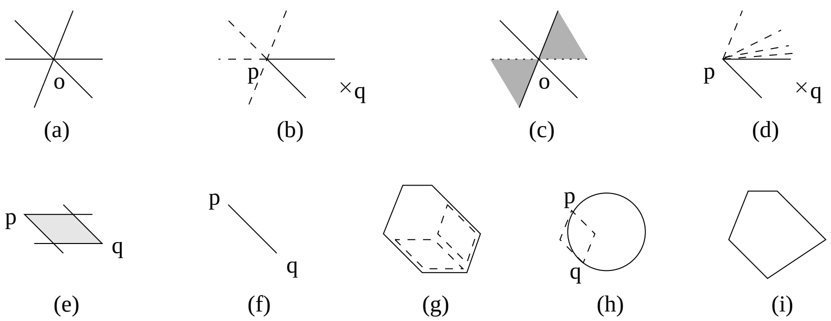

Fig. 2.9. Planar strong $\mathcal{O}$-convexity

2.3 Strongly $\mathcal{O}$-Convex Sets

We now consider an alternative generalization of convexity, called **strong $\mathcal{O}$-convexity,** which also stems from the notion of an orientation set. Rawlins introduced planar strong $\mathcal{O}$-convexity in his doctoral dissertation [35], as part of his research on restricted-orientation visibility. Rawlins and Wood studied the properties of strongly $\mathcal{O}$-convex sets in two dimensions [39,41], and demonstrated that strong $\mathcal{O}$-convexity generalizes not only standard convexity but also the notion of C-oriented polygons [16, 18].

The definition of strong $\mathcal{O}$-convexity is based on a characterization of convex sets in terms of visibility. Recall that a set is standard convex if and only if every two of its points are visible to each other. In other words, for every two points of a standard convex set, the line segment joining them is wholly in the set. We introduce a new type of visibility by replacing line segments with different objects, called **$\mathcal{O}$-blocks,** and define strong convexity in terms of this new visibility.

Definition 2.3 ($\mathcal{O}$-blocks). *If the orientation set $\mathcal{O}$ is nonempty, then the* **$\mathcal{O}$-block** *of two points is the intersection of all halfplanes, whose boundaries are $\mathcal{O}$-lines, that contain both points. If $\mathcal{O}$ is empty, then the $\mathcal{O}$-block of any two points is the entire plane.*

To construct the $\mathcal{O}$-block of two points p and q, we draw all $\mathcal{O}$-rays with endpoint p and choose the two that are closest to q, as illustrated in Fig. 2.9b. The two selected rays, with common endpoint p, form the boundary of an angle with vertex p that contains q.

If $\mathcal{O}$ is an infinite set, it may not be closed; thus, we may be unable to choose the ray closest to q. We give an example of a nonclosed orientation set in Fig. 2.9c; all lines in the shaded area are elements of this set, whereas the dotted horizontal line is not in the set. If $\mathcal{O}$ is not closed, we choose two rays with common endpoint p such that, for each of the two selected rays, (1) there is a sequence of $\mathcal{O}$-rays convergent to this ray and (2) there are no $\mathcal{O}$-rays with endpoint p between this ray and the point q, as shown in Fig. 2.9d. The two selected rays again form the boundary of an angle with vertex p.

Similarly, we draw the $\mathcal{O}$-rays from q closest to p and obtain the angle with vertex q whose boundary is formed by these rays. The $\mathcal{O}$-block of p and q is the intersection of the two angles, shown by the shaded parallelogram in Fig. 2.9e. In particular, if the line through p and q is an $\mathcal{O}$-line, then the $\mathcal{O}$-block of p and q is the line segment joining p and q, as shown in Fig. 2.9f.

Definition 2.4 (Strong $\mathcal{O}$-convexity). *A set is* **strongly $\mathcal{O}$-convex** *if, for every two of its points, their $\mathcal{O}$-block is contained in the set.*

We denote the orientation set in Fig. 2.9a by $\mathcal{O}_a$ and that in Fig. 2.9c by $\mathcal{O}_c$. The polygon in Fig. 2.9g is strongly $\mathcal{O}_a$-convex and strongly $\mathcal{O}_c$-convex; two $\mathcal{O}_a$-blocks contained in this polygon are shown by dashed lines. On the other hand, the circle in Fig. 2.9h is neither strongly $\mathcal{O}_a$-convex nor strongly $\mathcal{O}_c$-convex, because the dashed block is not in the circle. Finally, the polygon in Fig. 2.9i is strongly $\mathcal{O}_c$-convex, but not strongly $\mathcal{O}_a$-convex.

Lemma 2.9.

1. *Every translate of a strongly $\mathcal{O}$-convex set is strongly $\mathcal{O}$-convex.*
2. *If C is a collection of strongly $\mathcal{O}$-convex sets, the intersection $\bigcap C$ of these sets is also strongly $\mathcal{O}$-convex.*
3. *For every orientation set $\mathcal{O}$, each strongly $\mathcal{O}$-convex set is standard convex.*
4. *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every strongly $\mathcal{O}_1$-convex set is strongly $\mathcal{O}_2$-convex.*
5. *For two orientation sets $\mathcal{O}_1$ and $\mathcal{O}_2$ through the same point o, strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity if and only if the closure of $\mathcal{O}_1$ is identical to the closure of $\mathcal{O}_2$.*
6. *For a closed orientation set $\mathcal{O}$, a polygon is strongly $\mathcal{O}$-convex if and only if it is convex and its edges are parallel to elements of $\mathcal{O}$.*

Proof.

(1) Since translation preserves $\mathcal{O}$-lines, it also preserves $\mathcal{O}$-blocks, which implies that translates of strongly $\mathcal{O}$-convex sets are strongly $\mathcal{O}$-convex.

(2) If C is a collection of strongly $\mathcal{O}$-convex sets, then, for every two points of the intersection $\bigcap C$, their $\mathcal{O}$-block is a subset of every element of C; hence, this $\mathcal{O}$-block is contained in $\bigcap C$.

(3) For every two points, the line segment joining them is contained in their $\mathcal{O}$-block. Therefore, for every two points of a strongly $\mathcal{O}$-convex set, the segment joining them is wholly in the set.

(4) Suppose that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. The definition of $\mathcal{O}$-blocks readily implies that, for every two points, their $\mathcal{O}_2$-block is a subset of their $\mathcal{O}_1$-block. If P is strongly $\mathcal{O}_1$-convex, then, for every two points of P, their $\mathcal{O}_2$-block is in P, which means that P is strongly $\mathcal{O}_2$-convex.

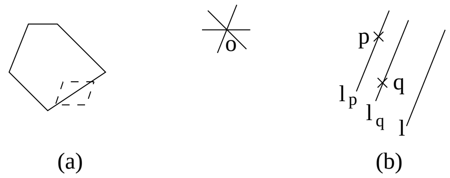

Fig. 2.10. Proof of Lemma 2.9

(5) Let $\mathcal{O}_{\mathrm{cl1}}$ be the closure of $\mathcal{O}_1$, and $\mathcal{O}_{\mathrm{cl2}}$ be the closure of $\mathcal{O}_2$. By definition, the notions of $\mathcal{O}_1$-blocks and $\mathcal{O}_{\mathrm{cl1}}$-blocks are equivalent, which implies that strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_{\mathrm{cl1}}$-convexity. Similarly, strong $\mathcal{O}_2$-convexity is identical to strong $\mathcal{O}_{\mathrm{cl2}}$-convexity. If $\mathcal{O}_{\mathrm{cl1}} = \mathcal{O}_{\mathrm{cl2}}$, then strong $\mathcal{O}_1$-convexity is equivalent to strong $\mathcal{O}_2$-convexity. Suppose, conversely, that $\mathcal{O}_{\mathrm{cl1}} \neq \mathcal{O}_{\mathrm{cl2}}$; without loss of generality, we assume that $\mathcal{O}_{\mathrm{cl1}}$ is not a subset of $\mathcal{O}_{\mathrm{cl2}}$. We consider two distinct points such that the line through them is an $\mathcal{O}_{\mathrm{cl1}}$-line and not an $\mathcal{O}_{\mathrm{cl2}}$-line. Then, the segment joining these two points is strongly $\mathcal{O}_1$-convex but not strongly $\mathcal{O}_2$-convex.

(6) If a polygon P is not convex, it is not strongly $\mathcal{O}$-convex by Part 3 of the proof. If some edge of P is not parallel to any element of $\mathcal{O}$, then, for any two distinct points of this edge, their $\mathcal{O}$-block of is not in P, as shown in Fig. 2.10a; hence, we again conclude that P is not strongly $\mathcal{O}$-convex.

Now suppose that P is a convex polygon and all its edges are parallel to elements of $\mathcal{O}$. Then, P is the intersection of several halfplanes whose boundaries are $\mathcal{O}$-lines. To prove that P is strongly $\mathcal{O}$-convex, we demonstrate that each of these halfplanes is strongly $\mathcal{O}$-convex. Specifically, we show that, for every halfplane whose boundary l is an $\mathcal{O}$-line, and every two points p and q of this halfplane, the $\mathcal{O}$-block of p and q is in the halfplane.

Let l_p be the line through p parallel to l, and l_q be the line through q parallel to l, as shown in Fig. 2.10b. Since l_p and l_q are $\mathcal{O}$-lines, the $\mathcal{O}$-block of p and q is contained in the "strip" between l_p and l_q; hence, this $\mathcal{O}$-block is in the halfplane.

We conclude that P is the intersection of several strongly $\mathcal{O}$-convex halfplanes; therefore, P is strongly $\mathcal{O}$-convex by Part 2 of the proof. $\square$

Finally, we introduce the notion of the **strong $\mathcal{O}$-hull** of a geometric object, which is the intersection of all strongly $\mathcal{O}$-convex sets containing the object; in Fig. 2.11, we show a standard convex hull and a strong $\mathcal{O}$-hull.

Summary

We have introduced two notions of generalized convexity, called $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity, and presented their basic properties. The important properties of $\mathcal{O}$-convex sets include the separation and decomposition

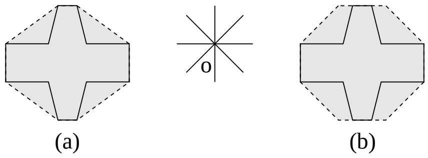

Fig. 2.11. Standard convex hull **(a)** and strong $\mathcal{O}$-hull **(b)**

results (Theorems 2.3 and 2.4), and the characterization of $\mathcal{O}$-convex sets in terms of $\mathcal{O}$-halfplane intersections (Lemma 2.6). The main result for strong $\mathcal{O}$-convexity is the comparison of convexities induced by different orientation sets (Lemma 2.9).