15-889e
Policy Search: Gradient Methods
Emma Brunskill

All slides from David Silver (with EB adding minor modifications), unless otherwise noted

Outline

1 Introduction

2 Finite Difference Policy Gradient

3 Monte-Carlo Policy Gradient

4 Actor-Critic Policy Gradient

Policy-Based Reinforcement Learning

- In past lectures we approximated the value or action-value function using parameters θ,

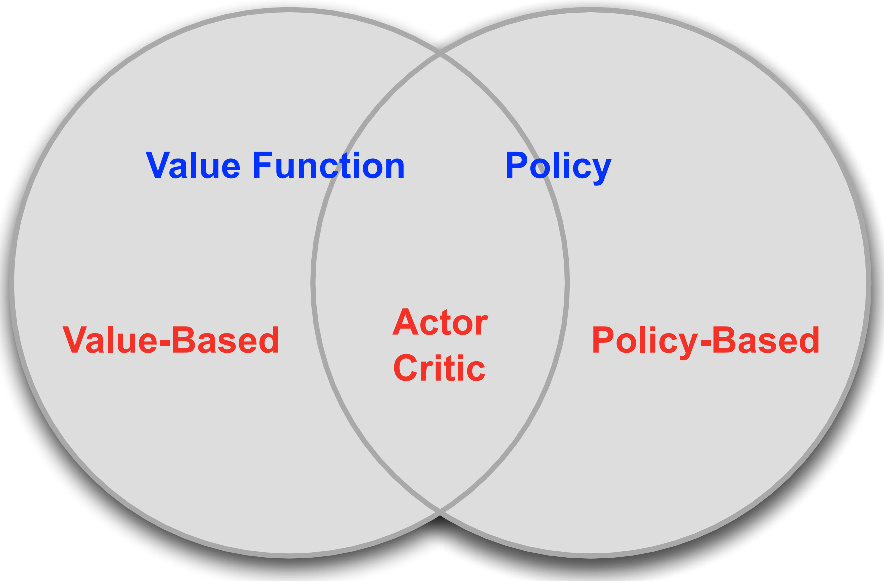
$$V_\theta(s) \approx V^\pi(s)$$
$$Q_\theta(s, a) \approx Q^\pi(s, a)$$

- A policy was generated directly from the value function
  - e.g. using epsilon-greedy
- In this lecture we will directly parametrise the policy

$$\pi_\theta(s, a) = \mathbb{P}[a \mid s, \theta]$$

Value-Based and Policy-Based RL

- Value Based
  - Learnt Value Function
  - Implicit policy
    (e.g. epsilon-greedy)
- Policy Based
  - No Value Function
  - Learn Policy
- Actor-Critic
  - Learn Value Function
  - Learn Policy



Advantages of Policy-Based RL

Advantages:

- Better convergence properties
- Effective in high-dimensional or continuous action spaces
- Value/Q function may be much more complicated to represent than optimal policy
  - (Q(s,up)=0.9872,Q(s,down)=.5894. action: go up!
- Can learn stochastic policies
  - When is this important? When is this not important?

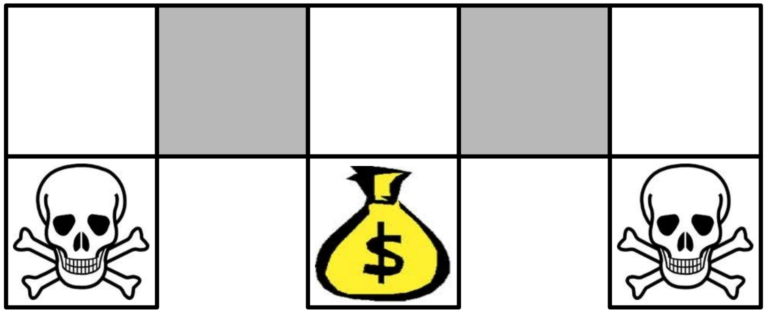
Disadvantages:

- Typically converge to a local rather than global optimum
- Evaluating a policy is typically inefficient and high variance

Example: Rock-Paper-Scissors

- Two-player game of rock-paper-scissors
  - Scissors beats paper
  - Rock beats scissors
  - Paper beats rock
- Consider policies for iterated rock-paper-scissors
  - A deterministic policy is easily exploited
  - A uniform random policy is optimal (i.e. Nash equilibrium)

Example: Aliased Gridworld (1)



- The agent cannot differentiate the grey states
- Consider features of the following form (for all N, E, S, W)

$$\phi(s, a) = \mathbf{1}(\text{wall to N}, a = \text{move E})$$

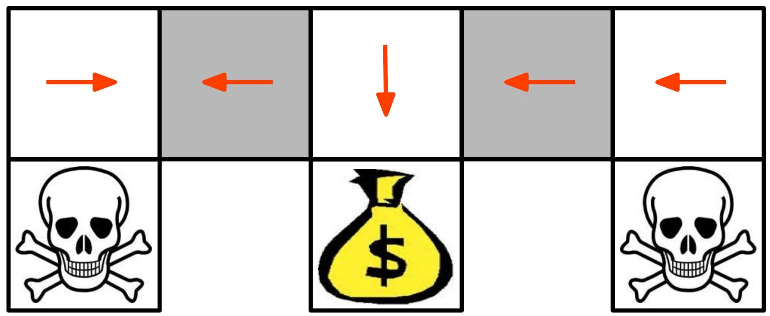
- Compare value-based RL, using an approximate value function

$$Q_\theta(s, a) = f(\phi(s, a), \theta)$$

- To policy-based RL, using a parametrised policy

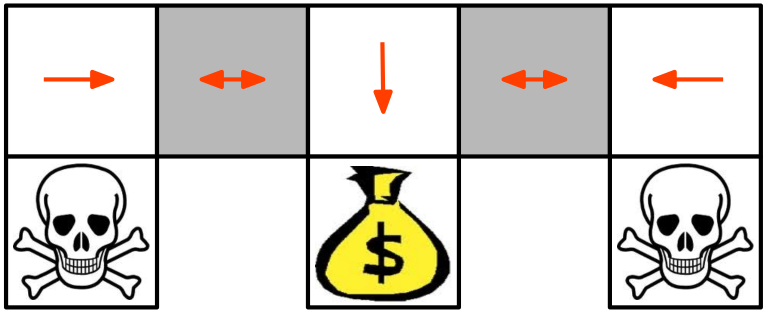
$$\pi_\theta(s, a) = g(\phi(s, a), \theta)$$

Example: Aliased Gridworld (2)



- Under aliasing, an optimal deterministic policy will either
  - move W in both grey states (shown by red arrows)
  - move E in both grey states
- Either way, it can get stuck and never reach the money
- Value-based RL learns a near-deterministic policy
  - e.g. greedy or -greedy
- So it will traverse the corridor for a long time

Example: Aliased Gridworld (3)



- An optimal stochastic policy will randomly move E or W in grey states

$$\pi_\theta(\text{wall to N and S, move E}) = 0.5$$
$$\pi_\theta(\text{wall to N and S, move W}) = 0.5$$

- It will reach the goal state in a few steps with high probability
- Policy-based RL can learn the optimal stochastic policy

Policy Objective Functions

- Goal: given policy $\pi_\theta(s, a)$ with parameters θ, find best θ
- But how do we measure the quality of a policy π_θ?
- In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$$

- In continuing environments we can use the average value

$$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

- Or the average reward per time-step

$$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a)\mathcal{R}_s^a$$

- where $d^{\pi_\theta}(s)$ is stationary distribution of Markov chain for π_θ

Policy Optimisation

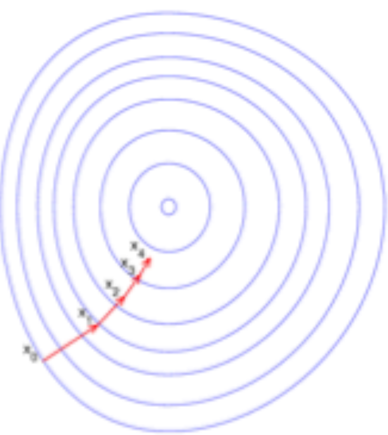
- Policy based reinforcement learning is an optimization problem
- Find θ that maximises $J(\theta)$
- Some approaches do not use gradient
  - Hill climbing
  - Simplex / amoeba / Nelder Mead
  - Genetic algorithms
- Greater efficiency often possible using gradient
  - Gradient descent
  - Conjugate gradient
  - Quasi-newton
- We focus on gradient descent, many extensions possible
- And on methods that exploit sequential structure

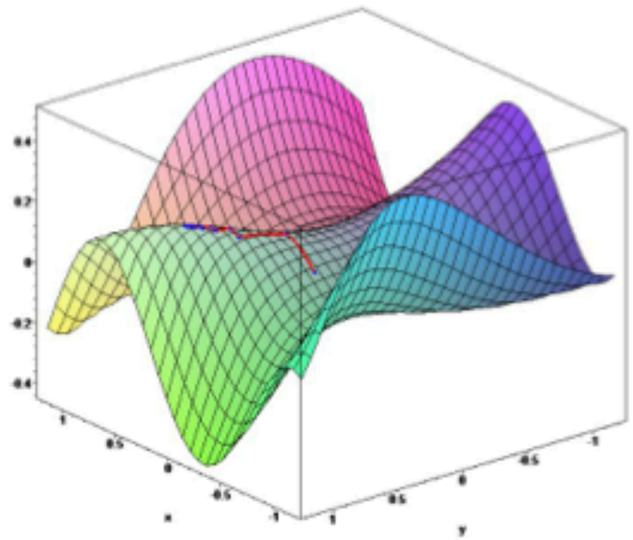
Method	Order optimizer	Stochasticity assumption	Global optimizer	Re-usability evaluations
Grid Search	Zero-order	No*	Global	Limited
Pure Random Search	Zero-order	No*	Global	Yes
Gradient-descent family	First-order	No*	Local	No
Bayesian Optimization	Zero-order	Yes	Global	Yes
Evolutionary Algorithms	Zero-order	No*	Global	No
Particle Swarm	Zero-order	No*	Global	No

Table 1: Optimization methods in robotics: Properties of various optimization methods commonly used for optimization in robotics. As discussed in Section 2.1, the ideal optimizer for robotic applications should be global, zero-order, and assuming stochasticity.
(*) Extensions exist for the stochastic case, but they increase the number of experiments required.

Table from Calandra, Seyfarth, Peters & Deisenroth, 2015

Policy Gradient





- Let $J(\theta)$ be any policy objective function
- Policy gradient algorithms search for a *local* maximum in $J(\theta)$ by ascending the gradient of the policy, w.r.t. parameters θ

$$\Delta\theta = \alpha \nabla_\theta J(\theta)$$

- Where $\nabla_\theta J(\theta)$ is the policy gradient

$$\nabla_\theta J(\theta) = \begin{pmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial J(\theta)}{\partial \theta_n} \end{pmatrix}$$

- and α is a step-size parameter

Computing Gradients By Finite Differences

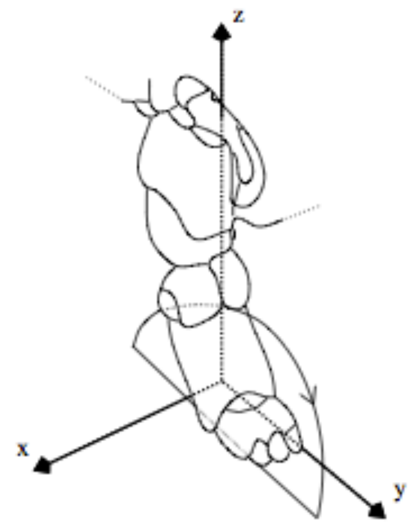
- To evaluate policy gradient of $\pi_\theta(s, a)$
- For each dimension $k \in [1, n]$
  - Estimate kth partial derivative of objective function w.r.t. θ
  - By perturbing θ by small amount ϵ in kth dimension

$$\frac{\partial J(\theta)}{\partial \theta_k} \approx \frac{J(\theta + \epsilon u_k) - J(\theta)}{\epsilon}$$

where u_k is unit vector with 1 in kth component, 0 elsewhere

- Uses n evaluations to compute policy gradient in n dimensions
  → **Scales linearly with number of parameters in policy!**
- Simple, noisy, inefficient - but sometimes effective
- Works for arbitrary policies, even if policy is not differentiable

Training AIBO to Walk by Finite Difference Policy Gradient



- Goal: learn a fast AIBO walk (useful for Robocup)
- AIBO walk policy is controlled by 12 numbers (elliptical loci)
- Adapt these parameters by finite difference policy gradient
- Evaluate performance of policy by field traversal time

For more details, see paper: Kohl and Stone, ICRA 2004

AIBO Walk Policies

- Before training
- During training
- After training

Videos at: http://www.cs.utexas.edu/users/AustinVilla/?p=research/learned_walk

Score Function

- We now compute the policy gradient directly
- Assume policy π_θ is differentiable whenever it is non-zero
- and we can compute the gradient $\nabla_\theta \pi_\theta(s, a)$
- Likelihood ratios exploit the following identity

$$\begin{aligned}\nabla_\theta \pi_\theta(s,a) &= \pi_\theta(s,a)\frac{\nabla_\theta \pi_\theta(s,a)}{\pi_\theta(s,a)} \\ &= \pi_\theta(s,a)\nabla_\theta \log \pi_\theta(s,a)\end{aligned}$$

- The score function is $\nabla_\theta \log \pi_\theta(s, a)$

Softmax Policy

- We will use a softmax policy as a running example
- Weight actions using linear combination of features $\phi(s, a)^{\top}\theta$
- Probability of action is proportional to exponentiated weight

$$\pi_\theta(s, a) \propto e^{\phi(s,a)^T \theta}$$

- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \phi(s, a) - \mathbb{E}_{\pi_\theta}[\phi(s, \cdot)]$$

Gaussian Policy

- In continuous action spaces, a Gaussian policy is natural
- Mean is a linear combination of state features $\mu(s) = \phi(s)^{\top}\theta$
- Variance may be fixed σ^2, or can also parametrised
- Policy is Gaussian, $a \sim \mathcal{N}(\mu(s), \sigma^2)$
- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \frac{(a - \mu(s))\phi(s)}{\sigma^2}$$

One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state $s \sim d(s)$
  - Terminating after one time-step with reward $r = \mathcal{R}_{s,a}$
- Use likelihood ratios to compute the policy gradient

$$
\begin{aligned}
J(\theta) &= \mathbb{E}_{\pi_\theta}[r] \\
&= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \mathcal{R}_{s,a} \\
\nabla_\theta J(\theta) &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) \mathcal{R}_{s,a} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) r]
\end{aligned}
$$

Policy Gradient Theorem

- The policy gradient theorem generalises the likelihood ratio approach to multi-step MDPs
- Replaces instantaneous reward r with long-term value $Q^{\pi}(s, a)$
- Policy gradient theorem applies to start state objective, average reward and average value objective

Theorem

For any differentiable policy $\pi_\theta(s, a)$,

for any of the policy objective functions $J = J_1, J_{avR}, \text{ or } \frac{1}{1-\gamma} J_{avV}$, the policy gradient is

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ Q^{\pi_\theta}(s, a)\right]$$

See board derivation.

Reference: https://inst.eecs.berkeley.edu/~cs294-40/fa08/scribes/lecture16.pdf

Benefit of Likelihood Ratio Approach:

- Number of samples need to approximate no longer depends on the # of policy parameters
- Gradient calculation is independent of the underlying system dynamics (ratio cancels): don't need to know dynamics!

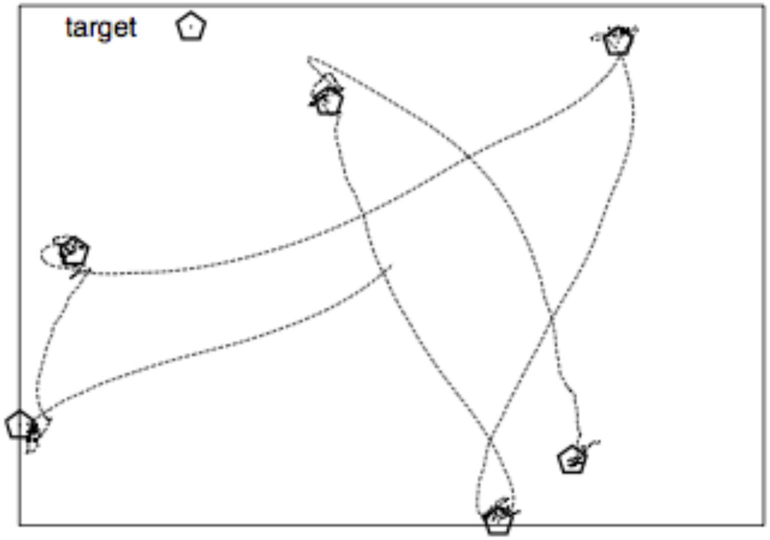
Monte-Carlo Policy Gradient (REINFORCE)

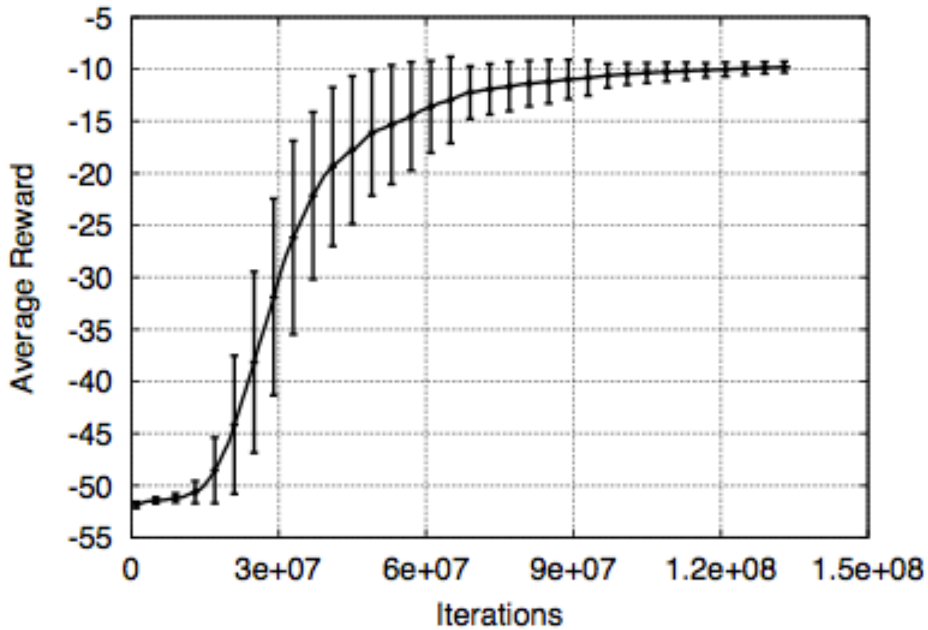
- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return v_t as an unbiased sample of $Q^{\pi_\theta}(s_t, a_t)$

$$\Delta\theta_t = \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

function REINFORCE
  Initialise θ arbitrarily
  for each episode $\{s_1, a_1, r_2, ..., s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ do
    for $t = 1$ to $T - 1$ do
      $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$
    end for
  end for
  return θ
end function

Puck World Example





- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of Monte-Carlo policy gradient

Reducing Variance Using a Critic

- Monte-Carlo policy gradient still has high variance
- We use a critic to estimate the action-value function,

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$

- Actor-critic algorithms maintain two sets of parameters

  Critic Updates action-value function parameters w

  Actor Updates policy parameters θ, in direction suggested by critic

- Actor-critic algorithms follow an approximate policy gradient

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\; Q_w(s, a)\right]$$

$$\Delta\theta = \alpha \nabla_\theta \log \pi_\theta(s, a)\; Q_w(s, a)$$

Estimating the Action-Value Function

- The critic is solving a familiar problem: policy evaluation
- How good is policy π_θ for current parameters θ?
- Policy evaluation problem. See earlier lectures.
  - Monte-Carlo policy evaluation
  - Temporal-Difference learning
- Could also use e.g. least-squares policy evaluation

Action-Value Actor-Critic

- Simple actor-critic algorithm based on action-value critic
- Using linear value fn approx. $Q_w(s, a) = \phi(s, a)^\top w$
  - **Critic** Updates w by linear TD(0)
  - **Actor** Updates θ by policy gradient

function QAC
  Initialise s, θ
  Sample $a \sim \pi_\theta$
  for each step do
    Sample reward $r = \mathcal{R}^a_s$; sample transition $s' \sim \mathcal{P}^a_{s,\cdot}$
    Sample action $a' \sim \pi_\theta(s', a')$
    $\delta = r + \gamma Q_w(s', a') - Q_w(s, a)$
    $\theta = \theta + \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$
    $w \leftarrow w + \beta \delta \phi(s, a)$
    $a \leftarrow a', s \leftarrow s'$
  end for
end function

Bias in Actor-Critic Algorithms

- Approximating the policy gradient introduces bias
- A biased policy gradient may not find the right solution
  - e.g. if $Q_w(s, a)$ uses aliased features, can we solve gridworld example?
- Luckily, if we choose value function approximation carefully
- Then we can avoid introducing any bias
- i.e. We can still follow the exact policy gradient

Compatible Function Approximation

Theorem (Compatible Function Approximation Theorem)

If the following two conditions are satisfied:

1. *Value function approximator is compatible to the policy*

$$\nabla_w Q_w(s, a) = \nabla_\theta \log \pi_\theta(s, a)$$

2. *Value function parameters w minimise the mean-squared error*

$$\varepsilon = \mathbb{E}_{\pi_\theta}\left[\left(Q^{\pi_\theta}(s, a) - Q_w(s, a)\right)^2\right]$$

Then the policy gradient is exact,

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ Q_w(s, a)\right]$$

Proof of Compatible Function Approximation Theorem

If w is chosen to minimise mean-squared error, gradient of ε w.r.t. w must be zero,

$$
\begin{aligned}
\nabla_w \varepsilon &= 0 \\
\mathbb{E}_{\pi_\theta}\left[(Q^\theta(s,a) - Q_w(s,a))\nabla_w Q_w(s,a)\right] &= 0 \\
\mathbb{E}_{\pi_\theta}\left[(Q^\theta(s,a) - Q_w(s,a))\nabla_\theta \log \pi_\theta(s,a)\right] &= 0 \\
\mathbb{E}_{\pi_\theta}\left[Q^\theta(s,a)\nabla_\theta \log \pi_\theta(s,a)\right] &= \mathbb{E}_{\pi_\theta}\left[Q_w(s,a)\nabla_\theta \log \pi_\theta(s,a)\right]
\end{aligned}
$$

So $Q_w(s,a)$ can be substituted directly into the policy gradient,

$$
\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) Q_w(s,a)\right]
$$

Reducing Variance Using a Baseline

- We subtract a baseline function $B(s)$ from the policy gradient
- This can reduce variance, without changing expectation

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) B(s)\right] &= \sum_{s \in \mathcal{S}} d^{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s,a) B(s) \\
&= \sum_{s \in \mathcal{S}} d^{\pi_\theta} B(s) \nabla_\theta \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \\
&= 0
\end{aligned}$$

- A good baseline is the state value function $B(s) = V^{\pi_\theta}(s)$
- So we can rewrite the policy gradient using the advantage function $A^{\pi_\theta}(s,a)$

$$\begin{aligned}
A^{\pi_\theta}(s,a) &= Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s) \\
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ A^{\pi_\theta}(s,a)\right]
\end{aligned}$$

Estimating the Advantage Function (1)

- The advantage function can significantly reduce variance of policy gradient
- So the critic should really estimate the advantage function
- For example, by estimating both $V^{\pi_\theta}(s)$ and $Q^{\pi_\theta}(s, a)$
- Using two function approximators and two parameter vectors,

$$
\begin{aligned}
V_v(s) &\approx V^{\pi_\theta}(s) \\
Q_w(s, a) &\approx Q^{\pi_\theta}(s, a) \\
A(s, a) &= Q_w(s, a) - V_v(s)
\end{aligned}
$$

- And updating both value functions by e.g. TD learning

Estimating the Advantage Function (2)

- For the true value function $V^{\pi_\theta}(s)$, the TD error δ^{π_θ}

$$\delta^{\pi_\theta} = r + \gamma V^{\pi_\theta}(s') - V^{\pi_\theta}(s)$$

- is an unbiased estimate of the advantage function

$$\begin{aligned} \mathbb{E}_{\pi_\theta}[\delta^{\pi_\theta} | s, a] &= \mathbb{E}_{\pi_\theta}\left[r + \gamma V^{\pi_\theta}(s') | s, a\right] - V^{\pi_\theta}(s) \\ &= Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s) \\ &= A^{\pi_\theta}(s, a) \end{aligned}$$

- So we can use the TD error to compute the policy gradient

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s, a)\, \delta^{\pi_\theta}]$$

- In practice we can use an approximate TD error

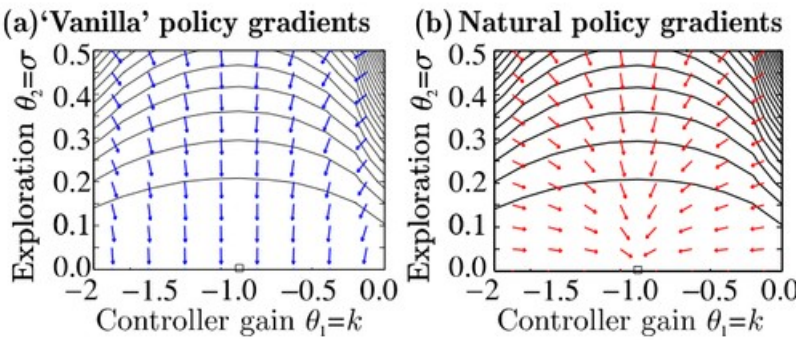
$$\delta_v = r + \gamma V_v(s') - V_v(s)$$

- This approach only requires one set of critic parameters v

Alternative Policy Gradient Directions

- Gradient ascent algorithms can follow any ascent direction
- A good ascent direction can significantly speed convergence
- Also, a policy can often be reparametrised without changing action probabilities
- For example, increasing score of all actions in a softmax policy
- The vanilla gradient is sensitive to these reparametrisations

Natural Policy Gradient



- The natural policy gradient is parametrisation independent
- It finds ascent direction that is closest to vanilla gradient, when changing policy by a small, fixed amount

$$\nabla_\theta^{nat} \pi_\theta(s,a) = G_\theta^{-1} \nabla_\theta \pi_\theta(s,a)$$

- where G_θ is the Fisher information matrix

$$G_\theta = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a)^T\right]$$

Natural Actor-Critic

- Using compatible function approximation,

$$\nabla_w A_w(s, a) = \nabla_\theta \log \pi_\theta(s, a)$$

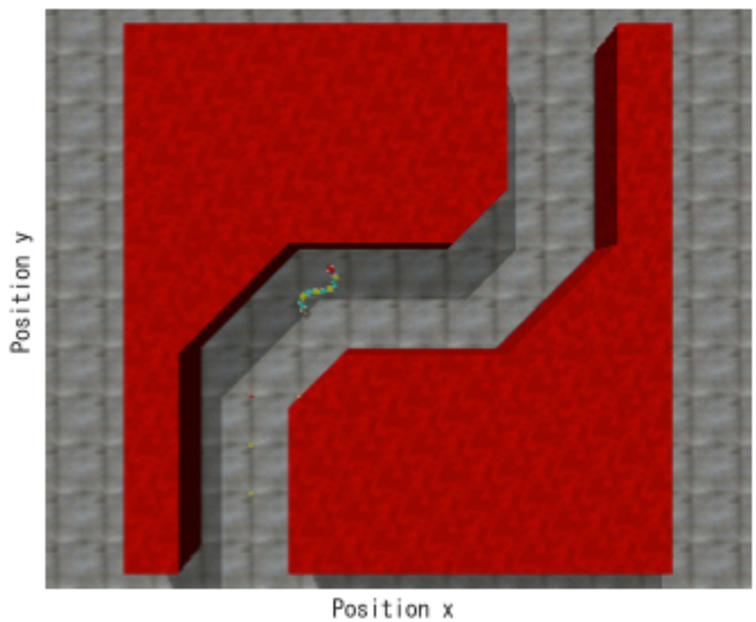
- So the natural policy gradient simplifies,

$$\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a) A^{\pi_\theta}(s, a)\right] \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)^T w\right] \\
&= G_\theta w
\end{aligned}$$

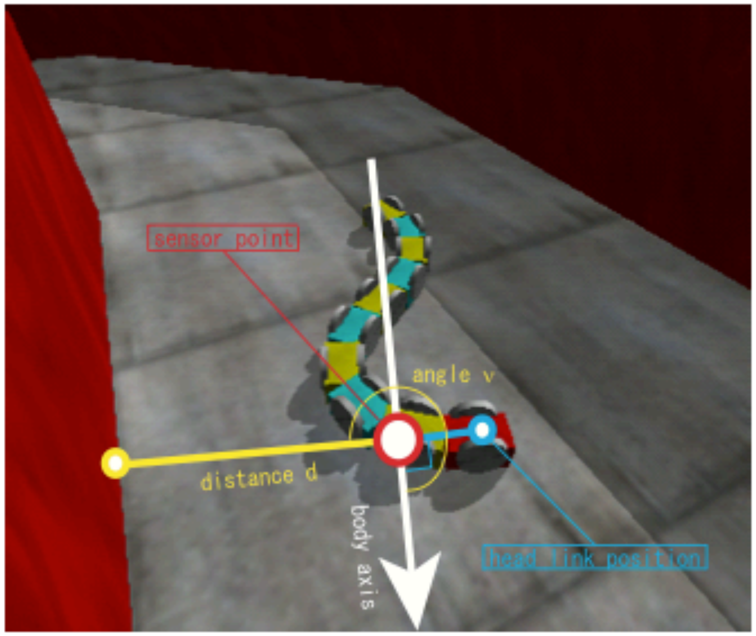
$$\nabla_\theta^{nat} J(\theta) = w$$

- i.e. update actor parameters in direction of critic parameters

Natural Actor Critic in Snake Domain

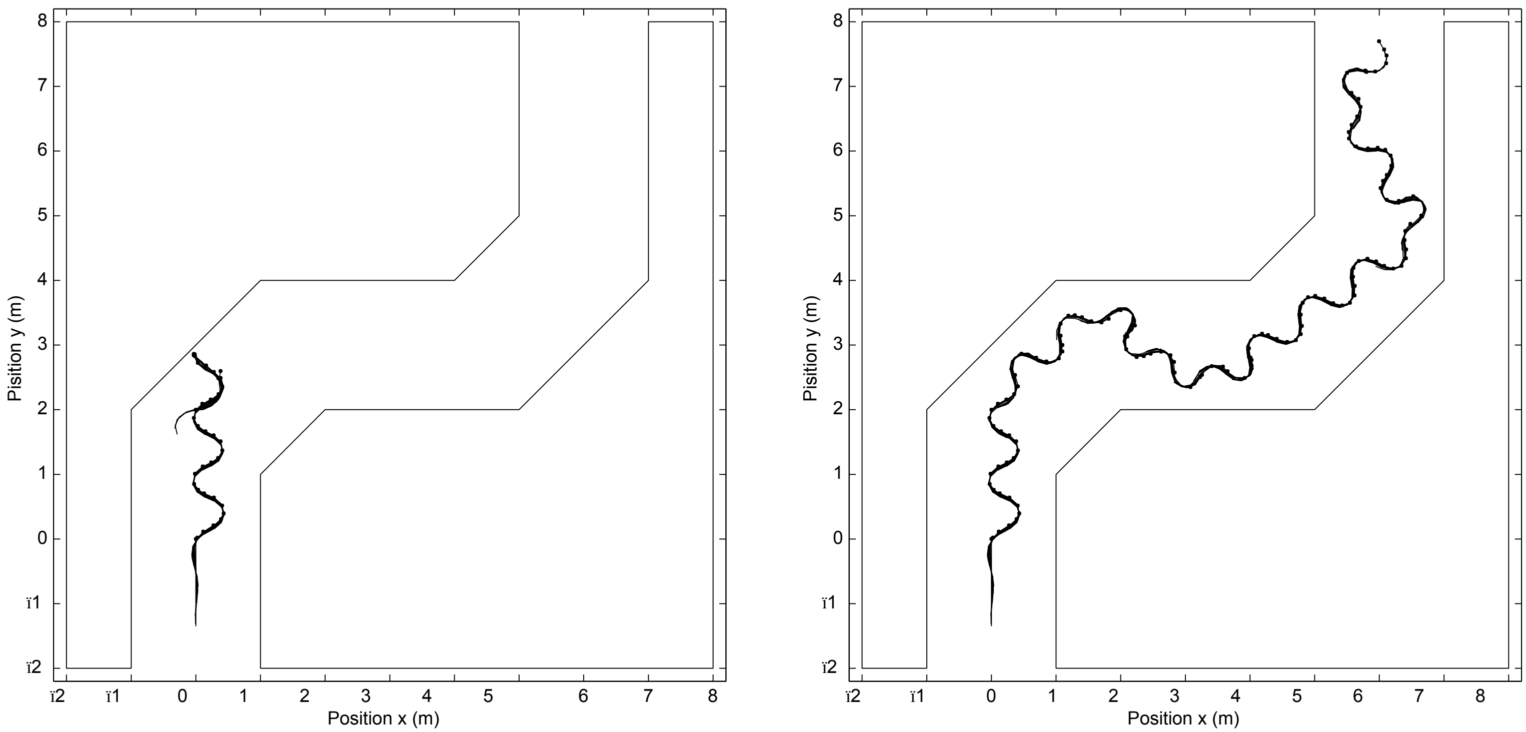


(a) Crank course



(b) Sensor setting

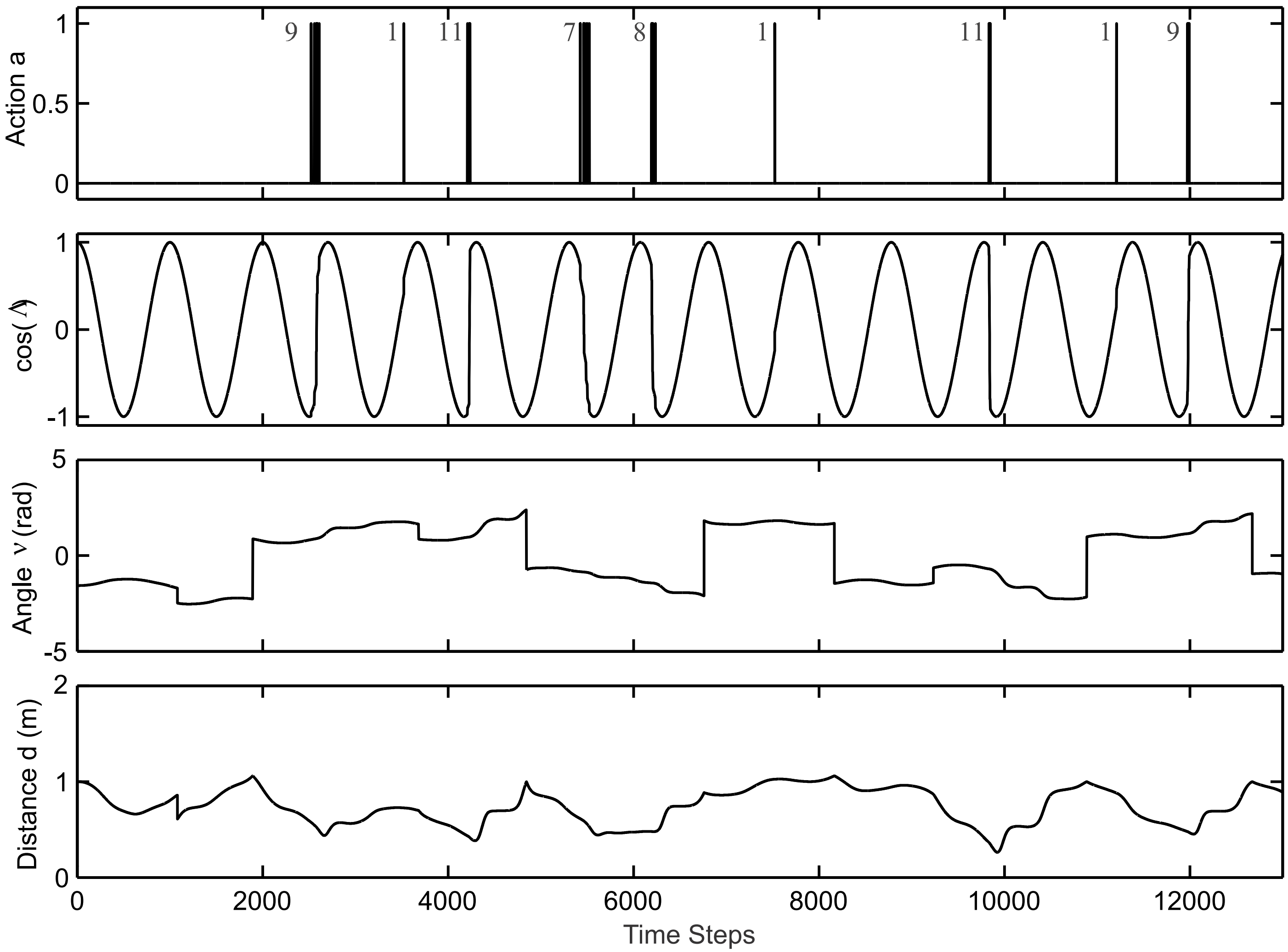
Natural Actor Critic in Snake Domain (2) θ^b



(a) Before learning (b) After learning

Figure 3: Behaviors of snake-like robot

Natural Actor Critic in Snake Domain (3)



Summary of Policy Gradient Algorithms

- The policy gradient has many equivalent forms

$$
\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; v_t\right] && \text{REINFORCE} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; Q^w(s,a)\right] && \text{Q Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; A^w(s,a)\right] && \text{Advantage Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; \delta\right] && \text{TD Actor-Critic} \\
G_\theta^{-1}\nabla_\theta J(\theta) &= w && \text{Natural Actor-Critic}
\end{aligned}
$$

- Each leads a stochastic gradient ascent algorithm
- Critic uses policy evaluation (e.g. MC or TD learning) to estimate $Q^\pi(s,a)$, $A^\pi(s,a)$ or $V^\pi(s)$

EB: Key Ideas

- Gradient approaches only guaranteed to find a local optima
- Finite-difference methods scale with # of parameters needed to represent the policy, but don't require differentiable policy
- Likelihood ratio gradient approaches
    - Cost independent of # params
    - Don't need to know dynamics model
    - Benefit from using a baseline to reduce variance
    - Natural gradient approaches are more robust
    - Be able to implement at least 1 gradient method which leverages info (from a critic / baseline)

Critics at Different Time-Scales

- Critic can estimate value function $V_\theta(s)$ from many targets at different time-scales From last lecture...
  - For MC, the target is the return v_t
  $$\Delta\theta = \alpha(v_t - V_\theta(s))\phi(s)$$
  - For TD(0), the target is the TD target $r + \gamma V(s')$
  $$\Delta\theta = \alpha(r + \gamma V(s') - V_\theta(s))\phi(s)$$
  - For forward-view TD(λ), the target is the λ-return v_t^λ
  $$\Delta\theta = \alpha(v_t^\lambda - V_\theta(s))\phi(s)$$
  - For backward-view TD(λ), we use eligibility traces
  $$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$
  $$e_t = \gamma\lambda e_{t-1} + \phi(s_t)$$
  $$\Delta\theta = \alpha\delta_t e_t$$

Actors at Different Time-Scales

- The policy gradient can also be estimated at many time-scales

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\, A^{\pi_\theta}(s, a)\right]$$

- Monte-Carlo policy gradient uses error from complete return

$$\Delta\theta = \alpha(v_t - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- Actor-critic policy gradient uses the one-step TD error

$$\Delta\theta = \alpha(r + \gamma V_v(s_{t+1}) - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

Policy Gradient with Eligibility Traces

- Just like forward-view TD(λ), we can mix over time-scales

$$\Delta\theta = \alpha(v_t^{\lambda} - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- where $v_t^{\lambda} - V_v(s_t)$ is a biased estimate of advantage fn
- Like backward-view TD(λ), we can also use eligibility traces
  - By equivalence with TD(λ), substituting $\phi(s) = \nabla_\theta \log \pi_\theta(s, a)$

$$\delta = r_{t+1} + \gamma V_v(s_{t+1}) - V_v(s_t)$$

$$e_{t+1} = \lambda e_t + \nabla_\theta \log \pi_\theta(s, a)$$

$$\Delta\theta = \alpha\delta e_t$$

- This update can be applied online, to incomplete sequences