# 15896/15483 Truth, Justice, and Algorithms

# Fall 2018 Procaccia and Psomas HW 1

# Due Thursday, October 18 at 11:59PM

1. **PoA of Awesome Games** (30 points: 5/15/10)

   Let an *awesome game* consist of a set $N$ of $n$ players and a set $M$ of $m$ activities. Each player $i$ chooses a subset of activities $S_i \subseteq M$ from a collection of allowable subsets $\Pi_i \subseteq 2^M$, and the cost of activity $j$ is $n_j$, where $n_j$ is the number of people participating in activity $j$. The cost of a player $i$ is the sum of her costs over activities she participates in; the social cost is the sum of player costs.

   (a) Prove that any awesome game has a pure-strategy Nash equilibrium.
   **Hint**: This problem has potential.

   (b) Prove that the price of anarchy (with respect to social cost) in awesome games is at most $5/2$.
   **Hint:** You may use without proof that for every pair of nonnegative integers $x$ and $y$, $y(x+1) \leq \frac{1}{3}x^2 + \frac{5}{3}y^2$.

   (c) Prove that that the upper bound of part (b) is tight for any $n \geq 3$, by constructing an appropriate family of awesome games.

   **Answer:**

   (a) We show that this game is an exact potential game; therefore, it has a pure NE. The potential function is

   $$\Phi(s) = \sum_j \sum_{i=1}^{n_j(s)} i.$$

   When someone changes from activity $j$ to $j'$, they give up $n_j(s)$ and receive $n_{j'}(s)+1$; this is exactly how much the potential function changes as well.

   (b) Let $A$ be a Nash equilibrium and $P$ be the optimal allocation. The cost of player $i$ in $A$ is $c_i(A) = \sum_{e \in A_i} n_e(A)$, where $n_e(A)$ denotes the number of players that participate in activity $e$ in $A$. Note that the social cost in $A$ is $\sum_i \sum_{e \in A_i} n_e(A) = \sum_{e \in M} n_e^2(A)$, and the social cost in $P$ is $\sum_i \sum_{e \in P_i} n_e(P) = \sum_{e \in M} n_e^2(P)$.
   Now, note that by definition of a Nash equilibrium, the cost of player $i$ does not decrease when this player switches to strategy $P_i$ (i.e., what she would play in the optimal allocation):

   $$c_i(A) = \sum_{e \in A_i} n_e(A) \leq \sum_{e \in P_i} n_e(P_i, A_{-i}) \leq \sum_{e \in P_i} (n_e(A)+1)$$

   because this adds at most one agent to each strategy in $P_i$ (it doesn't add one agent to activities in $A_i \cap P_i$).

Now we sum over all players $i$ to bound the social cost:

$$\sum_{i\in N} c_i(A) \leq \sum_{i\in N}\sum_{e\in P_i}(n_e(A)+1)$$
$$= \sum_{e\in M} n_e(P)(n_e(A)+1).$$

Note that the left hand side is the social welfare in the Nash equilibrium. We then apply the hint and conclude.

(c) Consider $n \geq 3$ players and $2n$ activities, where we further divide these activities into subsets $M_1 = \{e_1,\ldots,e_n\}$ and $M_2 = \{f_1,\ldots,f_n\}$. Each player $i$ has two choices: either choose $\{e_i, f_i\}$ or $\{e_{i+1}, f_{i-1}, f_{i+1}\}$. The optimal thing to do is for each player to choose the subset with two activities, which yields a social cost of $2n$; however, it is a NE for all players to choose the subset with three activities, which yields a social cost of $5n$.

2. **PoA of Special Games** (30 points: 20/10)

Consider the following class of *special games*, which are related to awesome games. Given a set $N$ of $n$ players, each player $i$ has an associated weight $w_i$. There is also a set $M$ of $m$ activities, and each player would like to participate in exactly one activity. Each player chooses an activity from the set of all activities, that is, the strategy space of each agent is $M$. Furthermore, the cost of each player $i$ is the sum of weights of players who choose the same activity as she did. The objective in special games is to minimize the maximum cost of any player (equivalently, minimize the maximum total weight in any activity). It is known that special games also always have pure Nash equilibria.

(a) Prove that the price of anarchy of special games (with respect to maximum cost) is at most $2-2/(m+1)$.

(b) Prove that the upper bound of part (a) is tight, by constructing an appropriate family of special games for each $m \in \mathbb{N}$.

**Answer:**

(a) Let $j^*$ be the activity with highest weight under a given Nash equilibrium $A$, and let $i^*$ be the smallest weight player on this activity. WLOG, there are two players on $j^*$; otherwise, the cost of $A$ is the cost of the optimal assignment, and the bound trivially follows. This means that $w_{i^*} \leq \mathrm{cost}(A)/2$, as $i^*$ is the smallest weight player on the bottleneck activity.

Now, suppose there is an activity $j \neq j^*$ with total cost less than $\ell_{j^*} - w_{i^*}$, where $\ell_{j^*}$ is the total cost of activity $j^*$. Then, moving $i^*$ to $j$ would decrease the social cost. Therefore, because $A$ is a Nash equilibrium, it must be that

$$\ell_j \geq \ell_{j^*} - w_{i^*} \geq \mathrm{cost}(A)/2.$$

Furthermore, note that the cost of an optimal assignment cannot be smaller than the average cost across all activities:

$$\mathrm{opt}(G) \geq \frac{\sum_i w_i}{m} = \frac{\sum_j \ell_j}{m} \geq \frac{\mathrm{cost}(A)+\mathrm{cost}(A)(m-1)/2}{m} = \frac{(m+1)\mathrm{cost}(A)}{2m}.$$

(b) Consider a game $G$ with $n = 2m$, where $m$ players have weight 1 and $m$ players have weight $m$. The optimal assignment is to assign one small player and one big player to each activity; this yields a social cost of $m+1$. However, it is also a NE to have two big players to one activity, all small players to another, and one big player per each other activity; this yields a social cost of $2m$.

3. **Lemke-Howson Revisited** (15 points)

   Recall that the Lemke-Howson algorithm as presented in class was formulated for finding symmetric equilibria in 2-player symmetric games. In this problem, you will prove that this is, in fact, without loss of generality.

   Let NASH be the problem of, given any game, finding a Nash equilibrium.

   Define a symmetric game as a game in which changing the identity of players does not change the game from their point of view. Formally, in the special case of two-player games, a symmetric game is one in which both players have the same set of strategies and the matrix of payoffs to player one is the transpose of the matrix of payoffs to player two (Lecture 8, Slide 16). Furthermore, define SYMMETRIC NASH to be the problem of, given a symmetric game, finding a symmetric Nash equilibrium (i.e., a Nash equilibrium in which all players use the same (possibly mixed) strategy).

   Prove that there exists a polynomial-time reduction from NASH to SYMMETRIC NASH for two-player games. That is, given a black box solution for SYMMETRIC NASH, prove that there exists a polynomial-time algorithm that, on any given two-player game, uses a single query to the black box and outputs a solution to NASH.

   **Answer:** Suppose we are given a two-player game described by matrices $A$ and $B$, and WLOG assume that all of the entries of these matrices are positive (adding a constant to all entries changes nothing). Consider the symmetric game represented by matrix $\begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}$. Let $(x,y)$ be a symmetric equilibrium of this game, where $x$ denotes the first $m$ components of this game, where $m$ is the number of rows of $A$. It is easy to see that, for $(x,y)$ to be a best response to itself, $y$ must be a best response to $x$ and vice versa; hence, $x$ and $y$ are a Nash equilibrium of the original game.

4. **Equilibrium Computation in Anonymous Games** (25 points)

   Define an *anonymous game* as a game in which each player's utility does not depend on the identity of the other players. Formally, an anonymous game $G$ consists of a set $N$ of $n$ players, a set $S$ of $m$ strategies available to all players, and a set of utility functions $u_\ell^i$, each of which represents the payoff to agent $i$ for playing strategy $\ell$, where the payoff to each player only depends on her choice of action and the number of other players who chose each strategy. Furthermore, throughout this problem, assume that all utilities are normalized to take values in $[0,1]$.

   Now, define the *total variation distance* between two distributions $P$ and $Q$ supported on a finite set $A$ as

   $$\|P-Q\|_{TV} = \frac{1}{2}\sum_{\alpha\in A} |P(\alpha)-Q(\alpha)|.$$

   For anonymous games where the strategy space is of constant size, it is possible to show that there exists a PTAS (polynomial-time approximation scheme) for mixed Nash equilibrium. The idea is similar to the Lipton-Markakis-Metha algorithm we saw in class: show that there is a relatively small set of candidates ("discretized" strategy profiles), and then check them one-by-one through enumeration.

The computational component is a bit tedious in this case, so this problem focuses on the existence component. Moreover, to make things simple, assume that there are two strategies per player, which means that we can represent a mixed strategy profile as a vector of probabilities $(q_1,\ldots,q_n)$, where $q_i$ is the probability that player $i$ plays strategy 1.

We will use the following (highly nontrivial) theorem. Intuitively, it asserts that any arbitrary sequence of probabilities can be well-approximated by another sequence of discretized probabilities in a way that preserves certain desirable properties.

**Theorem 1.** *Let $p_1,\ldots,p_n$ be $n$ arbitrary probabilities, that is, $p_i \in [0,1]$ for $i = 1,\ldots,n$. Furthermore, let $X_1,\ldots,X_n$ be $n$ independent indicator random variables such that $\mathbb{E}[X_i] = p_i$ for all $i$. Finally, let $k$ be a positive integer. Then there exists another set of probabilities $q_1,\ldots,q_n$ that satisfy the following properties.*

- $|q_i - p_i| = O(1/k)$ *for all* $i \in [n]$*;*
- $q_i$ *is an integer multiple of* $1/k$ *for all* $i \in [n]$*;*
- *given independent indicator random variables* $Y_1,\ldots,Y_n$ *such that* $\mathbb{E}[Y_i] = q_i$ *for all* $i$*, we have that for all* $j = 1,\ldots,n$*,*

$$\left\| \sum_{i\neq j} X_i - \sum_{i\neq j} Y_i \right\|_{TV} = O\left(\frac{1}{\sqrt{k}}\right);$$

- *for all* $i$*, if* $p_i = 0$*, then* $q_i = 0$ *and if* $p_i = 1$*, then* $q_i = 1$ *(i.e., for all* $i$*, the support of* $Y_i$ *is contained in the support of* $X_i$*).*

Note that, say, $\sum_{i=1}^n X_i$ can naturally be interpreted as a probability distribution over $\{0,1,\ldots,n\}$.

Using the theorem, show that, for every two-strategy anonymous game with $n$ players and all $k \in \mathbb{N}$, there exists a mixed strategy profile $(q_1,\ldots,q_n)$, where each $q_i$ is a multiple of $1/k$, that constitutes an $O(1/\sqrt{k})$-approximate mixed Nash equilibrium for the game; i.e., no player can gain more than $O(1/\sqrt{k})$ by deviating.

**Answer:**

Given a MNE $\vec{p} = (p_1,\ldots,p_n)$, we show that the distribution $vecq = (q_1,\ldots,q_n)$ guaranteed to exist by Theorem 1 is a $O(1/\sqrt{k})$-approximate MNE. Now, let $Y$ be a sequence of independent indicator random variables such that $\mathbb{E}[Y_i] = q_i$ for all $i$.

Now, we first examine the change in utility for each player $i$ when going from $\vec{p}$ to $\vec{q}$. In particular, we first show that

$$\left| \mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} X_j\right)\right] - \mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} Y_j\right)\right] \right| \leq 2 \left\| \sum_{j\neq i} X_j - \sum_{j\neq i} Y_j \right\|_{TV}.$$

This is because

$$\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} X_j\right)\right] = \sum_{x\in\Pi_{n-1}} u_\ell^i(x)\cdot \Pr\left[\sum_{j\neq i} X_j = x\right]$$

and

$$\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} Y_j\right)\right] = \sum_{x\in\Pi_{n-1}} u_\ell^i(x)\cdot \Pr\left[\sum_{j\neq i} Y_j = x\right],$$

which means that

$$\begin{aligned}\left|\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} X_j\right)\right] - \mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} Y_j\right)\right]\right| &= \left|\sum_{x\in\Pi_{n-1}} u_\ell^i(x)\cdot \Pr\left[\sum_{j\neq i} X_j = x\right] - \sum_{x\in\Pi_{n-1}} u_\ell^i(x)\cdot \Pr\left[\sum_{j\neq i} Y_j = x\right]\right| \\ &= \sum_{x\in\Pi_{n-1}} u_\ell^i(x)\cdot \left|\Pr\left[\sum_{j\neq i} X_j = x\right] - \Pr\left[\sum_{j\neq i} Y_j = x\right]\right| \\ &\leq 2\left\|\sum_{j\neq i} X_j - \sum_{j\neq i} Y_j\right\|_{TV},\end{aligned}$$

where the last transition follows from the definition of the total variation distance and the fact that utilities are normalized to between 0 and 1; i.e., $u_\ell^i(x) \leq 1$.

Therefore, because the total variation distance is in $O(1/\sqrt{k})$ by Theorem 1, the change in utility for each player $i$ is also at most $O(1/\sqrt{k})$.

We now show that given two strategy profiles $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ such that $X$ is a MNE of a game, if for all $i$ (1) the support of $Y_i$ is a subset of the support of $X_i$ and (2) $\left\|\sum_{j\neq i} X_j - \sum_{j\neq i} Y_j\right\| \leq \varepsilon$ for some $\varepsilon > 0$, then $Y$ is a $4\varepsilon$-approximate MNE.

In order to prove this, we must show that every strategy $\ell$ in the support of each $Y_i$ is an approximate best response to $(Y_j)_{j\neq i}$. Note that Theorem 1 guarantees that the support of each $Y_i$ is a subset of the support of each $X_i$, and further note that because $X$ is a MNE, every $\ell$ in the support of each $Y_i$ is a best response to $(X_j)_{j\neq i}$ (since every strategy in the support of each $X_i$ is a best response to $(X_j)_{j\neq i}$).

By the definition of a MNE

$$\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} X_j\right)\right] \geq \mathbb{E}\left[u_{\ell'}^i\left(\sum_{j\neq i} X_j\right)\right].$$

However, by the first part above, we know that

$$\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} Y_j\right)\right] \geq \mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} X_j\right)\right] - 2\varepsilon,$$

and

$$\mathbb{E}\left[u_{\ell'}^i\left(\sum_{j\neq i} X_j\right)\right] \geq \mathbb{E}\left[u_{\ell'}^i\left(\sum_{j\neq i} Y_j\right)\right] - 2\varepsilon.$$

Putting everything together yields

$$\mathbb{E}\left[u_\ell^i\left(\sum_{j\neq i} Y_j\right)\right] \geq \mathbb{E}\left[u_{\ell'}^i\left(\sum_{j\neq i} Y_j\right)\right] - 4\varepsilon,$$

as desired, so $Y$ is a $4\varepsilon$-approximate MNE. Putting the two parts together completes the argument.