15896/15483 Truth, Justice, and Algorithms
Fall 2018 Procaccia and Psomas HW 1

Due October 1, 2018, at 11:59pm

1. **Cake Cutting** (30 points: 20/10)

   Consider the cake cutting problem with n players and valuation functions $v_1, \ldots, v_n$ satisfying additivity, normalization, and divisibility. Denote the *social welfare* of an allocation $\boldsymbol{A}$ by $\mathrm{sw}(\boldsymbol{A}) = \sum_{i=1}^{n} v_i(A_i)$.

   (a) Show that for all valuation functions $v_1, \ldots, v_n$,

   $$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = O(\sqrt{n}).$$

   (b) Give a family of examples of $v_1, \ldots, v_n$ (one example for each value of n) such that

   $$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = \Omega(\sqrt{n}).$$

   Answer:

   (a) To prove the upper bound, let $V_1, \ldots, V_n$ be the players' valuation functions, and let $\mathbf{A}^*$ be the optimal allocation. Let $L = \{i \in N : V_i(A_i^*) \geq 1/\sqrt{n}\}$ be the set of "large" players, and $S = N \setminus L$ be the set of "small" players. We consider two cases.

   Case 1: $|L| \geq \sqrt{n}$. It follows from the assumption that $|S| \leq n - \sqrt{n}$. Define an allocation $\mathbf{A}$ as follows. For each $i \in S$, reallocate A_i^* among the players in S so that for each $j \in S$, $V_j(A_j \cap A_i^*) \geq V_j(A_i^*)/|S|$; this can even be done algorithmically (although only existence is required), e.g., using (a slight variation of) the Even-Paz protocol. For each $i \in L$, we reallocate A_i^* among the players in $\{i\} \cup S$ so that

   $$V_i(A_i \cap A_i^*) \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{\sqrt{n} + |S|},$$

   and for all $j \in S$,

   $$V_j(A_j \cap A_i^*) \geq \frac{V_j(A_i^*)}{\sqrt{n} + |S|}.$$

   This can be done, e.g., by creating $\sqrt{n} - 1$ copies of player i with identical valuations and running the Even-Paz algorithm with the $\sqrt{n}$ identical players and the players in S.

   Note that the allocation $A_1, \ldots, A_n$ is proportional, because for all $i \in L$,

   $$V_i(A_i) \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{\sqrt{n} + |S|} \geq \frac{1}{\sqrt{n} + |S|} \geq \frac{1}{n},$$

and for all $i \in S$,

$$V_i(A_i) \geq \sum_{j\in L} \frac{V_i(A_j^*)}{\sqrt{n}+|S|} + \sum_{j\in S} \frac{V_i(A_j^*)}{|S|} \geq \frac{\sum_{j\in N} V_i(A_j^*)}{n} = \frac{1}{n}.$$

Moreover, for each $i \in N$, $V_i(A_i) \geq V_i(A_i^*)/\sqrt{n}$, hence it holds that $\mathrm{sw}(\mathbf{A}) \geq \mathrm{sw}(\mathbf{A}^*)/\sqrt{n}$; the ratio is at most $\sqrt{n}$.

Case 2: $|L| < \sqrt{n}$. Observe that $\mathrm{sw}(\mathbf{A}^*) \leq |L| + |S|/\sqrt{n} < 2\sqrt{n}$, while for any proportional allocation $\mathbf{A}$, $\mathrm{sw}(\mathbf{A}) \geq \sum_{i\in N} 1/n = 1$; the ratio is $O(\sqrt{n})$.

(b) To establish the lower bound, consider the following valuation functions. The set of players $L \subseteq N$ now contains exactly $\sqrt{n}$ players, each uniformly interested only in a single interval of length $1/\sqrt{n}$, such that for $i, j \in L$ the two desired intervals are disjoint. The set of players $S = N \setminus L$ contains $n - \sqrt{n}$ players that desire the entire cake uniformly.

The optimal allocation $\mathbf{A}^*$ gives each player in L its desired interval, hence $\mathrm{sw}(\mathbf{A}^*) = \sqrt{n}$. In contrast, any proportional allocation $\mathbf{A}$ would have to give an interval of length $1/n$ to each player in S, leaving only $1/\sqrt{n}$ by length to the players in L. With their density of $\sqrt{n}$, it must hold that $\sum_{i\in L} V_i(A_i) \leq \sqrt{n}/\sqrt{n} = 1$, while $\sum_{i\in S} V_i(A_i) \leq 1$. Thus, $\mathrm{sw}(\mathbf{A}) \leq 2$; the ratio is $\Omega(\sqrt{n})$.

2. **The Partial Nash Algorithm** (25 points: 5/10/10)

Consider a setting with a set M of m divisible goods and a set N of n players. Define an allocation $x \in \mathbb{R}^{n\times m}$ as an $n \times m$ matrix in which x_{ij} denotes the fraction of good j allocated to player i. Let $\mathscr{F} = \{x \,|\, x_{ij} \geq 0 \text{ and } \sum_i x_{ij} \leq 1\}$ denote the set of feasible allocations. Lastly, assume that each player i has a homogeneous valuation function $v_i : \mathbb{R}^{n\times m} \to \mathbb{R}$; i.e., each player i's valuation for the allocation $x' = c \cdot x$ satisfies $v_i(x') = c \cdot v_i(x)$ for any $c \geq 0$.

We define *Nash Fairness (NF)* as follows. An allocation x^* is Nash fair if, for any other allocation x', the total proportional change in valuations is not positive; i.e.,

$$\sum_{i\in N} \frac{v_i(x') - v_i(x^*)}{v_i(x^*)} \leq 0.$$

It is known that an NF allocation exists, and, in fact, it is the unique allocation that maximizes the Nash product $\prod_{i\in N} v_i(x)$; you may rely on this fact in your solution.

The Partial Nash (PN) algorithm first computes the NF allocation x^*, and then assigns each player i a fraction of x_i^* that depends on the extent to which the presence of i inconveniences the other players (i.e., decreases the value of other players).

Algorithm 1 Partial Nash (PN) algorithm

1: Compute the NF allocation x^* based on the reported bids.
2: For each player i, remove her and compute the NF allocation x_{-i}^* that would occur in her absence.
3: Allocate to each player i a fraction f_i of everything she receives according to x^*, where

$$f_i = \frac{\prod_{i'\neq i} v_{i'}(x^*)}{\prod_{i'\neq i} v_{i'}(x_{-i}^*)}.$$

(a) Show that the allocation produced by the PN algorithm is feasible.

(b) Prove that the PN algorithm is strategyproof; that is, no player can benefit by reporting untruthfully.

(c) Prove that the PN algorithm always yields an allocation such that, for every player i, $v_i(x) \geq \frac{1}{e} \cdot v_i(x^*)$; i.e., it provides a $1/e$ approximation of the optimal allocation.

Hint. *Given a sequence of n real numbers* $d_1, \ldots, d_n \geq -1$ *such that* $\sum_{i=1}^n d_i \leq 1$, $\prod_{i=1}^n (1+d_i) \leq (1+1/n)^n$.

Answer:

(a) Note that an NF allocation always exists, so it suffices to show that all $f \in [0,1]$. It is clear that each $f_i \geq 0$ because both the numerator and denominator are non-negative.

To show that each $f_i \leq 1$, note that each f_i is defined as

$$f_i = \frac{\prod_{i' \neq i} v_{i'}(x^*)}{\prod_{i' \neq i} v_{i'}(x^*_{-i})},$$

or the product of other players' values for their allocations in a world in which bidder i does not exist divided by the product of other players' values for their allocations in a world in which bidder i exists.

Because the NF solution without player i is $x^*_{-i} = \arg\max_x \prod_{i' \neq i} v_{i'}(x)$ by definition, all we need to show is that x^* without player i's share is a feasible allocation. However, this is clearly true, as we can remove player i's share and still have a feasible allocation.

(b) We prove that any arbitrary player i cannot benefit from misreporting her preferences. Let $v_i(\cdot)$ represent i's true preferences, and let $\tilde{v}_i(\cdot)$ represent a false valuation function that i reports to the mechanism. Furthermore, let $\tilde{v_{i'}}(\cdot)$ represent the valuation function that each player $i' \neq i$ reports to the mechanism; these reports may or may not be truthful.

Furthermore, let x_T denote the NF allocation that results when i (truthfully) reports $v_i(\cdot)$, and let x_F denote the NF allocation that results when i (falsely) reports $\tilde{v}_i(\cdot)$. Further, let f_T (resp. f_F) denote the fraction of i's allocation that i receives when she reports truthfully (resp. reports falsely).

It suffices to show that $f_T v_i(x_T) \geq f_F v_i(x_F)$. Note that the denominators of f_T and f_F are equal, as the denominators do not depend on i's report. Therefore, it suffices to show that

$$v_i(x_T) \cdot \prod_{i' \neq i} \tilde{v_{i'}}(x_T) \geq v_i(x_F) \cdot \prod_{i' \neq i} \tilde{v_{i'}}(x_F).$$

To show this, recall that the NF allocation maximizes the product of reported valuations, which means that

$$x_T = \arg\max_{x \in \mathscr{F}} \left(v_i(x) \cdot \prod_{i' \neq i} \tilde{v_{i'}}(x) \right).$$

Because x_F is also a feasible allocation, this implies that player i cannot benefit from misreporting her preferences.

(c) The final valuation for each player is $v_i(x) = f_i \cdot v_i(x^*)$; that is, each player gets a fraction of her NF allocation. Therefore, it suffices to show that $\min_i f_i \geq \frac{1}{e}$.

WLOG, let player i be $\arg\min_i f_i$; that is, player i has the lowest fraction f_i. Now, consider the allocation x^*_{-i} computed by the PN mechanism after removing i. In this allocation, every other player i' receives value $v_{i'}(x^*_{-i}) = (1+d_{i'})v_{i'}(x^*)$, where each $d_i \geq -1$. Plugging this into the expression for f_i yields

$$f_i = \frac{1}{\prod_{i' \neq i}(1+d_{i'})}.$$

Furthermore, note that because x^* is an NF allocation, we know that by the definition of NF,

$$\sum_{i' \in N} \frac{v_{i'}(x^*_{-i}) - v_{i'}(x^*)}{v_{i'}(x^*)} \leq 0 \iff \sum_{i' \neq i} d_{i'} + \frac{v_i(x^*_{-i}) - v_i(x^*)}{v_i(x^*)} \iff \sum_{i' \neq i} d_{i'} \leq 1,$$

where the last transition follows because $v_i(x^*_{-i}) = 0$. We can therefore apply the hint above to see that

$$f_i = \frac{1}{\prod_{i' \neq i}(1+d_{i'})} \geq \frac{1}{(1+1/n)^n} \geq \frac{1}{e}.$$

3. $1/2$-**EFX** (30 points: 20/10)

Consider a setting with n players and a set G of m indivisible goods. Let $\vec{v} = (v_1, \dots, v_n)$ represent the additive valuation functions of the n players and assume that all players have positive valuations for all items.

As mentioned in class, it is an open problem whether EFX allocations exist in settings with more than two players; therefore, in this problem, we consider a relaxation of EFX. Define an allocation to be $1/2$-EFX if, for any two players i and j, i's value for j's bundle minus any good is at most twice i's value for her own bundle.

Definition 1. *An allocation* $\boldsymbol{A}$ *is* $1/2$*-EFX if, for all* i *and* j, $\forall g \in A_j$, $v_i(A_i) \geq (1/2) \cdot v_i(A_j \setminus \{g\})$.

Consider the following algorithm for finding a $1/2$-EFX allocation for n players.

Algorithm 2 $1/2$-EFX Allocation

Require: $n, G, (v_1, \dots, v_n)$ ▷ Input: players, goods, and valuation functions
$P \leftarrow G$ ▷ Initialize: all goods in pool
for $i \in [n]$ **do**
  $A_i \leftarrow \emptyset$ ▷ Initialize: all players start with no goods
end for
while $P \neq \emptyset$ **do** ▷ Repeat while pool not empty
  $g^* \leftarrow \text{pop}(P)$ ▷ Remove an arbitrary good from the pool
  $j \leftarrow \text{FindUnenviedPlayer}(\boldsymbol{A})$ ▷ and give it to an unenvied player
  $A_j \leftarrow A_j \cup \{g^*\}$
  if $\exists i \in [n], g \in A_j$ such that $v_i(A_i) < \frac{1}{2} v_i(A_j \setminus \{g\})$ **then** ▷ if this breaks $1/2$-EFX
    $P \leftarrow P \cup A_i$ ▷ Return i's old allocation to the pool
    $A_j \leftarrow A_j \setminus \{g^*\}$
    $A_i \leftarrow \{g^*\}$ ▷ and give i $\{g^*\}$
  end if
  $A \leftarrow \text{RemoveEnvyCycles}(\boldsymbol{A})$ ▷ Ensure the envy graph is acyclic
end while

You may assume that FindUnenviedPlayer always returns an unenvied player (if one exists). Furthermore, given a $1/2$-EFX allocation with an envy graph that contains cycles, RemoveEnvyCycles returns a $1/2$-EFX allocation with an acyclic envy graph. Note that, in this algorithm, RemoveEnvy-Cycles ensures that when FindUnenviedPlayer is called, an unenvied player does exist.

(a) Prove that at the beginning of each iteration of the while loop, the partial allocation is $1/2$-EFX.

(b) Prove that the algorithm terminates.

Hint. *Consider what happens to the social welfare in each round.*

Answer:

(a) We proceed by induction. Let A_k^ℓ be the bundle of player k at the beginning of round ℓ, and let B_k^ℓ be the bundle of player k at the end of round ℓ, right before RemoveEnvyCycles is run. Let $A^\ell = (A_1^\ell, \ldots, A_n^\ell)$ and $B^\ell = (B_1^\ell, \ldots, B_n^\ell)$. Throughout this proof, i and j refer to the players they represent in the while loop in round ℓ.

At the beginning of the program, all players have no items, which trivially satisfies $1/2$-EFX. Therefore, assume that the partial allocation at the beginning of round ℓ is $1/2$-EFX—we will show that the partial allocation at the beginning of round $\ell+1$ is also $1/2$-EFX. It suffices to show that B^ℓ is $1/2$-EFX because the partial allocation at the beginning of round $\ell+1$ is the output of RemoveEnvyCycles(B^ℓ), which preserves $1/2$-EFX-ness.

If the if-statement in lines 10-12 is not executed, then B^ℓ is $1/2$-EFX by definition.

If the if-statement is executed, then for all $k \neq i$, we have that $B_k^\ell = A_k^\ell$. This is because the only player whose bundle changed was i.

Recall that we assume that the partial allocation at the beginning of round ℓ is $1/2$-EFX. Therefore, all pairs (k, k') such that $k \neq i$ and $k' \neq i$ remain $1/2$-EFX. Furthermore, it cannot be the case that any player $k \neq i$ $1/2$-EFX-envies i because i only has one item. Therefore, it remains only to show that there is no pair (i, k) such that i $1/2$-EFX-envies k. We demonstrate this by showing that $v_i(B_i^\ell) > v_i(A_i^\ell)$; if i increases in value in this round and no one else changed, then i cannot be $1/2$-EFX-envious of anyone else.

Recall that j was unenvied at the beginning of round ℓ, so we know that $v_i(A_i^\ell) \geq v_i(A_j^\ell)$. Also, because the if-statement was executed, there must exist a $g \in A_j^\ell \cup \{g^*\}$ such that $v_i(A_i^\ell) < \frac{1}{2} v_i(A_j^\ell \cup \{g^*\} \setminus \{g\})$. Therefore, $v_i(A_i^\ell) < \frac{1}{2} v_i(A_j^\ell \cup \{g^*\})$. Therefore, we have

$$\begin{aligned}
v_i(A_i^\ell) &< \frac{1}{2} v_i(A_j^\ell \cup \{g^*\}) \\
&= \frac{1}{2} (v_i(A_j^\ell) + v_i(\{g^*\})) && \text{(additive values)} \\
&\leq \frac{1}{2} (v_i(A_i^\ell) + v_i(\{g^*\})). && (v_i(A_i^\ell) \geq v_i(A_j^\ell))
\end{aligned}$$

Moving terms around, we have

$$\begin{aligned}
v_i(A_i^\ell) - \frac{1}{2} v_i(A_i^\ell) &< \frac{1}{2} v_i(\{g^*\}) \\
v_i(A_i^\ell) &< v_i(\{g^*\}) \\
v_i(A_i^\ell) &< v_i(B_i^\ell).
\end{aligned}$$

Because i's value for her bundle increased, and she was not $1/2$-EFX-envious of any other player before this increase, she is still not $1/2$-EFX-envious of any other player. Therefore, this allocation satisfies $1/2$-EFX.

(b) Note: this allows for agents to have a value of 0 for goods; when valuations are strictly positive, a more straightforward argument works. Define a potential function for round ℓ as

$$\phi(\ell) = \sum_{k=1}^{n} v(A_k^\ell).$$

If the if-statement is not executed (denote this by Case 1), then j's bundle changes; it cannot decrease in value and therefore $\phi(\ell+1) \geq \phi(\ell)$. If the if-statement is executed (denote this by Case 2), then, as shown above, i's bundle strictly increases in value and therefore $\phi(\ell+1) > \phi(\ell)$. Further, note that the size of the pool of items decreases by at one every time that Case 1 occurs (i.e., no player dumps her partial allocation back into the pool). If m rounds pass without Case 2 occurring, then the algorithm must terminate. Therefore, Case 2 must occur at least once every m rounds, resulting in a strict increase in the potential function: $\phi(\ell+m) > \phi(\ell)$. However, there are only $(n+1)^m$ possible potential allocations (each good can be given to any player or be still in the pool), and so there can be at most $(n+1)^m$ distinct values for the potential function ϕ and ϕ can only increase that many times. Therefore, the algorithm terminates.

4. **Random EF Allocations** (20 points)

Consider a setting with m indivisible goods and n players with additive valuations, where each player has value drawn i.i.d. and uniformly at random from $[0,1]$ for each good. Prove that for any constant number of players $n \geq 2$, the probability that an EF allocation exists goes to 1 as the number of goods m goes to infinity (that is, an EF allocation exists *with high probability*).

Hint. ***k-th order statistics.*** *Given $X_1, \ldots, X_n$ i.i.d. random variables drawn uniformly at random from $[0,1]$, the expectation of the k-th smallest value is $\frac{k}{n+1}$.*

Hint. ***Hoeffding's inequality.*** *Hoeffding's inequality provides an upper bound on the probability that the sum of (bounded) independent random variables deviates from its expected value by more than a prescribed amount. Let $X_1, \ldots, X_n$ be independent random variables where each X_i is bounded by $[a_i, b_i]$. Now, define the random variable X as the sum of all X_i's: $X = X_1 + \cdots + X_n$. Hoeffding's inequality states that*

$$\Pr\left[|X - \mathbb{E}[X]| \geq t\right] \leq 2\exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).$$

Answer: Consider the allocation rule in which you give each item to the player who values it most. Consider the perspective an arbitrary player i. From the definition of k-order statistics, if i wins an item, i's expected value for that item is $n/(n+1)$, and we know that the expected value of each item to each player is $1/2$. We now establish i's expected value for an item that i doesn't win; in particular, we show that i's expected value for any item that i doesn't win is at most $1/2$. Let X_{ij} be the value of item j to player i, and let A_{ij} be the event that agent i receives item j.

$$\begin{aligned}
\mathbb{E}[X_{ij}] &= \mathbb{E}[X_{ij} \mid A_{ij}] \Pr[A_{ij}] + \mathbb{E}[X_{ij} \mid \neg A_{ij}] \Pr[\neg A_{ij}] \\
\frac{1}{2} &= \frac{n}{n+1} \cdot \frac{1}{n} + \mathbb{E}[X_{ij} \mid \neg A_{ij}] \cdot \frac{n-1}{n} \\
&\implies \mathbb{E}[X_{ij} \mid \neg A_{ij}] = \frac{1}{2} \cdot \frac{n}{n+1} < \frac{1}{2}.
\end{aligned}$$

Now, consider i's value for her own bundle and i's value for any other bundle. Let i's value for her own bundle be Z_{ii} and let i's value for the bundle belonging to player i' be $Z_{ii'}$. In expectation, i's value for her own bundle is

$$\begin{aligned}
\mathbb{E}[Z_{ii}] &= \mathbb{E}\left[\sum_{j=1}^{m} (X_{ij} \,|\, A_{ij}) \cdot A_{ij}\right] \\
&= \sum_{j=1}^{m} \mathbb{E}[(X_{ij} \,|\, A_{ij}) \cdot A_{ij}] \\
&= \sum_{j=1}^{m} \mathbb{E}[X_{ij} \,|\, A_{ij}] \cdot \mathbb{E}[A_{ij}] \\
&= m \cdot \frac{n}{n+1} \cdot \frac{1}{n} \\
&= \frac{m}{n+1}.
\end{aligned}$$

In expectation, i's value for the bundle of any player $i' \neq i$ is

$$\begin{aligned}
\mathbb{E}[Z_{ii'}] &= \mathbb{E}\left[\sum_{j=1}^{m} (X_{i'j} \,|\, A_{i'j}) \cdot \mathbb{1}\{A_{i'j}\}\right] \\
&= \sum_{j=1}^{m} \mathbb{E}[(X_{i'j} \,|\, A_{i'j}) \cdot \mathbb{1}\{A_{i'j}\}] \\
&= \sum_{j=1}^{m} \mathbb{E}[X_{i'j} \,|\, A_{i'j}] \cdot \mathbb{E}[A_{i'j}] \\
&< m \cdot \frac{1}{2} \cdot \frac{1}{n} \\
&= \frac{m}{2n}.
\end{aligned}$$

Note that for all $n \geq 2$, $\frac{m}{n+1} > \frac{m}{2n}$. Let $\varepsilon = \frac{1}{3} \cdot (\frac{m}{n+1} - \frac{m}{2n}) = \frac{m(n-1)}{6n(n+1)}$.

Let $Y_{ii'}$ denote the event that agent i envies agent i'. We first note that the probability that i does not envy i' is at least the probability that i's valuation for her own bundle is at least $\frac{m}{n+1} - \varepsilon$ times the probability that $i's$ valuation for the bundle belonging to i' is at most $\frac{m}{2n} + \varepsilon$, which in turn gives us an upper bound on the probability that i envies i'. Then, we apply a union bound over all n^2 pairs of agents to conclude our argument.

$$\begin{aligned}
\Pr[\neg Y_{ii'}] &\geq \Pr\left[Y_{ii} \geq \frac{m}{n+1} - \varepsilon\right] \cdot \Pr\left[Y_{ii'} \leq \frac{m}{2n} + \varepsilon\right] \\
&= \left(1 - \exp\left(\frac{-2\varepsilon^2}{m}\right)\right) \cdot \left(1 - \exp\left(\frac{-2\varepsilon^2}{m}\right)\right) && \text{(by Hoeffding)} \\
&= \left(1 - \exp\left(\frac{-2\varepsilon^2}{m}\right)\right)^2.
\end{aligned}$$

This yields an upper bound on the probability that i envies i':

$$\Pr[Y_{ii'}] \leq 1 - \left(1 - \exp\left(\frac{-2\varepsilon^2}{m}\right)\right)^2,$$

and applying a union bound yields

$$\Pr[\exists\, i, i' \text{ s.t. } Y_{ii'}] \leq n^2 \left(1 - \left(1 - \exp\left(\frac{-2\varepsilon^2}{m}\right)\right)^2\right).$$

Because $\frac{4\varepsilon^2}{m} = \frac{16m(n-1)^2}{36n^2(n+1)^2} \in O(m)$, we see that, as $m \to \infty$, $\exp\left(\frac{-2\varepsilon^2}{m}\right) \to 0$, and therefore the whole expression goes to 0.