Plan

Last Time

- Linear programming (LP) formulation
  - Problem description
  - Optimization representation
  - Graphical representation

Today

- LP: Continue graphical representation
- Solving linear programs
- Higher dimensions than just 2
- Integer programs

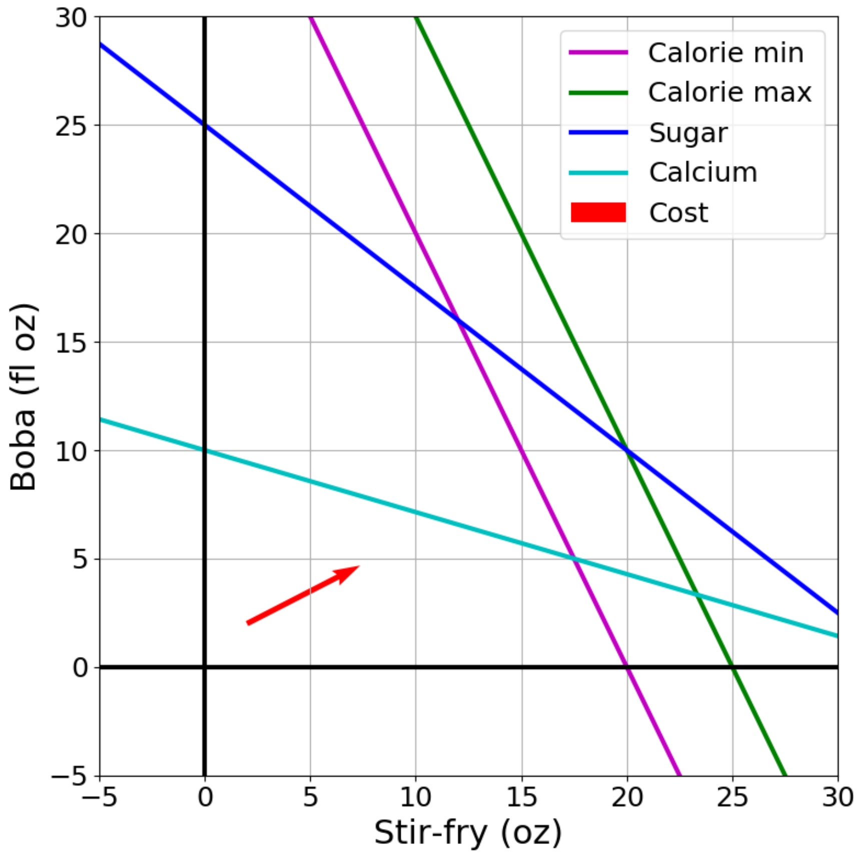
Optimization

Problem Description

Optimization Representation

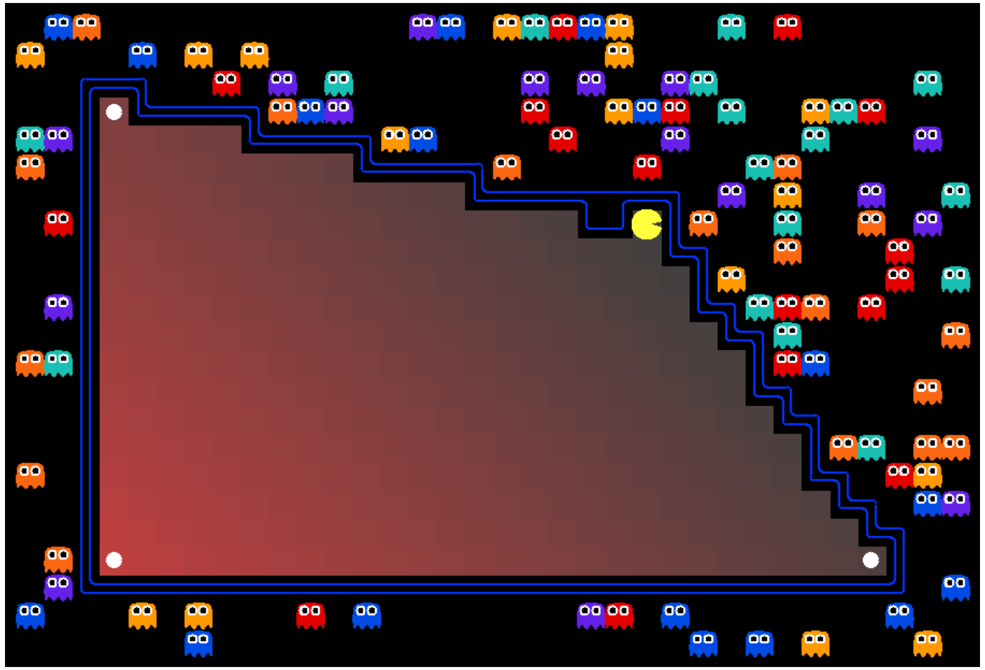
$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Graphical Representation



AI: Representation and Problem Solving

Linear and Integer Programming



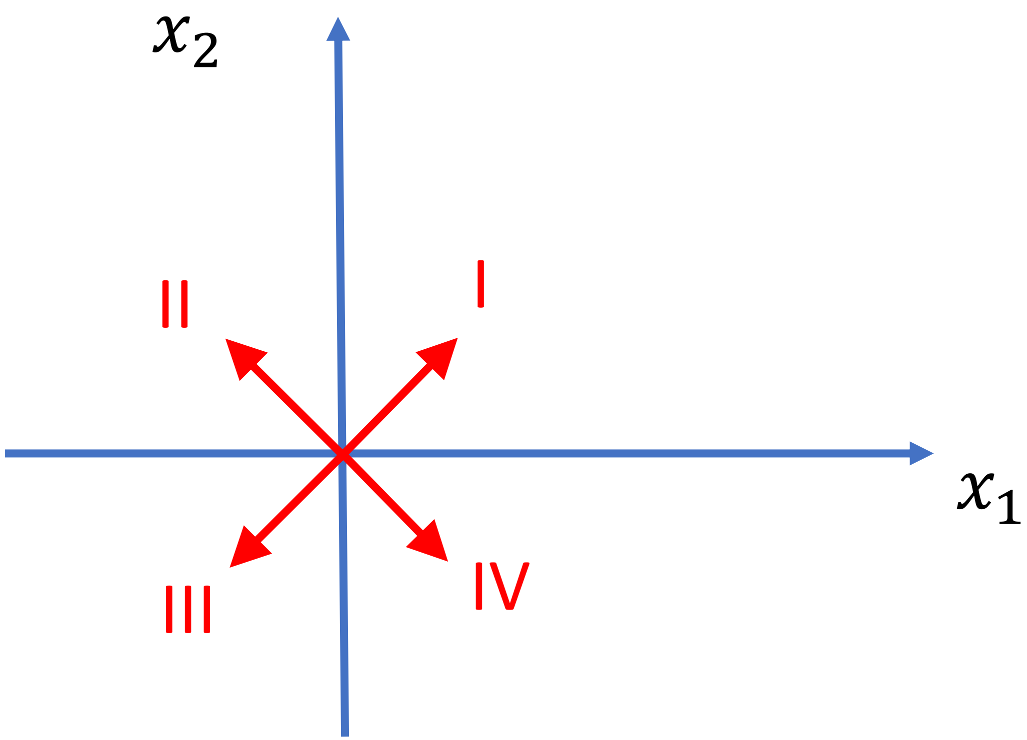
Instructor: Pat Virtue

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

Poll 1

Which of these points have cost $\mathbf{c}^\top \mathbf{x} = 0$?

for cost vector: $\mathbf{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



Question

Given the cost vector $[c_1, c_2]^\top$ where will

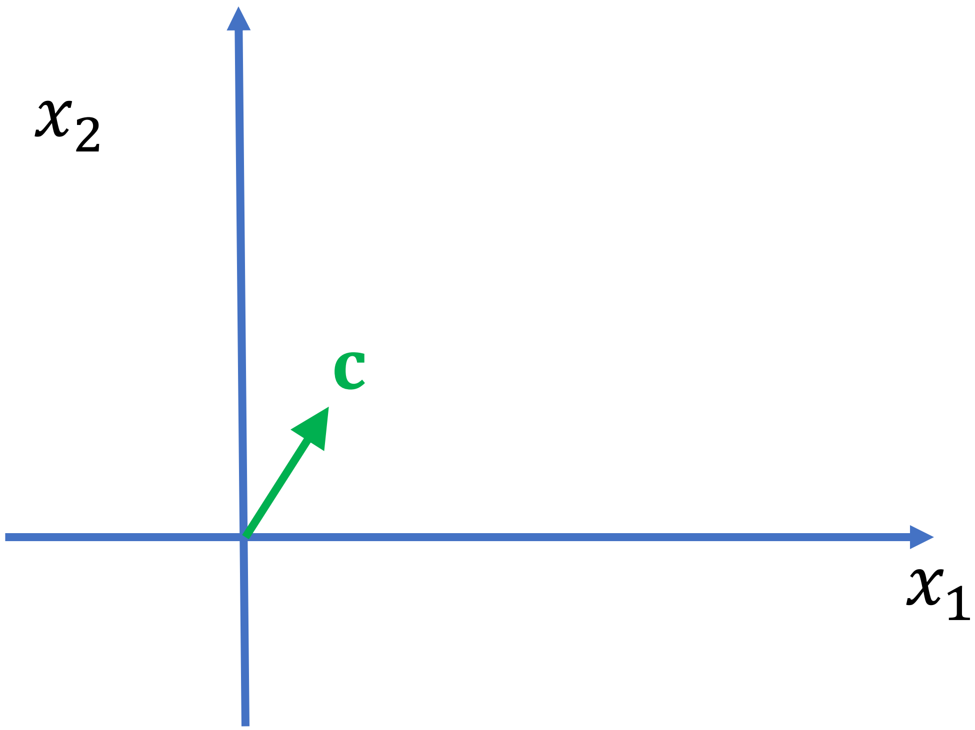
$\mathbf{c}^\top\mathbf{x} = 0$?

$\mathbf{c}^\top\mathbf{x} = 1$?

$\mathbf{c}^\top\mathbf{x} = 2$?

$\mathbf{c}^\top\mathbf{x} = -1$?

$\mathbf{c}^\top\mathbf{x} = -2$?



Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

$\mathbf{c}^\top\mathbf{x} = 0$?

$\mathbf{c}^\top\mathbf{x} = 1$?

$\mathbf{c}^\top\mathbf{x} = 2$?

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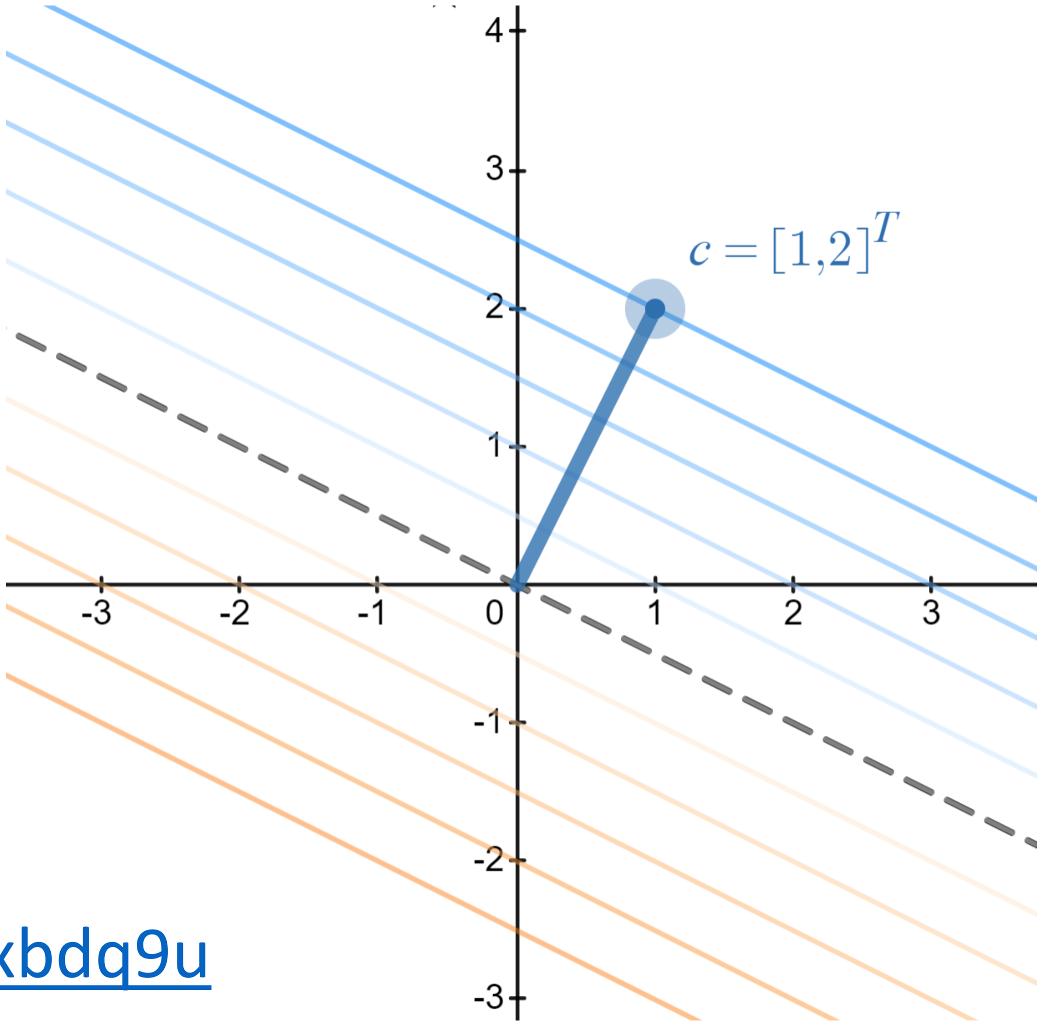


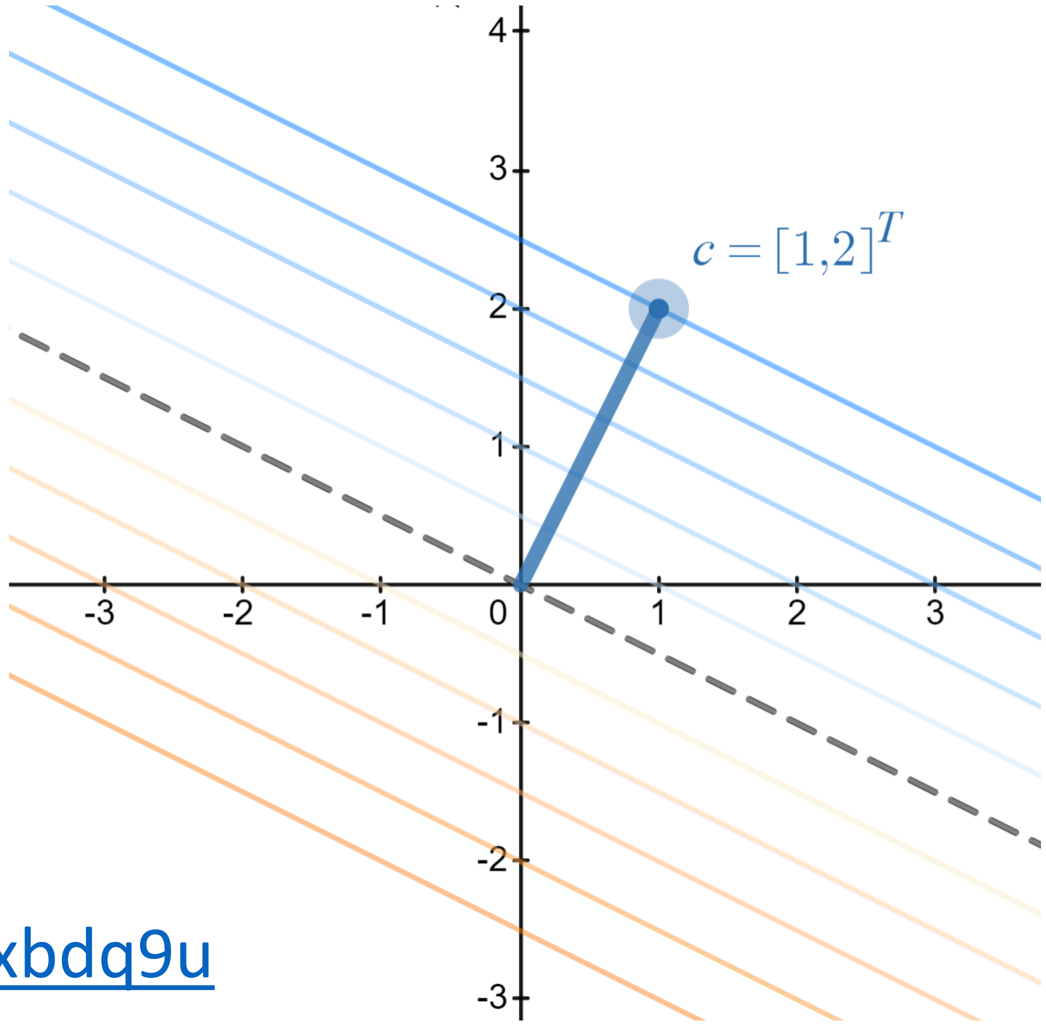
https://www.desmos.com/calculator/8d9kxbdq9u

Question

As the magnitude of $\mathbf{c}$ increases, the distance between the contours lines of the objective $\mathbf{c}^\top \mathbf{x}$:

A) Increases

B) Decreases



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Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \le b_1$$

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1 \, x_1 + a_2 \, x_2 = b_1$

2. $a_1 \, x_1 + a_2 \, x_2 \leq b_1$

3. $$\begin{aligned} a_{1,1} \, x_1 + a_{1,2} \, x_2 &\leq b_1 \\ a_{2,1} \, x_1 + a_{2,2} \, x_2 &\leq b_2 \\ a_{3,1} \, x_1 + a_{3,2} \, x_2 &\leq b_3 \\ a_{4,1} \, x_1 + a_{4,2} \, x_2 &\leq b_4 \end{aligned}$$

Feasible region:
All points x that satisfy the constraints

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\, x_1 + a_2\, x_2 = b_1$

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Feasible region:
All points x that satisfy the constraints

Graphics Representation

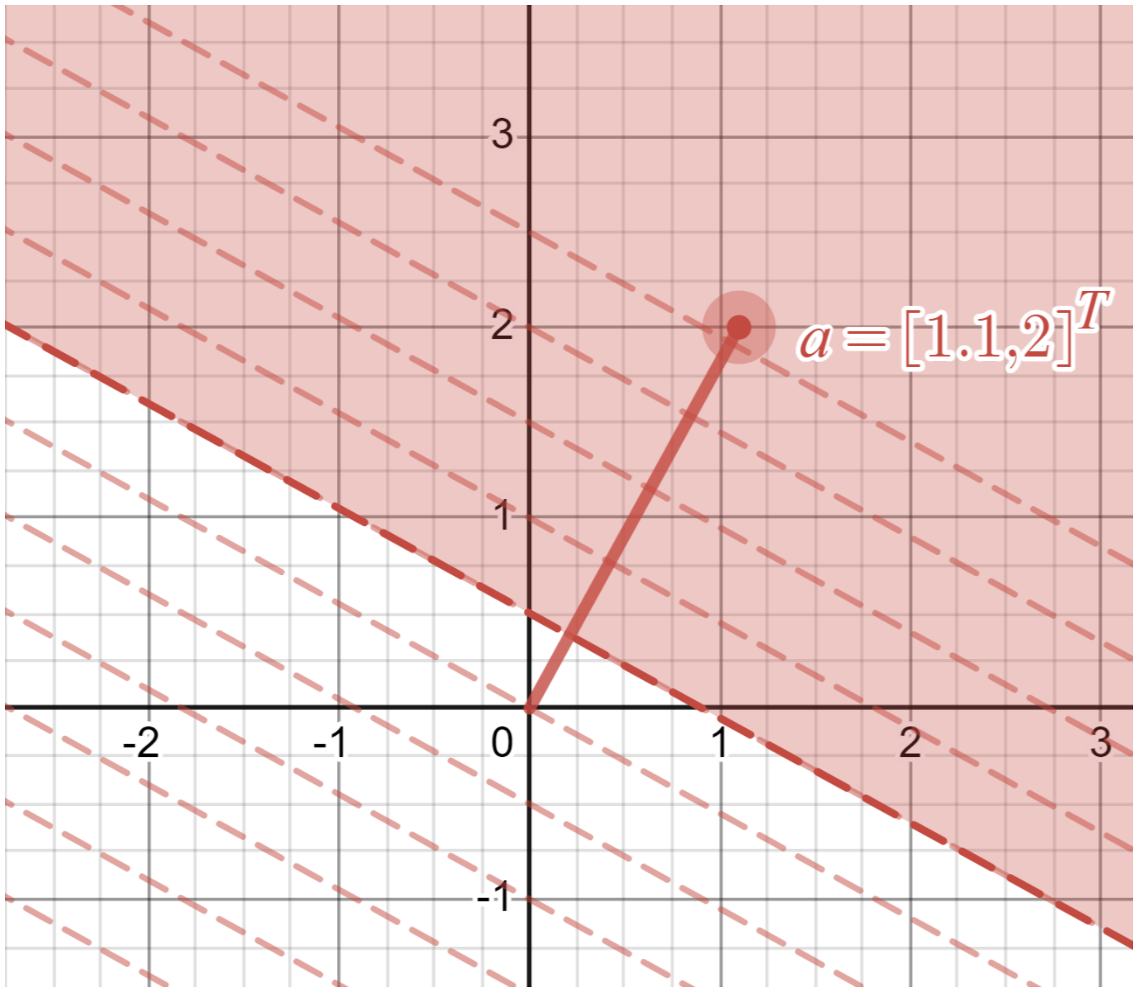
Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\, x_1 + a_2\, x_2 = b_1$

2. $a_1\, x_1 + a_2\, x_2 \leq b_1$

3.



https://www.desmos.com/calculator/lp0rqsb1w6

Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1\, x_1 + a_2\, x_2 = b_1$

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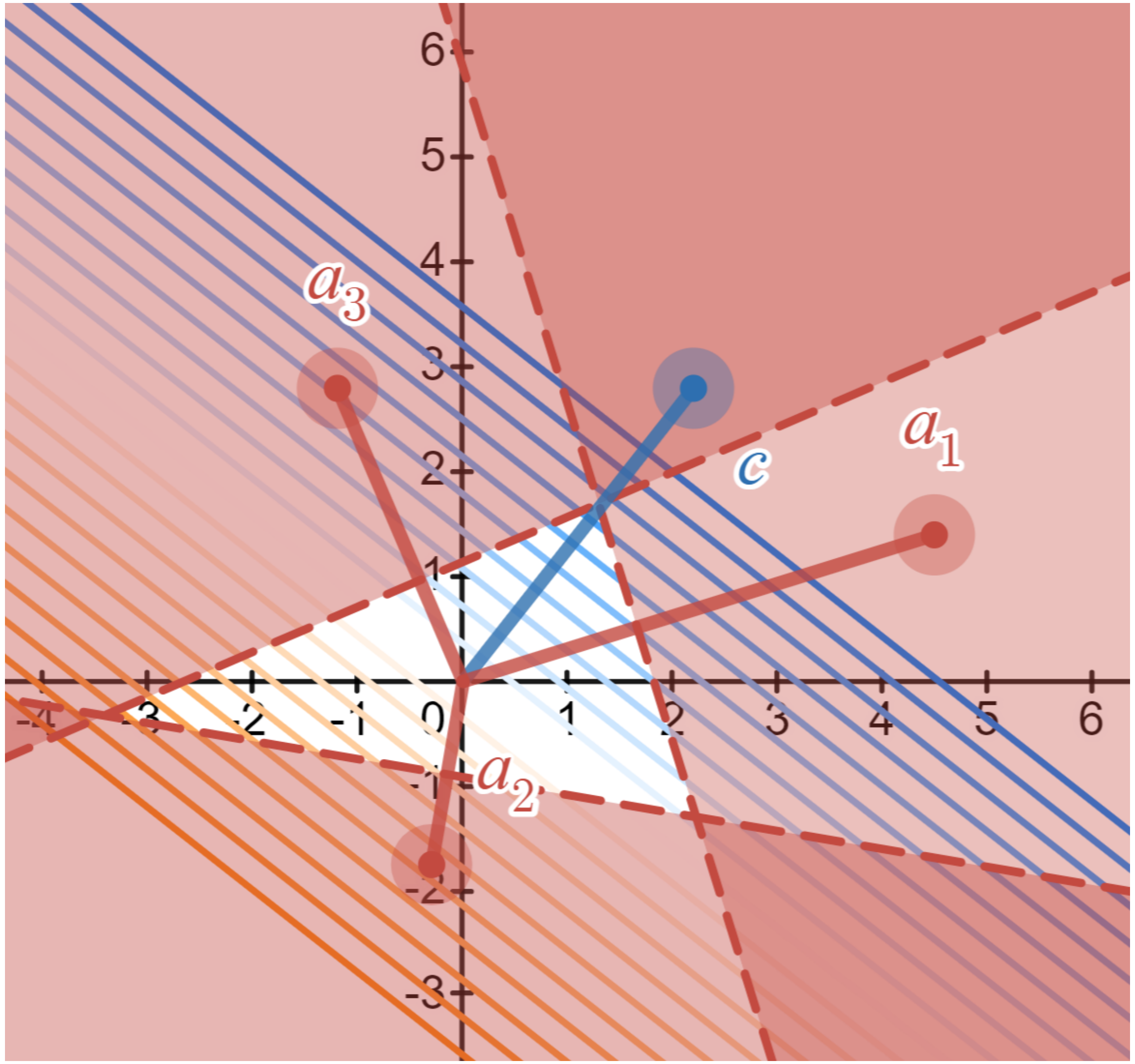

https://www.desmos.com/calculator/plp1thgsbh

Reminder: Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

$\mathbf{c}^\top \mathbf{x} = 0$?

$\mathbf{c}^\top \mathbf{x} = 1$?

$\mathbf{c}^\top \mathbf{x} = 2$?

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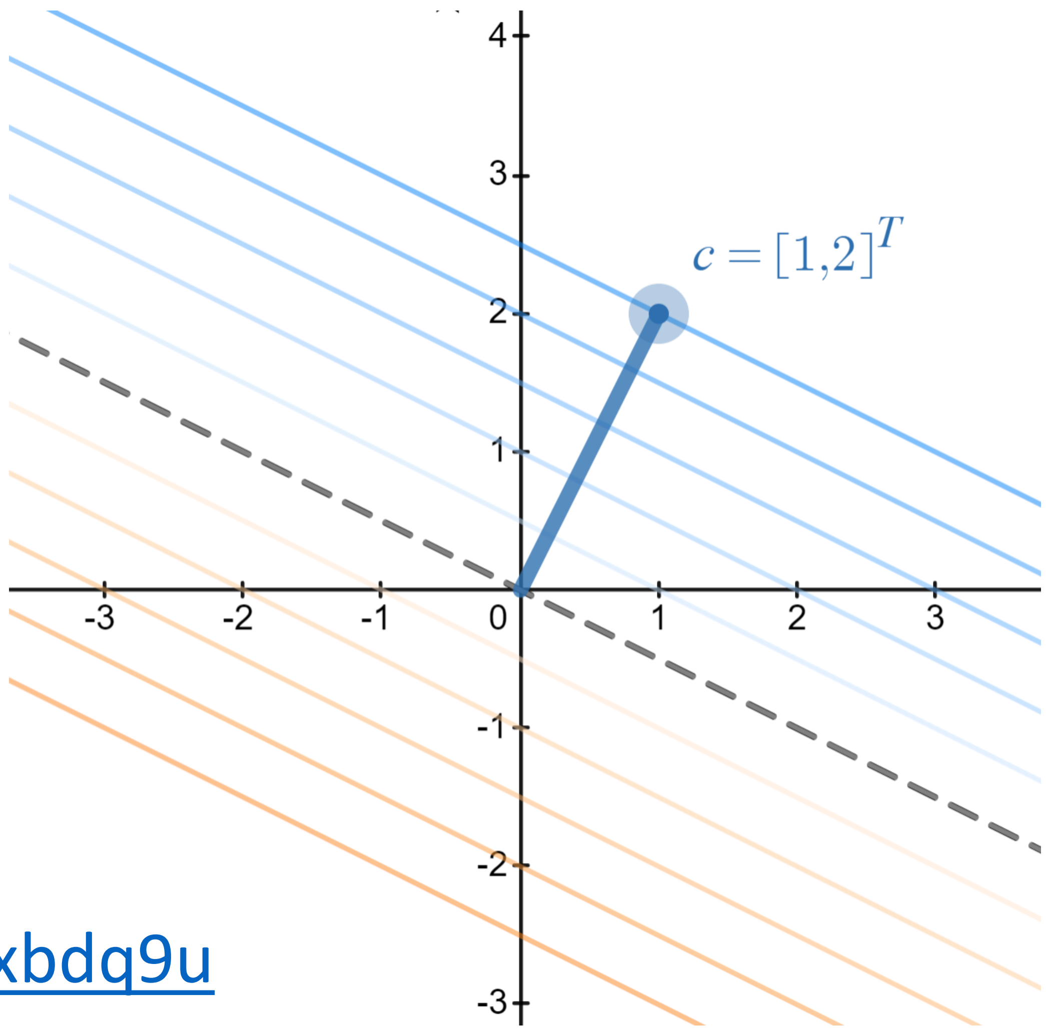
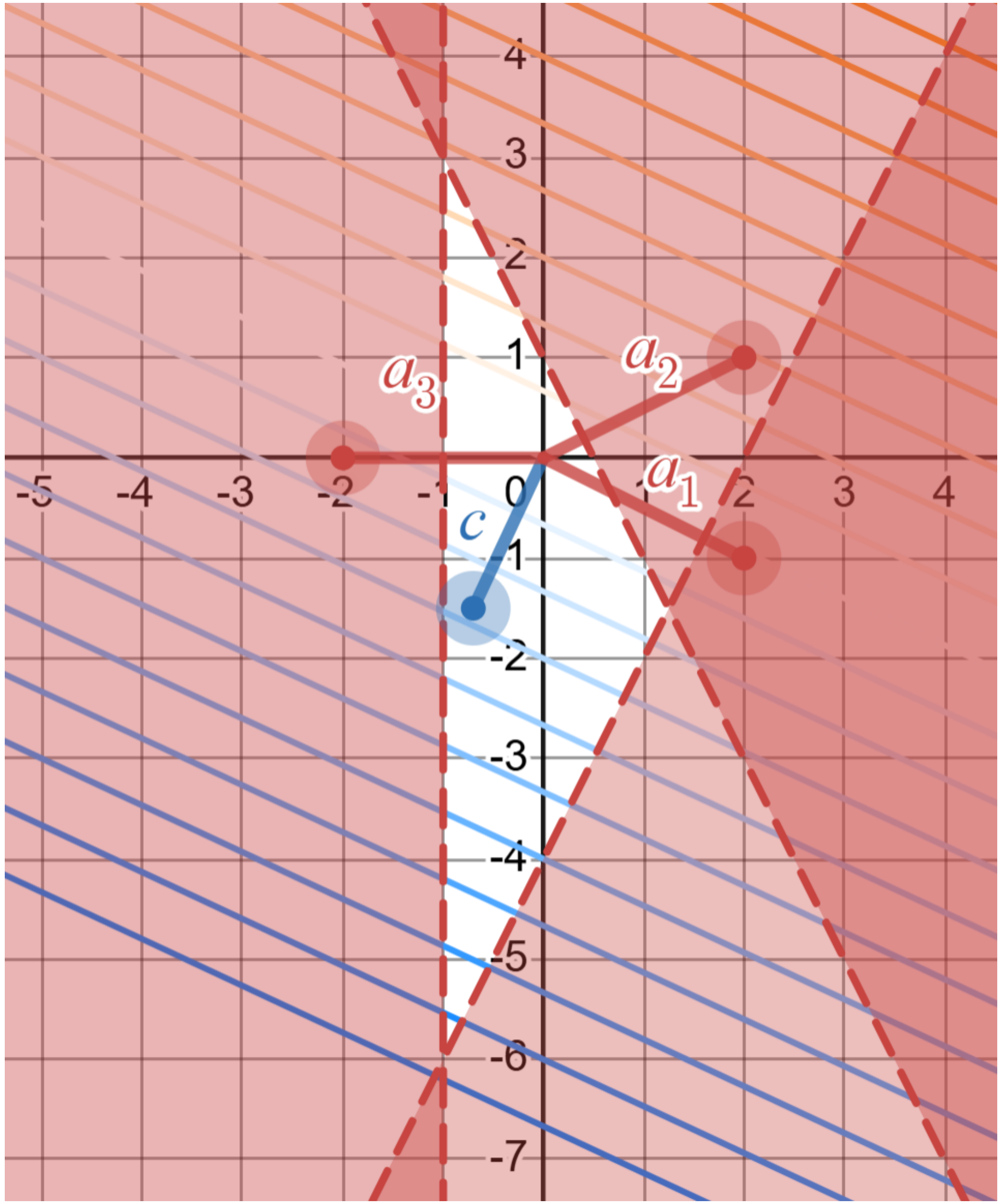


https://www.desmos.com/calculator/8d9kxbdq9u

Poll 2

What is the solution to this LP?



https://www.desmos.com/calculator/tnlo7p5plp

Solving a Linear Program

Inequality form, with no constraints

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

Solving a Linear Program

Inequality form, with one constraint

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad a_1 x_1 + a_2 x_2 \leq b$$

Poll 3

True or False: A minimizing LP with exactly on constraint, will always have a minimum objective at $-\infty$.

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 \leq b \end{aligned}$$

Solving an LP

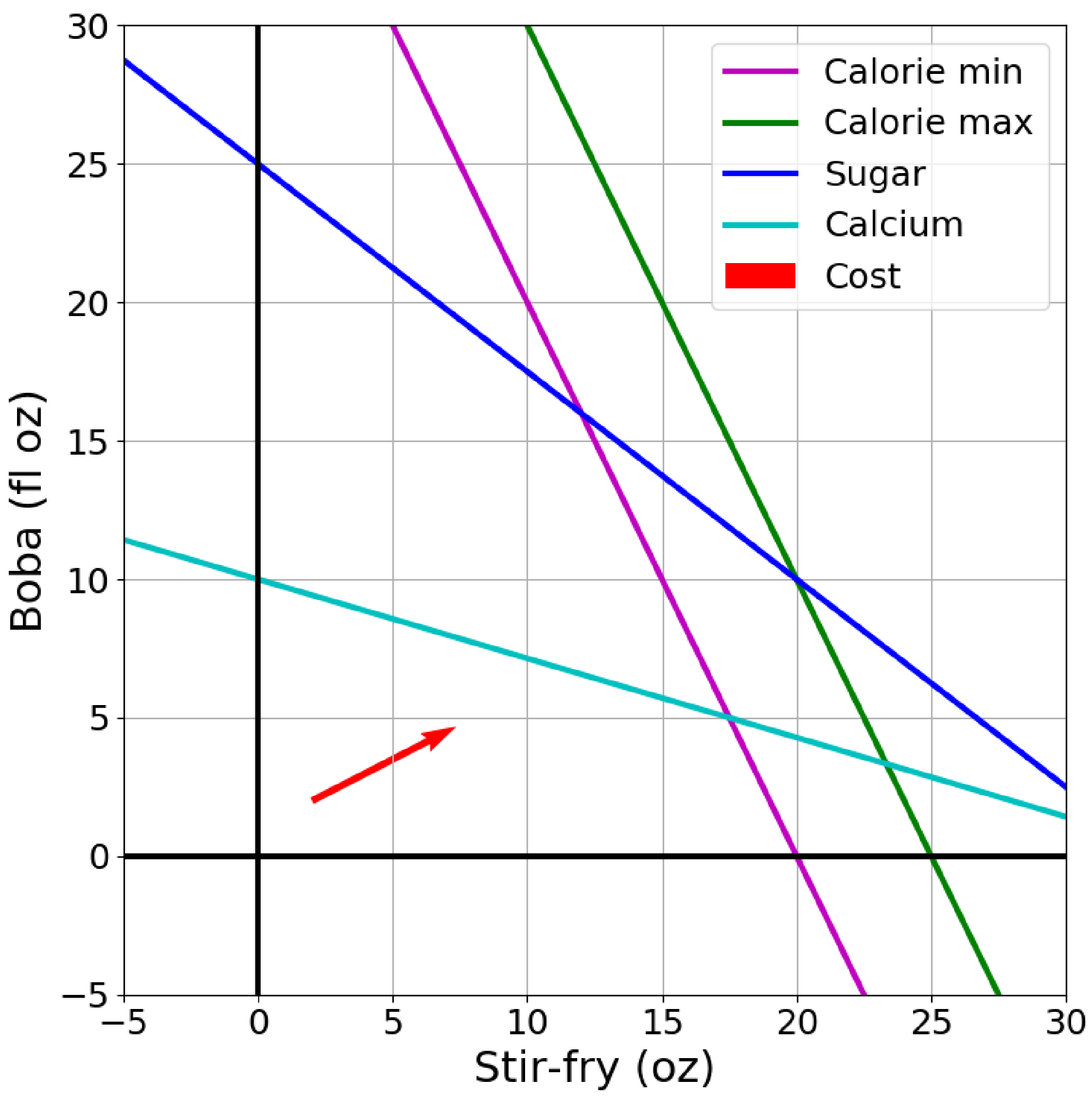
Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections

In more detail:

1. Enumerate all intersections
2. Keep only those that are feasible (satisfy *all* inequalities)
3. Return feasible intersection with the lowest objective value



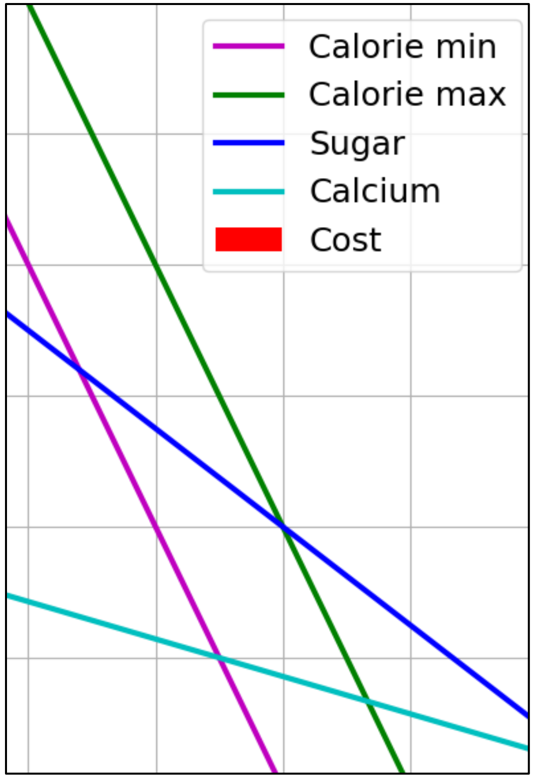
Solving an LP

But, how do we find the intersection between boundaries?

$$\min_{\mathbf{x}} \quad \mathbf{c}^{\top}\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

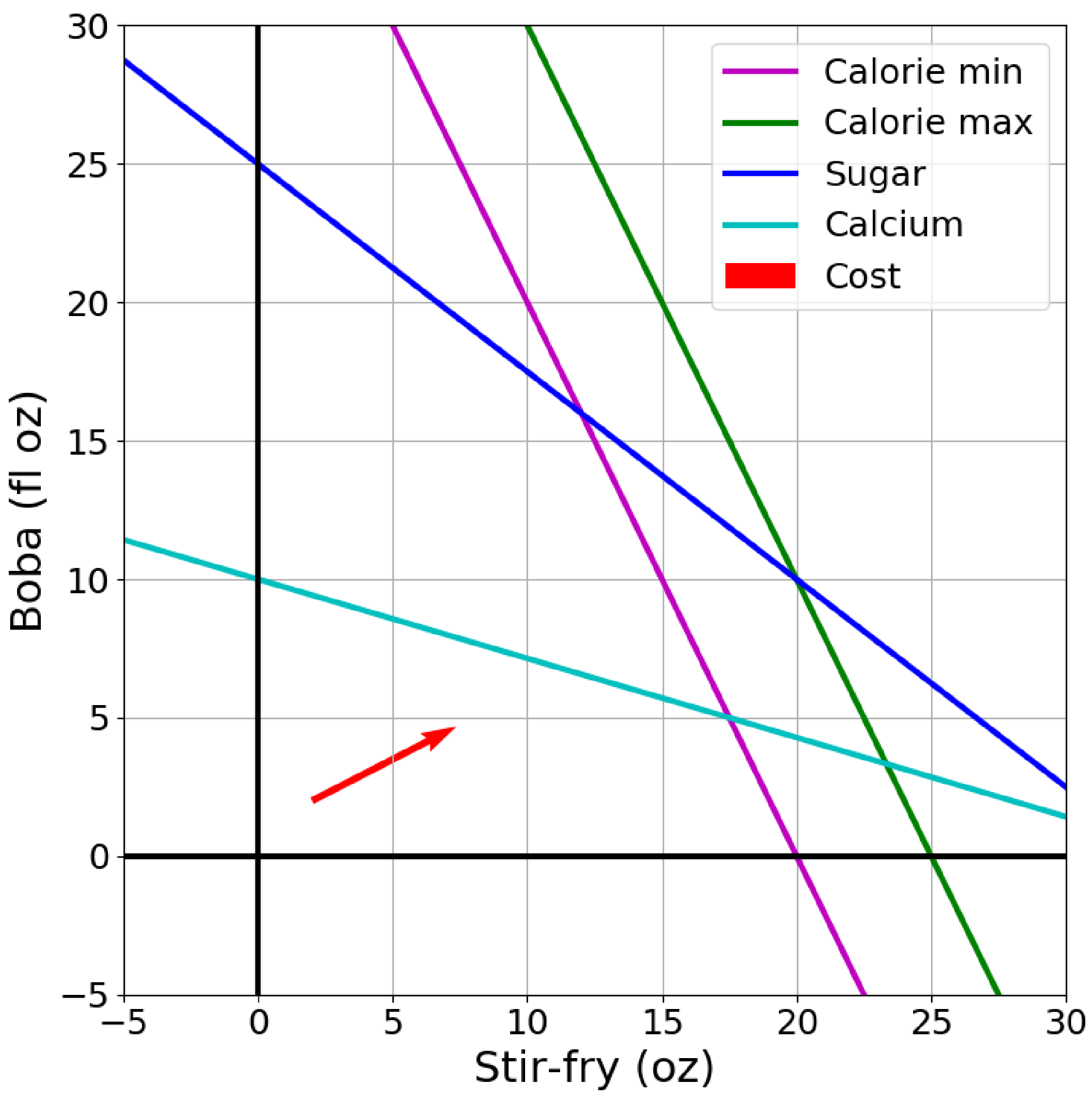


Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

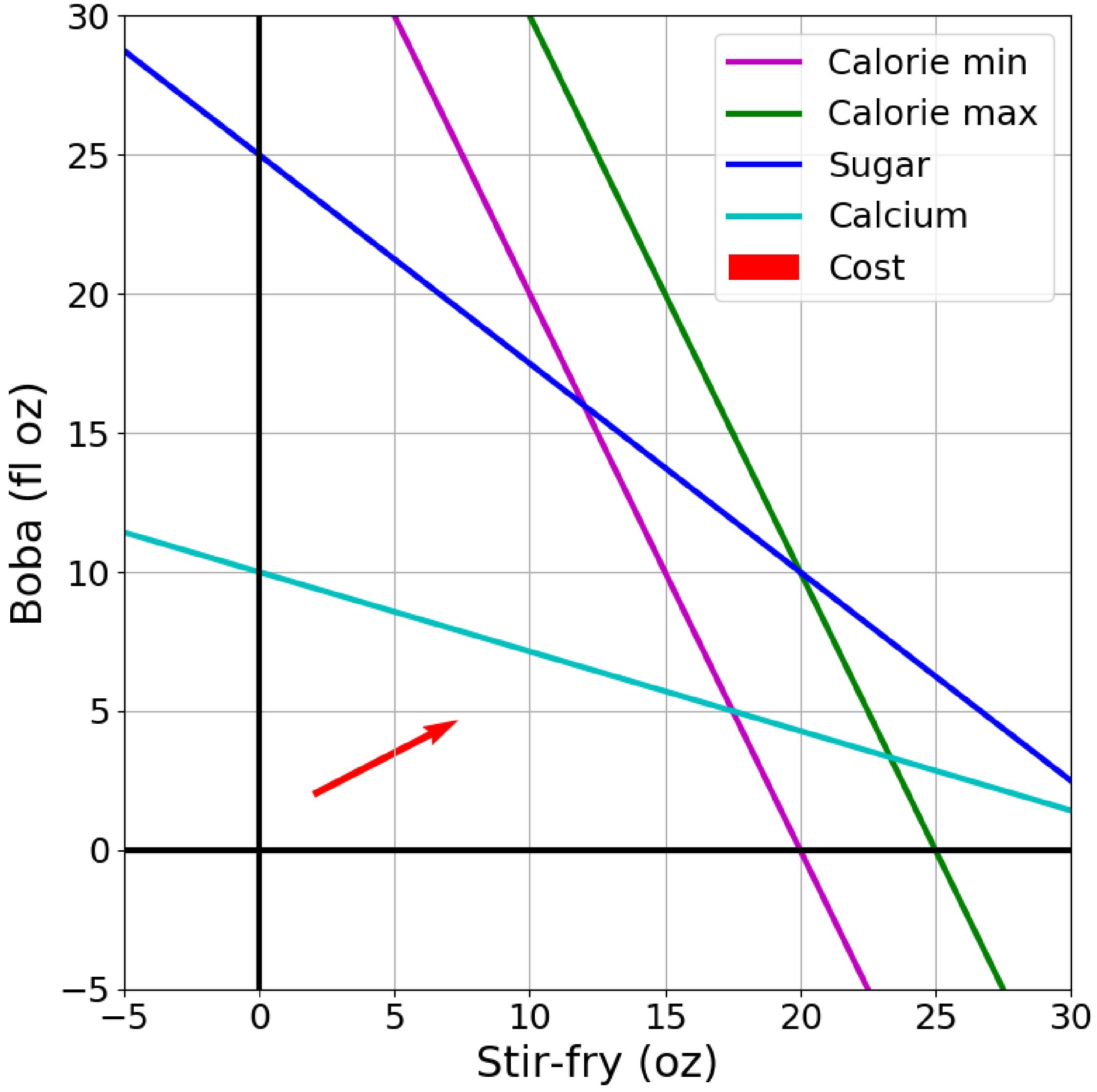
- Check objective at all feasible intersections
- Simplex



Solving an LP

Simplex algorithm

- Start at a feasible intersection (if not trivial, can solve another LP to find one)
- Define successors as “neighbors” of current intersection
  - i.e., remove one row from our square subset of A, and add another row not in the subset; then check feasibility
- Move to any successor with lower objective than current intersection
  - If no such successors, we are done



Greedy local hill-climbing search! … but always finds *optimal* solution

Solving an LP

Solutions are at feasible intersections of constraint boundaries!!

Algorithms

- Check objective at all feasible intersections
- Simplex
- Interior Point

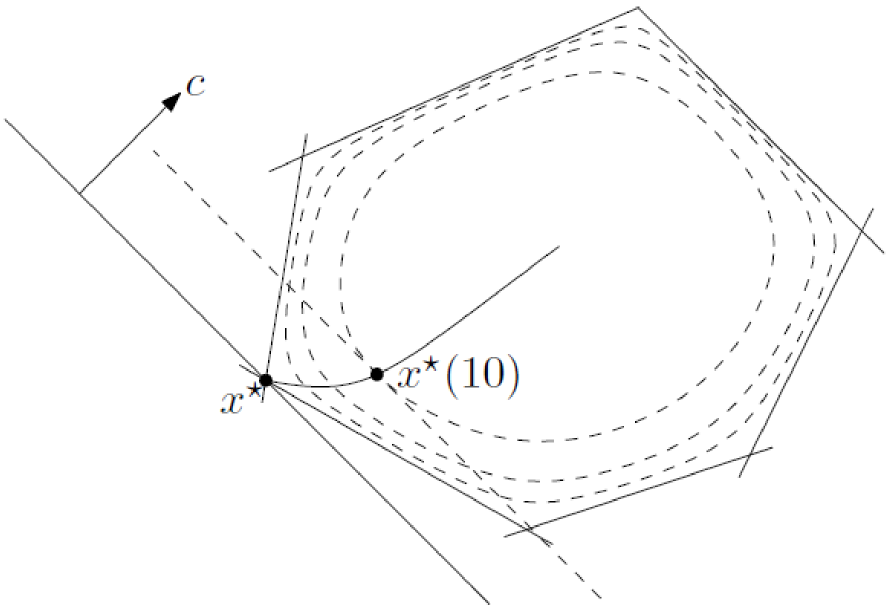


Figure 11.2 from Boyd and Vandenberghe, *Convex Optimization*

What about higher dimensions?

Problem Description

Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
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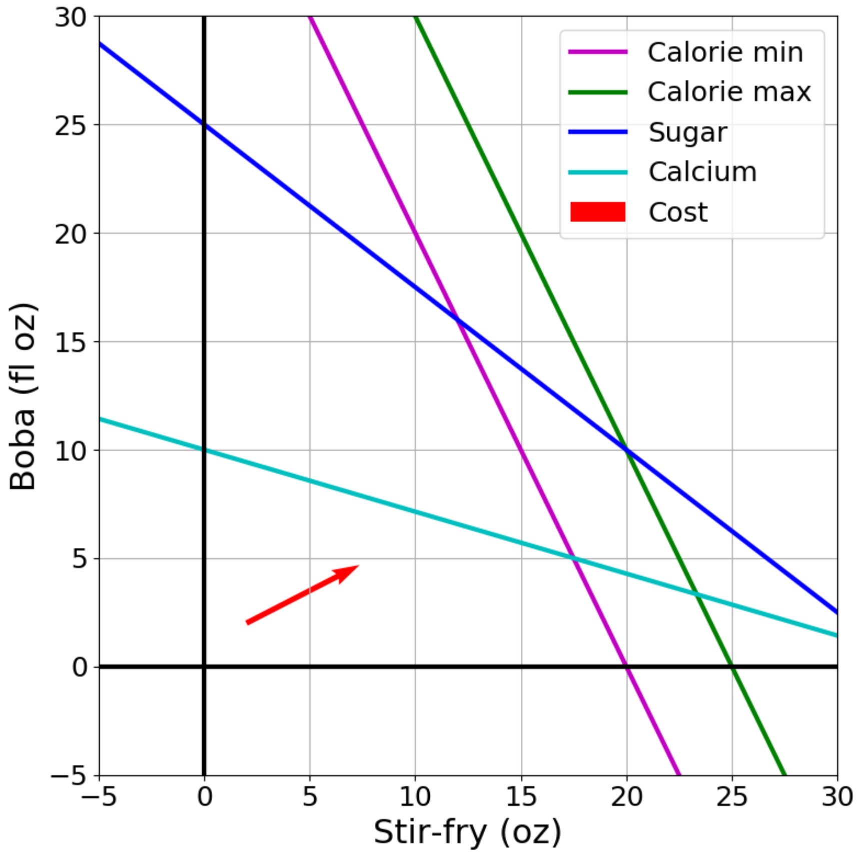
Graphical Representation



“Marty, you’re not thinking fourth-dimensionally”

https://www.youtube.com/watch?v=CUcNM7OsdsY

Shapes in higher dimensions

How do these linear shapes extend to 3-D, N-D?

$$a_1 \, x_1 + a_2 \, x_2 = b_1$$

$$a_1 \, x_1 + a_2 \, x_2 \leq b_1$$

$$a_{1,1} \, x_1 + a_{1,2} \, x_2 \leq b_1$$
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$$a_{3,1} \, x_1 + a_{3,2} \, x_2 \leq b_3$$
$$a_{4,1} \, x_1 + a_{4,2} \, x_2 \leq b_4$$

What are intersections in higher dimensions?

How do these linear shapes extend to 3-D, N-D?

$$\min_{\mathbf{x}} \quad \mathbf{c}^{\top}\mathbf{x}$$
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$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
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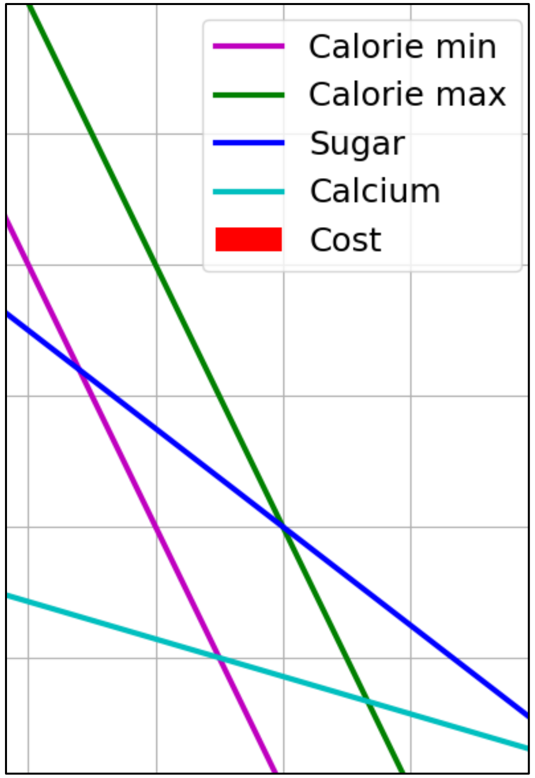


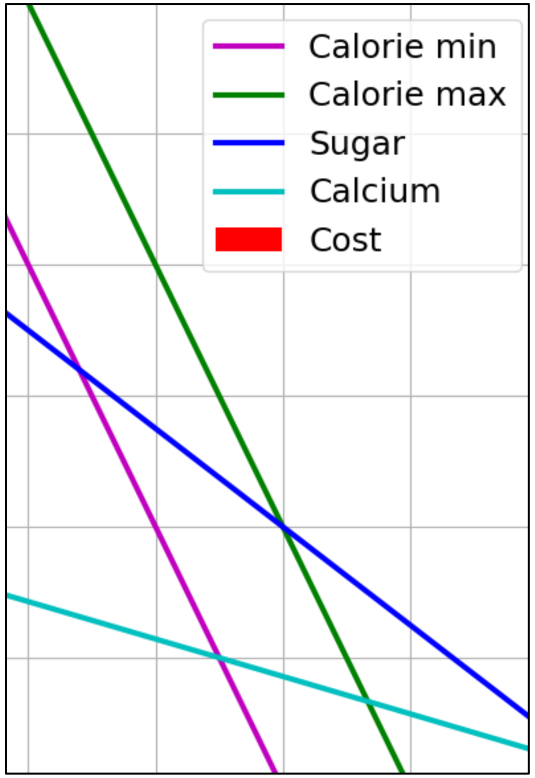
How do we find intersections in higher dimensions?

Still looking at subsets of A matrix

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium



Integer Programming

Linear Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming → Integer Programming

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (bowls) and boba (glasses).

<u>Healthy Squad Goals</u>

- $2000 \leq$ Calories ≤ 2500
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Food	Cost	Calories	Sugar	Calcium
Stir-fry (per bowl)	1	100	3	20
Boba (per glass)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Linear Programming vs Integer Programming

Linear objective with linear constraints, but now with additional constraint that all values in $\boldsymbol{x}$ must be integers

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

We could also do:

- Even more constrained: Binary Integer Programming
- A hybrid: Mixed Integer Linear Programming

Notation Alert!

Integer Programming: Graphical Representation

Just add a grid of integer points onto our LP representation

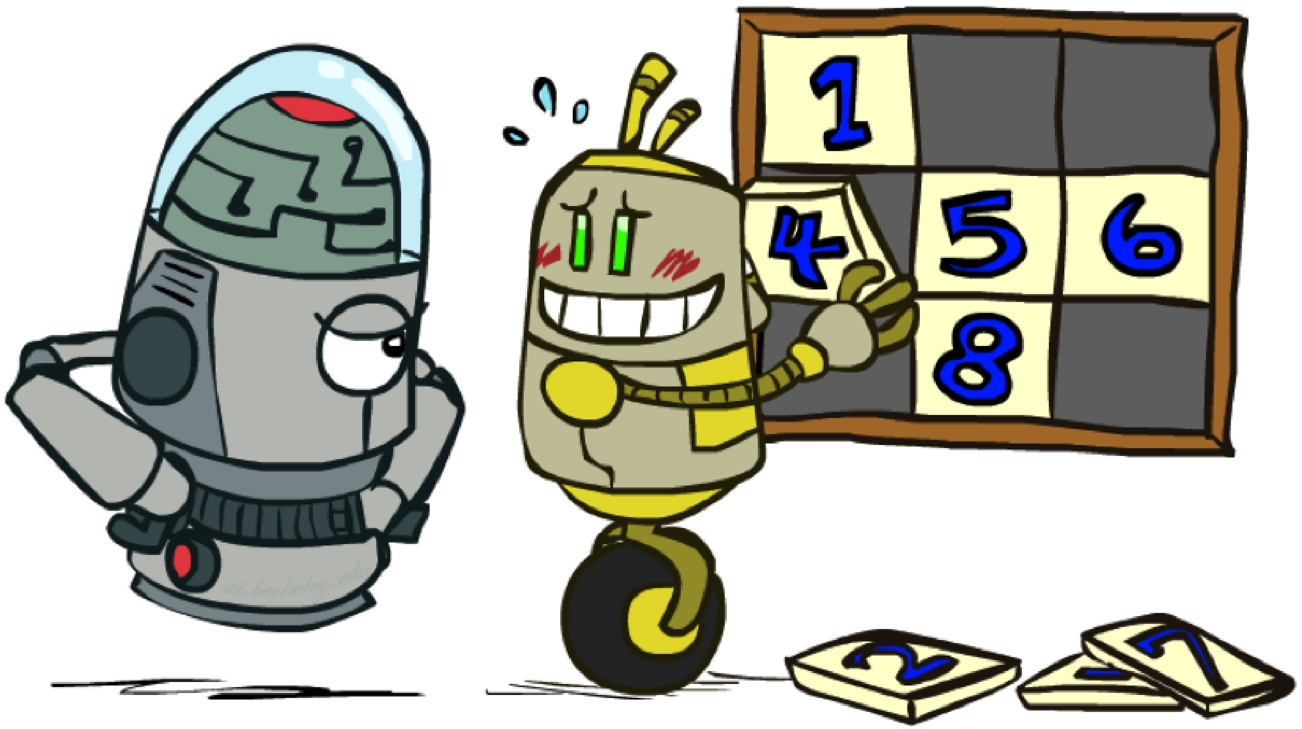
$$\begin{aligned} \min_{\mathbf{x}}. \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^N \end{aligned}$$

Relaxation

Relax IP to LP by dropping integer constraints

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\cancel{\mathbf{x} \in \mathbb{Z}^N}$$

Remember heuristics?



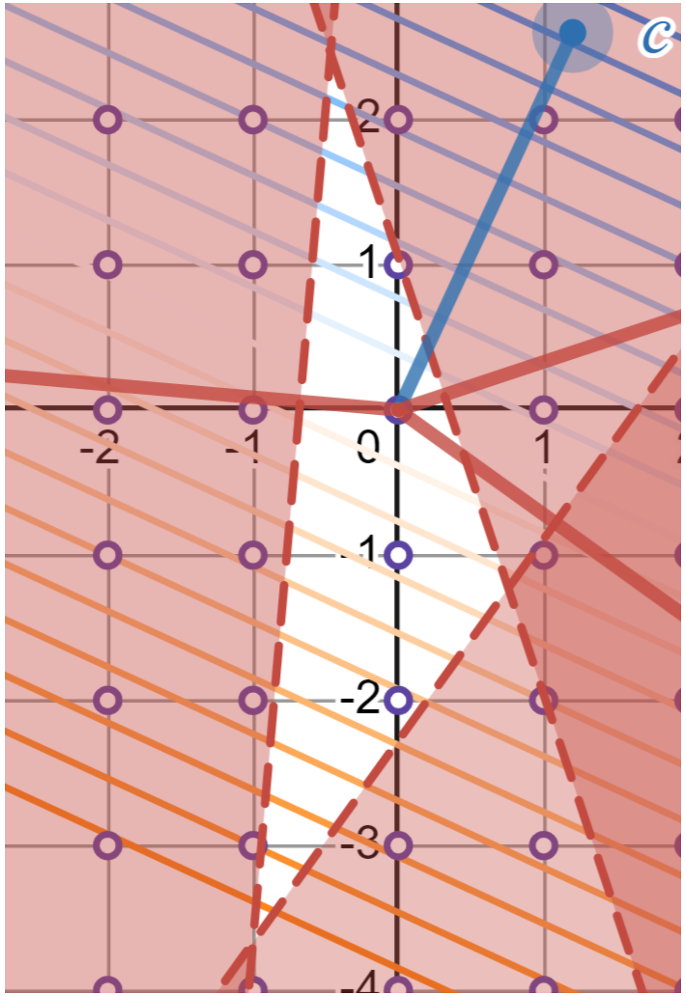
Notation Alert

Let y represent the objective value (total cost), $y = \mathbf{c}^\top \mathbf{x}$

Pay attention to argmin. vs just min.

$$x_{IP}^* = \underset{\mathbf{x}}{\text{argmin.}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \in \mathbb{Z}^N$$

$$y_{IP}^* = \underset{\mathbf{x}}{\text{min.}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$
$$\mathbf{x} \in \mathbb{Z}^N$$



Poll 4:

Let y_{IP}^* be the optimal objective of an integer program P.

Let $\mathbf{x}_{IP}^*$ be an optimal point of an integer program P.

Let y_{LP}^* be the optimal objective of the LP-relaxed version of P.

Let $\mathbf{x}_{LP}^*$ be an optimal point of the LP-relaxed version of P.

Assume that P is a minimization problem.

$$
\begin{aligned}
y_{IP}^* = \min_{\mathbf{x}}.\quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.}\quad & A\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \in \mathbb{Z}^N
\end{aligned}
$$

$$
\begin{aligned}
y_{LP}^* = \min_{\mathbf{x}}.\quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.}\quad & A\mathbf{x} \preceq \mathbf{b}
\end{aligned}
$$

Which of the following are true?

A) $\mathbf{x}_{IP}^* = \mathbf{x}_{\mathrm{LP}}^*$

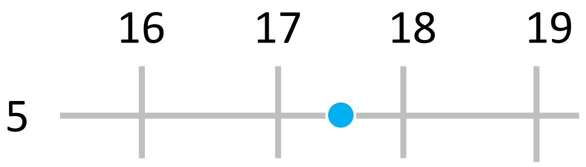
B) $y_{IP}^* \le y_{LP}^*$

C) $y_{IP}^* \ge y_{LP}^*$

Poll 5:

True/False: It is sufficient to consider the integer points around the corresponding LP solution?

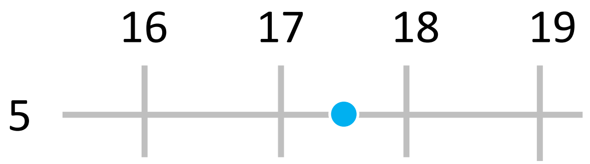
Solving an IP

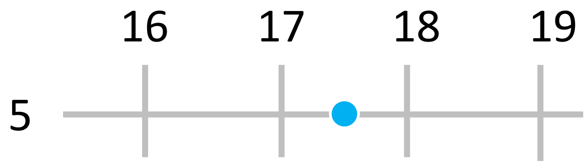


Branch and Bound algorithm

Core steps:

Use LP solver to find $\mathbf{x}_{LP}^{\star}$





If $\mathbf{x}_{LP}^{\star}$ is all integer valued,

return solution

Otherwise:

Create two branches

Left branch: Added constraint $x_i \leq floor(x_i)$

Right branch: Added constraint $x_i \geq ceil(x_i)$

Solving an IP

Branch and Bound algorithm

1. Push LP solution of problem into priority queue,

    ordered by objective value of LP solution

2. Repeat:
    - If queue is empty, return IP is infeasible
    - Pop candidate solution $\mathbf{x}^\star_{LP}$ from priority queue ()
    - If $\mathbf{x}^\star_{LP}$ is all integer valued, we are done; return solution
    - Otherwise, select a coordinate x_i that is not integer valued, and add two additional LPs to the priority queue:

        Left branch: Added constraint $x_i \leq floor(x_i)$

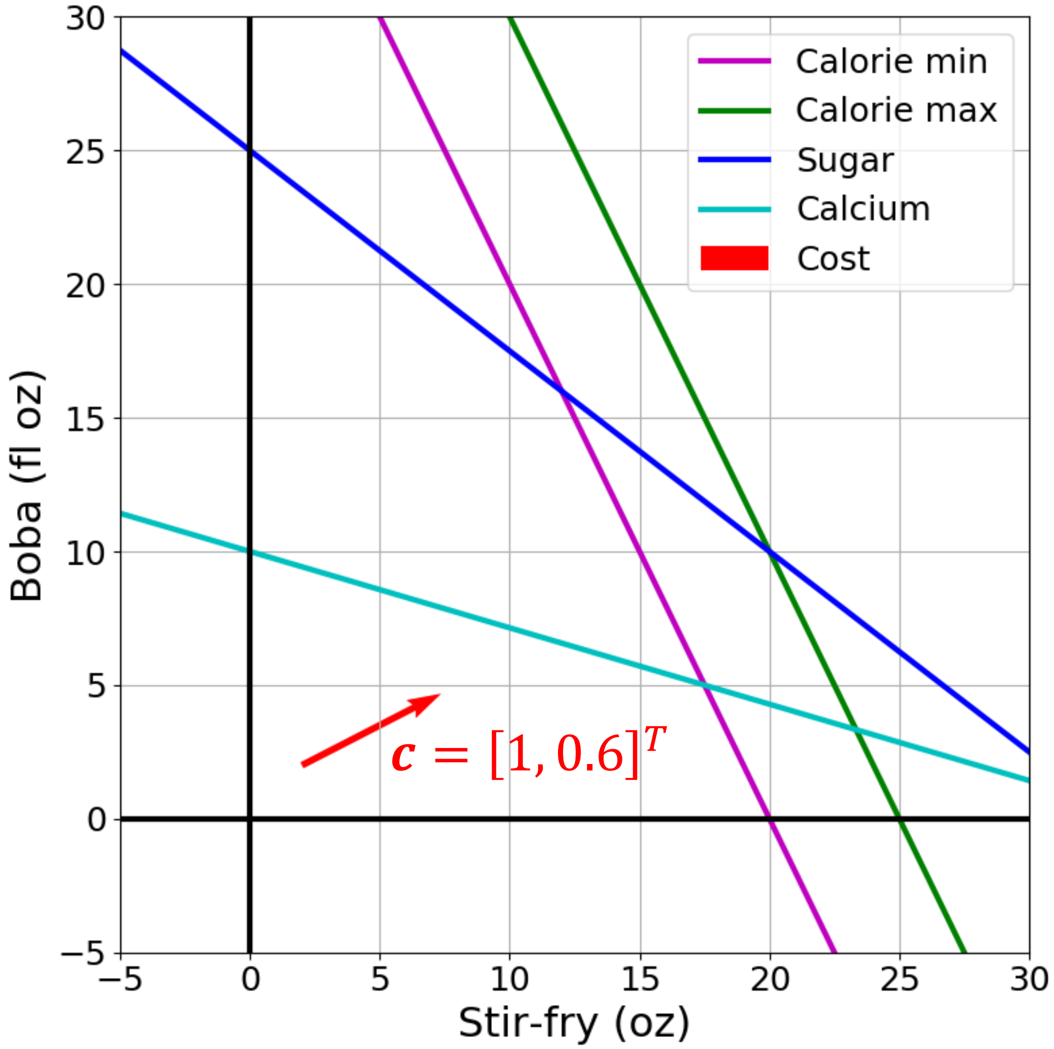
        Right branch: Added constraint $x_i \geq ceil(x_i)$

Note: Only add LPs to the queue if they are feasible

Branch and Bound Example

$\mathbf{x}^\star = (17.5, 5)$
$y^\star = 20.5$

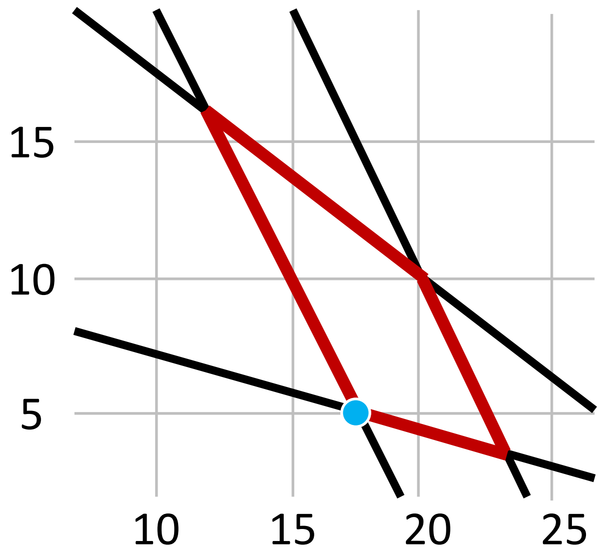
$x_1 \leq 17$ $\quad$ $x_1 \geq 18$



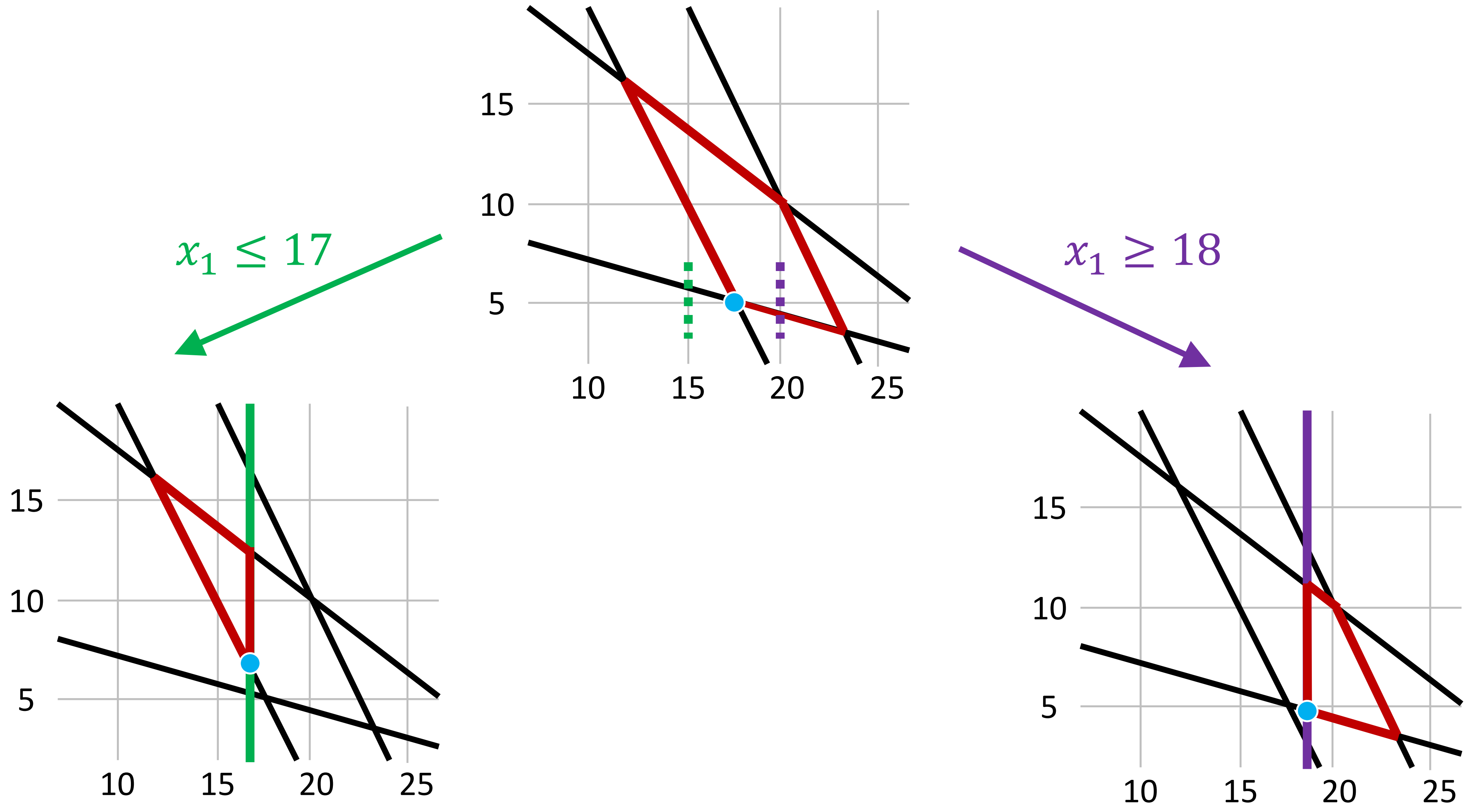
Priority Queue:

1. $\mathbf{x}^\star = (17.5, 5)$, $y^\star = 20.5$

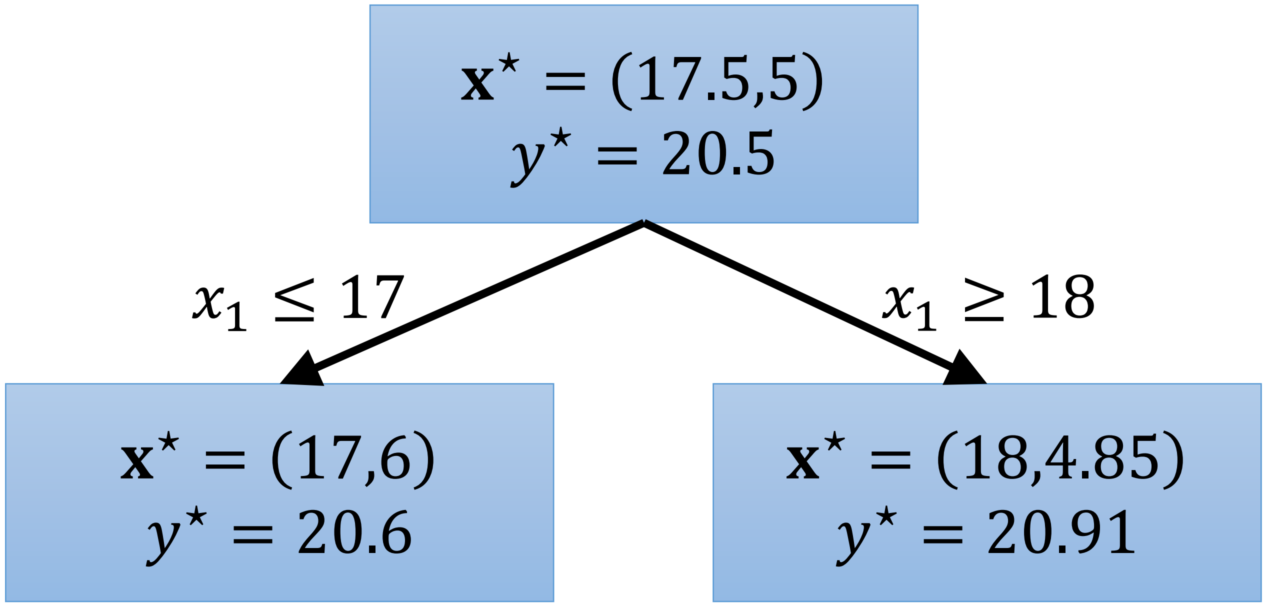
Branch and Bound Example



Branch and Bound Example



Branch and Bound Example



Priority Queue:

1. $\mathbf{x}^\star = (17,6)$, $y^\star = 20.6$
2. $\mathbf{x}^\star = (18,4.85)$, $y^\star = 20.91$